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Thème

Synchronization in fractional difference systems

Présenté Par :
DJENINA Noureddine

Devant le jury :

Mr. Hannachi Fareh	MCA	Univérsité Larbi Tébessi	Président
Mr. Ouannas Adel	MCA	Univérsité Larbi Tébessi	Rapporteur
Mme. Gasri Ahlem	MCB	Univérsité Larbi Tébessi	Examinateur

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Abstract

This thesis, studying synchronization of discrete fractional dynamical systems of incommensurate order, via to the introduction of new stability results for linear incommensurate fractional order difference systems.
Keywords: Fractional-order difference systems, stability, discrete Volterra equation, Z-transform method, synchronization.

Résumé

Cette memoire étudie la synchronisation de systèmes dynamiques discret fractionnaires d'ordre incommensurable, via l'introduction de nouveaux résultats de stabilité pour les systèmes de différence linéaire d'ordre fractionnaire incommensurable.
Les mots clés : Systèmes de différence d'ordre fractionnaire, stabilité, équation de Volterra discrète, méthode de Z-transformation, synchronisation.

ملخص

تتضمن هذه المذكرة، دراسة مزامنة أنظمة حركية متقطعة كسرية ذات رتب مختلفة، وذلك من خلال طرح نتائج استقرار جديدة تتعلق بأنظمة الفروق الكسرية الخطية ذات الرتب المختلفة.
الكلمات المفتاحية : أنظمة الفروق كسرية الرتب, نظرية الاستقرار, معادلة فولتيرا المتقطعة, طريقة تحويلات-Z, مزامنة.

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Introduction

Undoubtedly, it has been demonstrated, over the past few decades, that the non-integer calculus is a forceful mathematical argument for providing much and more dynamics for lots of ancient as well as modern models. For instant, there are several dynamical models and many real-world applications that were recently handled via this tool such as diffusion modeling [1], robot manipulators [2], economics [3], and many more. As a matter of fact, such branch of calculus, in its two structures (discrete and continuous), requires the order of the functional operator of calculus to be in its fractional-order case for a number of core concepts such as derivatives, integrals, and differences. This indeed allows creating appropriate mathematical models together with extremely rich and complex dynamics [4, 5]. In the area of the discrete fractional calculus, it has been turned that the explanation of its main concept is referred to Diaz and Olseronly in 1974 [6]. After that time for at least a decade or more, some serious efforts in this field began to appear. Among those efforts, to name a few, is what Miller and Ross proposed in [7]. They actually established some basic definitions, primary schemes and properties related to the fundamental theorem [8]. To date, a lot of researchers are competing in proposing impressive results and methods in regard to this field. Anyhow, for a complete comprehensive description about this branch of calculus, the reader may refer to [9, 10, 11].

Over the past few years, modeling several chemical and physical phenomena have broadly been carried out using the theory of Fractional-order Difference Systems (FoDSs) [12]. In fact, the FoDS is employed, with its particular digital data, for the purpose of approximating its corresponding fractional-order differential equations. This allows one to enter them into suitable computer programs and then simulate the obtained results [12].

Stability theory addresses the stability of solutions of difference (differential) equations and of trajectories of dynamical systems under small perturbations of initial conditions. However, it was reported in [12] that the Z-transform method can solve the linear FoDSs, as it can perform the same matter for the linear Fractional-order Difference Equation (FoDE). The Z-transform method can be, at the same time, employed as a powerful aid tool in discussing the stability analysis of such systems, as declared in [13, 14, 15, 16]. But, even so, there are formidable challenges, generally, in proposing a proper tool for performing this task [13]. Therefore; it might be said that the stability analysis of the FoDSs is not yet developed [17]. This has motivated, in recent years, many researchers to deal with this dilemma, and as a consequence of this, several other works have preferred it to be the main target of their investigations (see [15, 17, 18, 19]). In particular, Abu-Saris and Al-Mdallal developed in [15] a theory about the stability of linear FoDSs. But, unfortunately, it was shown that their theorem is difficult to be applied. That matter remained until 2015. This year, Čermák et al, developed another simplified and applicable theorem for the same purpose (see [17]).

It's not an easy task to directly extend the normal Lyapunov stability results to fractional cases since the Leibniz law becomes complicated and cannot hold generally. Fahd, J et al in [20] proposed a fractional Lyapunov direct method unfortunately, it was shown that their theorem is difficult to be applied, yet, so far there is no direct way to know the stability of nonlinear FoDS in advance time case, in delay time systems Dumitru B et al give a sufficient condition in [21].

As far as we know, discussing the stability analysis by providing novel simplified results for Linear FoDS with incommensurate orders remains up to date a recent and mostly unexamined topic. In light of this urgent need, this thesis present some simple applicable conditions for judging the stability of such system by first converting it into another equivalent form includes FoDEs of Volterra convolution-type as well as by using the properties of the Z-transform method.

The phenomenon of synchronization has attracted the interest of many researchers from various fields due to its potential applications in nonlinear sciences [22]. Synchronization is the process of controlling the output of a dynamical slave system in order to force its variables to match those of a corresponding master system in time [23]. Various kinds of control schemes have been introduced in the past to synchronize dynamical systems such as complete (anti-) synchronization [24], lag synchronization [25], function projective synchronization [26], generalized synchronization [27], and Q-S synchronization [28]. Recently, the topic of synchronization between dynamical systems described by fractional-order differece equations started to attract increasing attention [29]. Most of the research efforts have been devoted to the study of synchronization problems in commensurate FoDSs. FoDS with incommensurate orders remains unexamined topic.

However, this thesis is organized in the following order. Chapter 1 introduces some primary preliminaries associated with discrete fractional calculus, while Chapter 2 discusses some recently established results that have handled the stability of commensurate FoDS. Chapter 3 Show new results for linear incommensurate FoDS. In Chapter 4 we present an application of the results of Chapter 3 in synchronization.

Chapter 1

Discrete Fractional Calculus

In this chapter, we introduce the basic discrete fractional calculus that will be useful for our later studies. Beginning with integer order discrete calculus, then we introduce the fractional sum and difference of Rimann Liouville and Caputo operators.

1.1 Basic concepts

1.1.1 Integer order difference operator

Definition 1.1 *[30] Assume $f:\mathbb{N}\to\mathbb{R}$, then we define the **forward difference** operator Δ by:*

$$\Delta f(k) := f(k+1) - f(k), \quad \forall k \in \mathbb{N}. \tag{1.1}$$

Also, the operators Δ^N, $N = 1, 2, 3, \cdots$ is defined recursively by $\Delta^N f(k) = \Delta(\Delta^{N-1} f(k))$ for $k \in \mathbb{N}$. Finally, Δ^0 denotes the identity operator, i.e , $\Delta^0 f(k) = f(k)$.

Theorem 1.1 *[30] Assume $f, g : \mathbb{N} \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, then for $k \in \mathbb{N}$ we have:*

(i) $\Delta\alpha = 0$.

(ii) $\Delta\alpha f(k) = \alpha\Delta f(k)$.

(iii) $\Delta[f+g](k) = \Delta f(k) + \Delta g(k)$.

(iv) $\Delta\alpha^{k+\beta} = (\alpha-1)\alpha^{k+\beta}$.

(v) $\Delta[fg](k) = f(k+1)\Delta g(k) + \Delta f(k) g(k)$.

(vi) $\Delta\left(\frac{f}{g}\right)(k) = \frac{g(k)\Delta f(k) - f(k)\Delta g(k)}{g(k)g(k+1)}$,

*where in **(vi)** we assume $g(k) \neq 0$, $\forall k \in \mathbb{N}$.*

Proposition 1.1 *[30] Assume $f:\mathbb{N}\to\mathbb{R}$, then for $k_0\in\mathbb{N}$ we have:*

$$\Delta\sum_{j=k_0}^{k-1} f(j) = f(k), \quad \forall k\in\mathbb{N}, \tag{1.2}$$

and

$$\sum_{j=k_0}^{k-1}\Delta f(j) = f(k)-f(k_0), \quad \forall k\in\mathbb{N}. \tag{1.3}$$

Remark 1.1 *If $k\leq k_0$, we have the common convention:*

$$\sum_{j=k_0}^{k-1} f(j) := 0. \tag{1.4}$$

Theorem 1.2 ***(Summation by parts)****[30] Given two functions $u,v:\mathbb{N}\to\mathbb{R}$ and $a,b\in\mathbb{N}$, $a<b$, we have the summation by parts formulas:*

$$\sum_{j=a}^{b-1} u(j)\Delta v(j) = u(j)v(j)|_a^b - \sum_{j=a}^{b-1} v(j+1)\Delta u(j), \tag{1.5}$$

and

$$\sum_{j=a}^{b-1} u(j+1)\Delta v(j) = u(j)v(j)|_a^b - \sum_{i=a}^{b-1} v(j)\Delta u(j), \tag{1.6}$$

where $u(j)v(j)|_a^b = u(b)v(b)-u(a)v(a)$.

Proof. Let $u,v:\mathbb{N}\to\mathbb{R}$ be given, and $a,b\in\mathbb{N}$, $a<b$, then by Theorem 1.1 (properties **(v)**)

$$\sum_{j=a}^{b-1}\Delta(uv)(j) = \sum_{j=a}^{b-1}\left(v(j+1)\Delta u(j)+\Delta v(j)u(j)\right),$$

after simplification we find

$$u(b)v(b)-u(a)v(a) = \sum_{j=a}^{b-1} v(j+1)\Delta u(j) + \sum_{j=a}^{b-1} u(j)\Delta v(j),$$

this imply

$$\sum_{j=a}^{b-1} u(j)\Delta v(j) = u(j)v(j)|_a^b - \sum_{j=a}^{b-1} v(j+1)\Delta u(j),$$

and

$$\sum_{j=a}^{b-1} u(j+1)\Delta v(j) = u(j)v(j)|_a^b - \sum_{j=a}^{b-1} v(j)\Delta u(j).$$

■

1.1.2 Gamma function

Definition 1.2 *[30] The gamma function is defined by:*

$$\Gamma(z) := \int_0^\infty e^{-t}t^{z-1}dt, \tag{1.7}$$

for those complex numbers z for which the real part of z is positive (it can be shown that the above improper integral converges for all such z).

We have

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t}dt\\ &= 1,\end{aligned}$$

and by integration by parts for $x > 0 : \Gamma(x+1) = x\Gamma(x)$, thats give for any positive integer k

$$\Gamma(k) = (k-1)!.$$

We also have the important estimation:

$$\lim_{k\to\infty}\frac{\Gamma(k+\gamma)}{\Gamma(k)k^\gamma} = 1, \quad k \in \mathbb{N} \quad \text{and} \quad \gamma \in \mathbb{C}.$$

1.1.3 Falling function

Definition 1.3 *[30] For $r \in \mathbb{R}$, we define the falling function, $t^{(r)}$, read t to the r falling, by:*

$$t^{(r)} := \frac{\Gamma(t+1)}{\Gamma(t-r+1)} = t(t-1)(t-2)\cdots(t-r+1), \tag{1.8}$$

for those values of t and r such that the right-hand side of this last equation make sense. We then extend this definition by making the common convention that $t^{(r)} := 0$ when $t-r+1$ is a nonpositive integer.

Remark 1.2 *When $r = 0$, we set $t^{(0)} := 1$.*

Theorem 1.3 ***(Power Rules)****[30] For $r, a, t \in \mathbb{R}$, we have the following power rules:*

$$\Delta(t+a)^{(r)} = r(t+a)^{(r-1)}, \tag{1.9}$$

and

$$\Delta(a-t)^{(r)} = -r(a-t-1)^{(r-1)}, \tag{1.10}$$

hold, whenever the expressions in these two formulas are well defined.

1.1.4 Binomial coefficient

Definition 1.4 *[30] The* ***(generalized)*** *binomial coefficient* $\binom{t}{r}$ *is defined by:*

$$\binom{t}{r} := \frac{\Gamma(t+1)}{\Gamma(t-r+1)\Gamma(r+1)} = \frac{t^{(r)}}{\Gamma(r+1)}, \tag{1.11}$$

for those values of t *and* r *so that the right-hand side is well defined.*

Remark 1.3 *For* $0 < t \leq 1$ *and* $r \in \mathbb{N}$,

$$\begin{aligned} \binom{t+r-1}{r} &= \frac{(t+r-1)^{(r)}}{\Gamma(r+1)} = \frac{(t+r-1)(t+r-2)\dots(t)}{r!} \\ &= (-1)^r \frac{(-t)(-t-1)\dots(-t-r+1)}{r!} \\ &= (-1)^r \binom{-t}{r}. \end{aligned}$$

Theorem 1.4 ***(Pascal rule)****[17] For* $r, t \in \mathbb{R}$, *we heve:*

$$\binom{t-1}{r} + \binom{t-1}{r-1} = \binom{t}{r}. \tag{1.12}$$

Proof. Consider

$$\begin{aligned} \binom{t-1}{r} + \binom{t-1}{r-1} &= \frac{\Gamma(t)}{\Gamma(t-r)\Gamma(r+1)} + \frac{\Gamma(t)}{\Gamma(t-r+1)\Gamma(r)} \\ &= \Gamma(t)\left(\frac{t-r}{\Gamma(t-r+1)\Gamma(r+1)} + \frac{r}{\Gamma(t-r+1)\Gamma(r+1)}\right) \\ &= \Gamma(t)\frac{t}{\Gamma(t-r+1)\Gamma(r+1)} \\ &= \frac{\Gamma(t+1)}{\Gamma(t-r+1)\Gamma(r+1)} \\ &= \binom{t}{r}. \end{aligned}$$

■

Theorem 1.5 ***(Repeated summation rule)****[30] Let* $f : \mathbb{N} \to \mathbb{R}$ *be given and* $a \in \mathbb{N}$ *then:*

$$\sum_{\tau_1=a}^{t-1}\sum_{\tau_2=a}^{\tau_1-1}\cdots\sum_{\tau_p=a}^{\tau_{p-1}-1} f(\tau_p) = \sum_{j=a}^{t-1}\frac{(t-j-1)^{(p-1)}}{(p-1)!}f(j), \tag{1.13}$$

for $t = p+a, p+a+1, \cdots$.

Proof. We will prove this by induction on p for $p \geq 1$. The case $p = 1$ is trivially true. Assume (1.13) holds for some $p \geq 1$. It remains to show that (1.13) then holds when p is replaced by $p+1$. To this end, let

$$y(t) := \sum_{\tau_1=a}^{t-1}\sum_{\tau_2=a}^{\tau_1-1}\cdots\sum_{\tau_{p+1}=a}^{\tau_p-1} f(\tau_{p+1}).$$

Let $g(\tau_p) = \sum\limits_{\tau_{p+1}=a}^{\tau_p - 1} f(\tau_{p+1})$, then it follows from the induction assumption that

$$y(t) = \sum_{j=a}^{t-1} \frac{(t-j-1)^{(p-1)}}{(p-1)!} g(j),$$

using summation by parts formula

$$y(t) = -\frac{(t-j)^{(p)}}{(p)!} \sum_{i=a}^{j-1} f(i) \Big|_a^t + \sum_{j=a}^{t-1} \frac{(t-j-1)^{(p)}}{(p)!} f(j) = \sum_{j=a}^{t-1} \frac{(t-j-1)^{(p)}}{(p)!} f(j).$$

■

Motivated by (1.13), we define the p-th integer sum operator Δ_a^{-p} for positive integers p, by:

$$\Delta_a^{-p} f(t) := \sum_{j=a}^{t-1} \frac{(t-j-1)^{(p-1)}}{(p-1)!} f(j).$$

But, sence

$$\frac{(t-j-1)^{(p-1)}}{(p-1)!} = 0, \quad j = t-1, t-2, \cdots, t-p+1,$$

we obtaine

$$\Delta_a^{-p} f(t) = \sum_{j=a}^{t-p} \frac{(t-j-1)^{(p-1)}}{(p-1)!} f(j). \tag{1.14}$$

Remark 1.4 *We have for $t = p+a, p+a+1, \cdots$*

$$\Delta_a^{-p} \Delta^p f(t) = f(t) - \sum_{j=0}^{p-1} \frac{(t-a)^{(j)}}{j!} \Delta^j f(a),$$

and

$$\Delta^p \Delta_a^{-p} f(t) = f(t).$$

1.1.5 The Z-transform

Definition 1.5 *[31] The Z-transform of a sequence $(x(k))_{k\in\mathbb{Z}}$, which is identically zero for negative integers k (i.e., $x(k) = 0$ for $k = -1, -2, \cdots$), is defined by:*

$$\tilde{x}(z) = Z(x(k)) := \sum_{k=0}^{\infty} x(k) z^{-k}, \tag{1.15}$$

where z is a complex number.

The set of numbers z in the complex plane for which series (1.15) converges is called the region of convergence of $\tilde{x}(z)$. The most commonly used method to find the region of convergence of the series (1.15) is the ratio test. Suppose that:

$$\lim_{k\to\infty}\left|\frac{x(k+1)}{x(k)}\right| = R.$$

The series (1.15) converges in the region $|z| > R$ and diverges for $|z| < R$.

Properties of the Z-transform [31]

- **Linearity:**
  Let $\tilde{x}(z)$ be the Z-transform of $(x(k))_{k\in\mathbb{Z}}$ with radius of convergence R_1, and let $\tilde{y}(z)$ be the Z-transform of $(y(k))_{k\in\mathbb{Z}}$ with radius of convergence R_2. Then for any complex numbers α, β we have

  $$Z\left[\alpha x(k) + \beta y(k)\right] = \alpha\tilde{x}(z) + \beta\tilde{y}(z), \quad \text{for } |z| > max(R_1, R_2).$$

- **Shifting**:
  Let R be the radius of convergence of $\tilde{x}(z)$

  - Right-shifting: If $x(-i) = 0$ for $i = 1, 2, \cdots, j$, then

    $$Z\left[x(k-j)\right] = z^{-j}\tilde{x}(z) \quad \text{for } |z| > R.$$

  - Left-shifting:

    $$Z\left[x(k+j)\right] = z^{-j}\tilde{x}(z) - \sum_{r=0}^{j-1} x(r)z^{j-r} \quad \text{for } |z| > R.$$

- **Initial and final value:**

  - Initial value theorem:

    $$\lim_{|z|\to\infty}\tilde{x}(z) = x(0).$$

  - Final value theorem:

    $$x(\infty) = \lim_{k\to\infty} x(k) = \lim_{z\to 1}(z-1)\tilde{x}(z).$$

- **Convolution:**
  The convolution $*$ of two sequences $(x(k))_{k\in\mathbb{Z}}, (y(k))_{k\in\mathbb{Z}}$ is defined by

  $$x(k) * y(k) = \sum_{r=0}^{k} x(k-r)y(r) = \sum_{r=0}^{k} x(r)y(k-r).$$

Now,

$$Z\left[x(k) * y(k)\right] = \tilde{x}(z)\tilde{y}(z).$$

- **Multiplication by a^k proper:**
  Suppose that $\tilde{x}(z)$ is the Z-transform of $(x(k))_{k\in\mathbb{Z}}$ with radius of convergence R. Then

$$Z\left[a^k x(k)\right] = \tilde{x}\left(\frac{z}{a}\right), \quad \text{for } |z| > |a|\, R.$$

The inverse Z-transform (Integral Method) [31]

From the definition of the Z-transform, we have

$$\tilde{x}(z) = \sum_{j=0}^{\infty} x(j) z^{-j}.$$

Multiplying both sides of the above equation by z^{k-1}, $k \in \mathbb{N}$, we get

$$\tilde{x}(z) z^{k-1} = \sum_{j=0}^{\infty} x(j) z^{k-j-1}. \tag{1.16}$$

Equation (1.16) gives the Laurent series expansion of $\tilde{x}(z)z^{k-1}$ around $z = 0$.
Consider a circle C, centered at the origin of the z-plane, that encloses all poles of $\tilde{x}(z)z^{k-1}$. Since $x(k)$ is the coefficient of z^{-1}, it follows by the Cauchy integral formula that

$$x(k) = \frac{1}{2\pi i} \oint_C \tilde{x}(z) z^{k-1} dz,$$

and by the residue theorem we obtain

$$x(k) = \text{sum of residues of } \tilde{x}(z) z^{k-1}.$$

1.1.6 Volterra difference equations of convolution type

Definition 1.6 *[32] The Volterra difference equations of convolution type are of the form:*

$$x(k+1) = Ax(k) + \sum_{j=0}^{k} B(k-j) x(j), \tag{1.17}$$

where $x(k) \in \mathbb{R}^n, \forall k \in \mathbb{N}$, $A = (a_{ij})_{1\leq i,j\leq n}$ is a $n \times n$ real matrix and $B(k)$ is a $n \times n$ real matrix defined on $\mathbb{N}$.

If $(B(k))_{k\in\mathbb{N}} \in \ell^1(\mathbb{N}^{n\times n})$, the resolvent matrix $R(k)$ of (1.17) is defined as the unique solution of the matrix equation:

$$R(k+1) = AR(k) + \sum_{j=0}^{k} B(k-j)R(j), \quad R(0) = I_n, k \in \mathbb{N},$$

where I_n is the identity matrix. Let $(x(k))_{k\in\mathbb{N}}$ denote the solution of the equation

$$x(k+1) = Ax(k) + \sum_{j=0}^{k} B(k-j)x(j) + g(k).$$

Then by the variation of constants formula, we obtain

$$x(k) = R(k)x(0) + \sum_{j=0}^{k-1} R(k-j-1)g(j).$$

Let

$$h(k) = \sum_{j=0}^{\infty} \left| \sum_{i=0}^{k-1} R(k-i-1)B(i+j+1) \right|. \tag{1.18}$$

Theorem 1.6 *[32] If $(B(k))_{k\in\mathbb{N}} \in \ell^1(\mathbb{N}^{n\times n})$, for equation (1.17) the following statements are equivalent:*
(1) $\det(zI - A - \tilde{B}(z)) \neq 0$ for $|z| \geq 1$.
(2) $(R(k))_{k\in\mathbb{N}} \in \ell^1(\mathbb{N}^{n\times n})$.
(3) The zero solution of (1.17) is uniformly asymptotically stable.
(4) Both $R(k)$ and $h(k)$ of (1.18) tend to zero as $k \to \infty$.

1.1.7 Open mapping theorem

Definition 1.7 *[33] A map f from an open set $\Omega \subset \mathbb{C}$ to $\mathbb{C}$ is an open mapping if the image by f of any open subset of Ω is open.*

Theorem 1.7 ***(Open mapping theorem)[33]*** *If $f(z)$ is a nonconstant analytic on an open connected set Ω, then f is an open mapping.*

1.2 Fractional discrete operators

In this section, we define fractional sum and **Riemann Liouville** and **Caputo** difference operators, give some of their properties and the relation between them. Frequently, the functions we consider will be defined on a set of the form:

$$\mathbb{N}_a := \{a, a+1, a+2, ...\},$$

where $a \in \mathbb{R}$.

1.2.1 Fractional sum operator

First we begine with the fractonal sum, we use the (**Repeated Summation Rule)** and (**Gamma function)** properties, and replaced the natural number p bay a real positive number α we get the definition of fractional order sum.

Definition 1.8 *[34] Let $\alpha > 0$. Then, the $\alpha - th$ fractional sum of $f : \mathbb{N}_a \rightarrow \mathbb{R}$ is defined by:*

$$\Delta_a^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s), \tag{1.19}$$

for any $t \in \mathbb{N}_{a+\alpha}$.

Remark 1.5 *Note that $\Delta_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$.*

Lemma 1.1 *[30] Assume $\mu \geq 0$ and $\alpha > 0$. Then:*

$$\Delta_{a+\mu}^{-\alpha} (t-a)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-a)^{(\mu+\alpha)}, \tag{1.20}$$

for any $t \in \mathbb{N}_{a+\mu+\alpha}$.

Proof. Let

$$g_1(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-a)^{(\mu+\alpha)}$$

and

$$g_2(t) = \Delta_{a+\mu}^{-\alpha} (t-a)^{(\mu)} = \frac{1}{\Gamma(\alpha)} \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s-a)^{(\mu)},$$

for $t \in \mathbb{N}_{a+\mu+\alpha}$. To get advanced in this proof we will show that both of these functions satisfy the initial value problem

$$(t-a-(\mu+\alpha)+1)\Delta g(t) = (\mu+\alpha) g(t), \tag{1.21}$$

$$g(a+\mu+\alpha) = \Gamma(\mu+1). \tag{1.22}$$

Since

$$\begin{aligned} g_1(a+\mu+\alpha) &= \tfrac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(\mu+\alpha)^{(\mu+\alpha)} \\ &= \Gamma(\mu+1), \end{aligned}$$

and

$$\begin{aligned} g_2(a+\mu+\alpha) &= \tfrac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{a+\mu}(a+\mu+\alpha-s-1)^{(\alpha-1)}(s-a)^{(\mu)} \\ &= \tfrac{1}{\Gamma(\alpha)}(\alpha-1)^{(\alpha-1)}\mu^{(\mu)} \\ &= \Gamma(\mu+1), \end{aligned}$$

we have that $g_i(t), i = 1,2$ both satisfy the initial condition (1.22).
We next show that $g_1(t)$ satisfies the difference equation (1.21). Note that

$$\Delta g_1(t) = (\mu+\alpha)\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{(\mu+\alpha-1)}.$$

Multiplying both sides by $(t-a-(\mu+\alpha)+1)$ we obtain

$$\begin{aligned} (t-a-(\mu+\alpha)+1)\Delta g_1(t) &= (\mu+\alpha)\tfrac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}\left[t-a-(\mu+\alpha-1)\right](t-a)^{(\mu+\alpha-1)} \\ &= (\mu+\alpha)\tfrac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{(\mu+\alpha)} \\ &= (\mu+\alpha)g_1(t), \end{aligned}$$

for $t \in \mathbb{N}_{a+\mu+\alpha}$. That is, $g_1(t)$ is a solution of (2.21).
It remains to show that $g_2(t)$ satisfies (2.21). We have that

$$\begin{aligned} g_2(t) &= \tfrac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s-a)^{(\mu)} \\ &= \tfrac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}\left[(t-s-1)-(\alpha-2)\right](t-s-1)^{(\alpha-2)}(s-a)^{(\mu)} \\ &= \tfrac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}\left[(t-a-(\mu+\alpha)+1)-(s-a-\mu)\right](t-s-1)^{(\alpha-2)}(s-a)^{(\mu)} \\ &= \tfrac{t-a-(\mu+\alpha)+1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-2)}(s-a)^{(\mu)} \\ &\quad -\tfrac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-2)}(s-a-\mu)(s-a)^{(\mu)} \\ &= h(t)-k(t), \end{aligned}$$

where

$$h(t) = \frac{t-a-(\mu+\alpha)+1}{\Gamma(\alpha)}\sum_{r=a+\mu}^{t-\alpha}(t-r-1)^{(\alpha-2)}(r-a)^{(\mu)}$$

and

$$\begin{aligned} k(t) &= \tfrac{1}{\Gamma(\alpha)} \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-2)}(s-a-\mu)(s-a)^{(\mu)} \\ &= \tfrac{1}{\Gamma(\alpha)} \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-2)}(s-a)^{(\mu+1)}. \end{aligned}$$

So we get

$$\begin{aligned} \Delta g_2(t) &= \tfrac{1}{\Gamma(\alpha)} \Delta \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-1)}(s-a)^{(\mu)} \\ &= \tfrac{1}{\Gamma(\alpha)} \left[\sum_{s=a+\mu}^{t+1-\alpha} (t-s)^{(\alpha-1)}(s-a)^{(\mu)} - \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-1)}(s-a)^{(\mu)} \right] \\ &= \tfrac{1}{\Gamma(\alpha)} \left[\sum_{s=a+\mu}^{t-\alpha} \Delta_t (t-s-1)^{(\alpha-1)}(s-a)^{(\mu)} + (\alpha-1)^{(\alpha-1)}(t+1-\alpha-a)^{(\mu)} \right] \\ &= \tfrac{\alpha-1}{\Gamma(\alpha)} \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-2)}(s-a)^{(\mu)} + \tfrac{1}{\Gamma(\alpha)}(\alpha-1)^{(\alpha-1)}(t+1-\alpha-a)^{(\mu)} \\ &= \tfrac{\alpha-1}{\Gamma(\alpha)} \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-2)}(s-a)^{(\mu)} + (t+1-\alpha-a)^{(\mu)}. \end{aligned}$$

It follows that

$$(t-a-(\mu+\alpha)+1)\Delta g_2(t) = (\alpha-1)h(t) + (t+1-\alpha-a)^{(\mu+1)}.$$

Also, by summation by parts we get

$$\begin{aligned} k(t) &= \tfrac{1}{\Gamma(\alpha)} \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-2)}(s-a)^{(\mu+1)} \\ &= \frac{-(t-s)^{(\alpha-1)}(s-a)^{(\mu+1)}}{(\alpha-1)\Gamma(\alpha)} \Big|_{s=a+\mu}^{s=t+1-\alpha} \\ &\quad + \tfrac{\mu+1}{(\alpha-1)\Gamma(\alpha)} \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-1)}(s-a)^{(\mu)} \\ &= -\frac{(t+1-\alpha-a)^{(\mu+1)}}{(\alpha-1)} + \frac{\mu+1}{(\alpha-1)\Gamma(\alpha)} \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-1)}(s-a)^{(\mu)}. \end{aligned}$$

It follows that

$$(t+1-\alpha-a)^{(\mu+1)} = -(\alpha-1)k(t) + (\mu+1)g_2(t).$$

Finally, we get

$$\begin{aligned} (t-a-(\mu+\alpha)+1)\Delta g_2(t) &= (\alpha-1)h(t) + (t+1-\alpha-a)^{(\mu+1)} \\ &= (\alpha-1)h(t) - (\alpha-1)k(t) + (\mu+1)g_2(t) \\ &= (\mu+\alpha)g_2(t). \end{aligned}$$

This completes the proof. ∎

Theorem 1.8 *[34] Let $f : \mathbb{N}_a \to \mathbb{R}$ be given, and let $\mu, \nu > 0$. Then:*

$$\Delta_{a+\mu}^{-\nu}[\Delta_a^{-\mu} f(t)] = \Delta_a^{-(\mu+\nu)} f(t) = \Delta_{a+\nu}^{-\mu}[\Delta_a^{-\nu} f(t)], \tag{1.23}$$

for any $t \in \mathbb{N}_{a+\mu+\nu}$.

Proof. By definition of fractional sum, we have

$$\begin{aligned}
\Delta_{a+\nu}^{-\mu}[\Delta_a^{-\nu} f(t)] &= \frac{1}{\Gamma(\nu)} \Delta_{a+\nu}^{-\mu} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=a+\nu}^{t-\mu} (t-r-1)^{(\mu-1)} \sum_{s=a}^{r-\nu} (r-s-1)^{(\nu-1)} f(s) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=a+\nu}^{t-\mu} \sum_{s=a}^{r-\nu} (t-r-1)^{(\mu-1)} (r-s-1)^{(\nu-1)} f(s) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{s=a}^{t-(\mu+\nu)} \sum_{r=s+\nu}^{t-\mu} (t-r-1)^{(\mu-1)} (r-s-1)^{(\nu-1)} f(s) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{s=a}^{t-(\mu+\nu)} \left(\sum_{x=\nu-1}^{t-s-1-\mu} (t-x-s-2)^{(\mu-1)} x^{(\nu-1)} \right) f(s) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-(\mu+\nu)} \left(\Delta_{\nu-1}^{-\mu} (\tau)^{(\nu-1)} \big|_{\tau=t-s-1} \right) f(s) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-(\mu+\nu)} \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} (t-s-1)^{(\nu+\mu-1)} f(s) \quad \text{(by using Lemma 1.1)} \\
&= \Delta_a^{-(\mu+\nu)} f(t).
\end{aligned}$$

■

Theorem 1.9 *[34] Let $f : \mathbb{N}_a \to \mathbb{R}$ be given, for any $\alpha > 0$ and any positive integer p, we have:*

$$\Delta_a^{-\alpha} \Delta^p f(t) = \Delta^p \Delta_a^{-\alpha} f(t) - \sum_{r=0}^{p-1} \frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)} \Delta^r f(a), \tag{1.24}$$

for any $t \in \mathbb{N}_{a+\alpha}$.

Proof. We will prove this by induction on p for $p \geq 1$.

- The case where $p = 1$. We have by definition of fractional sum:

$$\begin{aligned}
\Delta_a^{-\alpha} \Delta f(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} (\Delta f)(s) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} (f(s+1) - f(s)) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+1) - \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t+1-\alpha}(t-s)^{(\alpha-1)}f(s) - \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s)\\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t+1-\alpha}(t+1-s-1)^{(\alpha-1)}f(s) - \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}f(a)\\
&= \Delta\left(\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s)\right) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}f(a)\\
&= \Delta\Delta_a^{-\alpha}f(t) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}f(a).
\end{aligned}
$$

- Assume (1.24) holds for some $p \geq 1$. It remains to show that (1.24) then holds when p is replaced by $p+1$. We have

$$
\begin{aligned}
\Delta_a^{-\alpha}\Delta^{p+1}f(t) &= \Delta_a^{-\alpha}\Delta^{p}(\Delta f)(t)\\
&= \Delta^p\Delta_a^{-\alpha}\Delta f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)}(\Delta^r\Delta f)(a) \quad \text{(from the induction assumption)}\\
&= \Delta^p\Delta_a^{-\alpha}\Delta f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)}\Delta^{r+1}f(a)\\
&= \Delta^p\left(\Delta_a^{-\alpha}\Delta f(t)\right) - \sum_{r=1}^{p}\frac{(t-a)^{(\alpha-p+r-1)}}{\Gamma(\alpha+r-p)}\Delta^r f(a)\\
&= \Delta^p\left(\Delta\Delta_a^{-\alpha}f(t) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}f(a)\right) - \sum_{r=1}^{p}\frac{(t-a)^{(\alpha-p+r-1)}}{\Gamma(\alpha+r-p)}\Delta^r f(a)\\
&= \Delta^{p+1}\Delta_a^{-\alpha}f(t) - \frac{(t-a)^{(\alpha-1-p)}}{\Gamma(\alpha-p)}f(a) - \sum_{r=1}^{p}\frac{(t-a)^{(\alpha-p+r-1)}}{\Gamma(\alpha+r-p)}\Delta^r f(a)\\
&= \Delta^{p+1}\Delta_a^{-\alpha}f(t) - \sum_{r=0}^{p}\frac{(t-a)^{(\alpha-p+r-1)}}{\Gamma(\alpha+r-p)}\Delta^r f(a).
\end{aligned}
$$

■

1.2.2 Rimann Liouville operator

Definition 1.9 *[34] Let $\alpha > 0$. Then, the $\alpha-$order* ***Rimann Liouville*** *fractional difference of a function f defined on $\mathbb{N}_a$ are defined by:*

$$
\Delta_a^{\alpha}f(t) := \Delta^p\Delta_a^{-(p-\alpha)}f(t) = \frac{1}{\Gamma(p-\alpha)}\Delta^p\sum_{s=a}^{t-(p-\alpha)}(t-s-1)^{(p-\alpha-1)}f(s), \quad \forall t \in \mathbb{N}_{a+p-\alpha}, \tag{1.25}
$$

where $p = [\alpha]+1$.

Remark 1.6 *If $\alpha \longrightarrow p \in \mathbb{N}$, then*

$$
\begin{aligned}
\lim_{\alpha\to p}\Delta_a^{\alpha}f(t) &= \Delta^p\Delta^{(0)}f(t)\\
&= \Delta^p f(t),
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{\alpha\to p-1}\Delta_a^{\alpha}f(t) &= \Delta^p\Delta^{-1}f(t)\\
&= \Delta^{p-1}f(t).
\end{aligned}
$$

We can say that the Δ_a^α operator is an interpolation of Δ^p operators. Also, it is clear that Δ_a^α maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+(p-\alpha)}$.
Note that:

$$\begin{aligned}\Delta_a^\alpha 1 &= \Delta^p \Delta_a^{-(p-\alpha)} 1\\ &= \tfrac{1}{\Gamma(p-\alpha)}\Delta^p \sum_{s=a}^{t-(p-\alpha)} (t-s-1)^{(p-\alpha-1)}\\ &= \tfrac{1}{\Gamma(1-\alpha)}(t-a)^{(-\alpha)}\\ &\neq 0,\end{aligned}$$

generally.

Theorem 1.10 *[34] Assume $\alpha > 0$ and f is defined on $\mathbb{N}_a$. Then:*

$$\Delta_{a+\alpha}^{\alpha}\Delta_a^{-\alpha} f(t) = f(t), \quad \forall t \in \mathbb{N}_{a+p}, \tag{1.26}$$

where $p = [\alpha] + 1$.

Proof. Simple by using Definition 1.9 and Theorem 1.8, for any $t \in \mathbb{N}_{a+p}$:

$$\begin{aligned}\Delta_{a+\alpha}^{\alpha}\Delta_a^{-\alpha} f(t) &= \Delta^p \Delta_{a+\alpha}^{-(p-\alpha)} \Delta^{-\alpha} f(t)\\ &= \Delta^p \Delta^{-p} f(t)\\ &= f(t).\end{aligned}$$

■

Remark 1.7 *We see that $\Delta_{a+\alpha}^{\alpha}$ have a right inverse $(\Delta_a^{-\alpha})$, or $\Delta_{a+\alpha}^{-\alpha}$ have a left inverse (Δ_a^{α}).*

Theorem 1.11 *[34] Assume $\alpha > 0$ and f is defined on $\mathbb{N}_a$. Then:*

$$\Delta_{a+p-\alpha}^{-\alpha}\Delta_a^{\alpha} f(t) = f(t) - \sum_{r=0}^{p-1} \frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)} \Delta_a^{r-(p-\alpha)} f(a), \ \ \forall t \in \mathbb{N}_{a+p}, \tag{1.27}$$

where $p = [\alpha] + 1$.

Proof. By Definition 1.9, for any t from $\mathbb{N}_{a+p}$, we have

$$\Delta_{a+p-\alpha}^{-\alpha}\Delta_a^{\alpha} f(t) = \Delta_{a+p-\alpha}^{-\alpha}\Delta^p \Delta_a^{-(p-\alpha)} f(t),$$

using Theorem 1.9

$$\Delta_{a+p-\alpha}^{-\alpha}\Delta_a^{\alpha} f(t) = \Delta^p \Delta_{a+p-\alpha}^{-\alpha}\Delta_a^{-(p-\alpha)} f(t) - \sum_{r=0}^{p-1} \frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)} \Delta^r \Delta_a^{-(p-\alpha)} f(a),$$

by using Theorem 1.8

$$\begin{aligned}\Delta_{a+p-\alpha}^{-\alpha}\Delta_a^{\alpha}f(t) &= \Delta^p\Delta_a^{-p}f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)}\Delta_a^{r-(p-\alpha)}f(a)\\ &= f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)}\Delta_a^{r-(p-\alpha)}f(a).\end{aligned}$$

■

Remark 1.8 *We see that*

$$\Delta_{a+\alpha}^{-\alpha}\Delta_a^{\alpha}f(t) \neq f(t)$$

generally.

1.2.3 Caputo operator

Definition 1.10 *[34] Let $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then, the $\alpha-$order **Caputo** fractional difference of a function f defined on $\mathbb{N}_a$, are defined by:*

$$^C\Delta_a^{\alpha}f(t) := \Delta_a^{-(p-\alpha)}\Delta^p f(t) = \frac{1}{\Gamma(p-\alpha)}\sum_{s=a}^{t-(p-\alpha)}(t-s-1)^{(p-\alpha-1)}\Delta^p f(s), \quad \forall t \in \mathbb{N}_{a+p-\alpha}, \tag{1.28}$$

where $p = [\alpha] + 1$.

Remark 1.9 *When $\alpha \to p-1 \in \mathbb{N}$, we get*

$$\begin{aligned}\lim_{\alpha\to p-1} {}^C\Delta_a^{\alpha}\ f(t) &= \lim_{\alpha\to p-1}\Delta_a^{-(p-\alpha)}\Delta^p f(t)\\ &= \Delta_a^{-1}\Delta^p f(t)\\ &= \Delta^{p-1}f(t) - \Delta^{p-1}f(a).\end{aligned}$$

So $^C\Delta_a^{\alpha}$ is not an interpolation of Δ^p. However, it is used in modeling because it is consistent with the classical initial and boundary conditions. To not be confused we defined $^C\Delta_a^p$ by $^C\Delta_a^p f(t) := \Delta^p f(t)$, where $p \in \mathbb{N}$. Also, it is clear that $^C\Delta_a^{\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+(p-\alpha)}$.

Remark 1.10 *The fractional Caputo difference of a constant $c \in \mathbb{R}$ is*

$$^C\Delta_a^{\alpha}c = \Delta_a^{-(p-\alpha)}\Delta^p c = 0.$$

Theorem 1.12 *[34] Assume $\alpha > 0$ and f is defined on $\mathbb{N}_a$. Then:*

$$\Delta_{a+(p-\alpha)}^{-\alpha}\ {}^C\Delta_a^{\alpha}\ f(t) = f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(r)}}{r!}\Delta^r f(a), \quad \forall t \in \mathbb{N}_a, \tag{1.29}$$

where $p = [\alpha] + 1$.
In particular, if $0 < \alpha \leq 1$ *then*

$$\Delta_{a+(1-\alpha)}^{-\alpha} \ {}^{C}\Delta_a^{\alpha} \ f(t) = f(t) - f(a), \quad \forall t \in \mathbb{N}_a. \tag{1.30}$$

Proof. By Definition 1.10, for any t from $\mathbb{N}_a$, we have:

$$\Delta_{a+(p-\alpha)}^{-\alpha} \ {}^{C}\Delta_a^{\alpha} \ f(t) = \Delta_{a+(p-\alpha)}^{-\alpha} \Delta_a^{-(p-\alpha)} \Delta^p f(t),$$

using Theorem 1.8

$$\begin{aligned} \Delta_{a+(p-\alpha)}^{-\alpha} \ {}^{C}\Delta_a^{\alpha} \ f(t) &= \Delta_a^{-p} \Delta^p f(t) \\ &= f(t) - \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a). \end{aligned}$$

■

Theorem 1.13 *[34] Let* $\alpha > 0$*, and* f *is defined on* $\mathbb{N}_a$*. Then:*

$${}^{C}\Delta_a^{\alpha} \ f(t) = \Delta_a^{\alpha} f(t) - \sum_{r=0}^{p-1} \frac{(t-a)^{(r-\alpha)}}{\Gamma(r-\alpha+1)} \Delta^r f(a), \quad \forall t \in \mathbb{N}_{a+p-\alpha}, \tag{1.31}$$

where $p = [\alpha] + 1$.
In particular, when $0 < \alpha < 1$*, we have*

$${}^{C}\Delta_a^{\alpha} \ f(t) = \Delta_a^{\alpha} f(t) - \frac{(t-a)^{(-\alpha)}}{\Gamma(1-\alpha)} f(a), \quad \forall t \in \mathbb{N}_{a-\alpha+1}. \tag{1.32}$$

Proof. By Definition 1.10

$${}^{C}\Delta_a^{\alpha} \ f(t) = \Delta_a^{-(p-\alpha)} \Delta^p f(t),$$

by using Theorem 1.9

$$\begin{aligned} {}^{C}\Delta_a^{\alpha} \ f(t) &= \Delta^p \Delta_a^{-(p-\alpha)} f(t) - \sum_{r=0}^{p-1} \frac{(t-a)^{(r-\alpha)}}{\Gamma(r-\alpha+1)} \Delta^r f(a) \\ &= \Delta_a^{\alpha} f(t) - \sum_{r=0}^{p-1} \frac{(t-a)^{(r-\alpha)}}{\Gamma(r-\alpha+1)} \Delta^r f(a). \end{aligned}$$

■

Theorem 1.14 ***(Discrete Taylor's Formula)****[35] Let* f *be defined on* $\mathbb{N}_a$*. Then, for all* $t \in \mathbb{N}_a$ *and* $p \in \mathbb{N}$ *withe* $p \geq 1$ *:*

$$f(t) = \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{(p-1)!} \sum_{r=a}^{t-p} (t-r-1)^{(p-1)} \Delta^p f(r). \tag{1.33}$$

Proof. The proof is by induction. For $p = 1$. (1.33) is the same as

$$f(t) = f(a) + \sum_{r=a}^{t-1} \Delta f(r) = f(a) + f(t) - f(a) = f(t).$$

Assume (1.33) holds for some $p \geq 1$. It remains to show that (1.33) then holds when p is replaced by $p + 1$. We have

$$\begin{aligned} f(t) &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{(p-1)!} \sum_{r=a}^{t-p} (t-r-1)^{(p-1)} \Delta^p f(r) \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) - \frac{1}{p!} (t-r)^{(p)} \Delta^p f(r) \mid_{r=a}^{r=t-p+1} + \frac{1}{p!} \sum_{r=a}^{t-p} (t-r-1)^{(p)} \Delta^{p+1} f(r) \\ &= \sum_{r=0}^{p} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{p!} \sum_{r=a}^{t-(p+1)} (t-r-1)^{(p)} \Delta^{p+1} f(r). \end{aligned}$$

■

Theorem 1.15 *[36] Let $\alpha > 0$, $\alpha \notin \mathbb{N}$ and f is defined on $\mathbb{N}_a$. Then:*

$$f(t) = \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{\Gamma(\alpha)} \sum_{r=a+p-\alpha}^{t-\alpha} (t-r-1)^{(\alpha-1)} \ {}^C\Delta_a^\alpha \ f(r), \quad \forall t \in \mathbb{N}_{a+p}, \tag{1.34}$$

where $p = [\alpha] + 1$.

Proof. *Notice that by (Discrete Taylor's Formula)*

$$\begin{aligned} f(t) &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{(p-1)!} \sum_{r=a}^{t-p} (t-r-1)^{(p-1)} \Delta^p f(r) \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \Delta_a^{-p} \Delta^p f(t) \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \Delta_a^{-(p-\alpha)-\alpha} \Delta^p f(t) \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \Delta_{a+p-\alpha}^{-\alpha} \Delta_a^{-(p-\alpha)} \Delta^p f(t) \quad \textit{(by using Theorem 1.8)} \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \Delta_{a+p-\alpha}^{-\alpha} \ {}^C\Delta_a^\alpha \ f(t) \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{\Gamma(\alpha)} \sum_{r=a+p-\alpha}^{t-\alpha} (t-r-1)^{(\alpha-1)} \ {}^C\Delta_a^\alpha \ f(r), \quad \forall t \in \mathbb{N}_{a+p}. \end{aligned}$$

■

Chapter 2

Stability of fractional difference systems

2.1 Introduction

In this chapter, we will present some stability theorems of fractional order difference systems. First, we will give a background on stability of integer order difference systems. Then we will move to the study of stability in fractional order case. The systems we will be concerned in studying their stability are written in the general form as follow:

$$\begin{cases} {}^{C}\Delta_{t_0}^{\alpha} x(t) = f(t+\alpha-1, x(t+\alpha-1)), \quad t \in \mathbb{N}_{a+1-\alpha}, \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}_1, t_0 \in \mathbb{N}_a, x(t) \in \mathbb{R}^n$, $f : \mathbb{N}_a \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $0 < \alpha \leq 1$.
A point $x_e \in \mathbb{R}^n$ is a fixed (equilibrium) point of (2.1) if and only if

$$f(t, x_e) = 0, \quad \forall t \in \mathbb{N}_a.$$

We will be concerned with the stability of the equilibrium point $x_e = 0$ (all cases can be transferred to be 0 the equilibrium point) of the discrete time systems (2.1).
Let $\|.\|$ be a norm on $\mathbb{R}^n$.

Definition 2.1 *[20] The trivial solution $x(t) = 0$ (or the equilibrium point $x = 0$) of (2.1) is said to be*
(i) *stable if, for each $\epsilon > 0$ and $t_0 \in \mathbb{N}_a$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that for any solution $x(t) = x(t, t_0, x_0)$, with $\|x_0\| < \delta$ one has $\|x(t)\| < \epsilon$, for all $t \in \mathbb{N}_{t_0} \subseteq \mathbb{N}_a$,*
(ii) *uniformly stable if it is stable and δ depends solely on ϵ,*
(iii) *asymptotically stable if it is stable and for all $t_0 \in \mathbb{N}_a$ there exists $\delta = \delta(t_0) > 0$ if $\|x_0\| < \delta$*

implies that $\lim_{t\to\infty} x(t, t_0, x_0) = 0$,

(iv) *uniformly asymptotically stable if it is uniformly stable and, for each* $\epsilon > 0$, *there exists* $T = T(\epsilon) \in \mathbb{N}$ *and* $\delta_0 > 0$ *such that* $\|x_0\| < \delta_0$ *implies* $\|x(t)\| < \epsilon$ *for all* $t \in \mathbb{N}_{t_0+T}$ *and for all* $t_0 \in \mathbb{N}_a$,

(v) *globally asymptotically stable if it is asymptotically stable for all* $x_0 \in \mathbb{R}^n$,

(vi) *globally uniformly asymptotically stable if it is uniformly asymptotically stable for all* $x_0 \in \mathbb{R}^n$.

2.2 Background on stability of integer order difference systems

2.2.1 Stability of linear systems

Consider the integer order difference system:

$$\begin{cases} \Delta x(k) = Ax(k), \quad k \in \mathbb{N}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.2}$$

where $x(k) = (x_1(k), x_2(k), \cdots, x_n(k))^T \in \mathbb{R}^n$ and A is an $n \times n$ constant matrix.
It has an equilibrium point at the origin $(x = 0)$. The solution of the linear system (2.2) starting from x_0 has the form

$$x(k) = (A + I_n)^k x_0, \quad \forall k \in \mathbb{N}, \tag{2.3}$$

where I_n is the identity matrix. We have the following result on the stability of linear system (2.2).

Theorem 2.1 *[31] If all the eigenvalues* λ_j *of* A *satisfies* $|\lambda_j + 1| < 1$, $1 \le j \le n$, *then the trivial solution of (2.2) is globally asymptotically stable on* $\mathbb{N}$.
Furthermore, if there is an eigenvalue λ *of* A *withe* $|\lambda + 1| > 1$, *then the trivial solution of (2.2) is unstable on* $\mathbb{N}$.

Example 2.1 *Consider the following linear system:*

$$\Delta x(k) = \begin{pmatrix} 0 & -5 \\ \frac{1}{4} & -2 \end{pmatrix} x(k), \quad k \in \mathbb{N}. \tag{2.4}$$

The characteristic equation for $A = \begin{pmatrix} 0 & -5 \\ \frac{1}{4} & -2 \end{pmatrix}$ *is* $\lambda^2 + 2\lambda + \frac{5}{4} = 0$ *and hence the eigenvalues of* A *are* $\lambda_1 = -1 + \frac{i}{2}$ *and* $\lambda_2 = -1 - \frac{i}{2}$. *Since*

$$|\lambda_1 + 1| = |\lambda_2 + 1| = \frac{1}{2} < 1,$$

then, by Theorem 2.1 the trivial solution of (2.4) is globally asymptotically stable on $\mathbb{N}$.

Example 2.2 *Consider the following linear system:*

$$\Delta x(k) = \begin{pmatrix} 1 & 6 \\ 0 & -2 \end{pmatrix} x(k), \quad k \in \mathbb{N}. \tag{2.5}$$

The characteristic equation for $A = \begin{pmatrix} 1 & 6 \\ 0 & -2 \end{pmatrix}$ *is* $\lambda^2 + 3\lambda + 2 = 0$ *and hence the eigenvalues of* A *are* $\lambda_1 = 1$ *and* $\lambda_2 = -2$. *Since*

$$|\lambda_2| = 2 > 1,$$

then, by Theorem 2.1 the trivial solution of (2.5) is unstable on $\mathbb{N}$.

Remark 2.1 *Let* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *be the eigenvalues of* A. *Assume* $|\lambda_j + 1| \leq 1, 1 \leq j \leq n$.
If whenever $|\lambda_j + 1| = 1$, *then* λ_j *is a simple eigenvalue of* A. *Then the trivial solution of (2.2) is stable on* $\mathbb{N}$.
If there is a non simple eigenvalue λ *of* A *satisfying* $|\lambda + 1| = 1$, *then we can't conclude.*

Example 2.3 *Consider the following system:*

$$\Delta x(k) = \begin{pmatrix} \cos\theta - 1 & \sin\theta \\ -\sin\theta & \cos\theta - 1 \end{pmatrix} x(k), \quad k \in \mathbb{N}, \tag{2.6}$$

where θ *is a real number. For each* θ *the eigenvalues of the coefficient matrix in (2.6) are* $\lambda_{1,2} = e^{\pm i\theta} - 1$. *Since* $|\lambda_1 + 1| = |\lambda_2 + 1| = 1$ *and both eigenvalues are simple, we have by Remark 2.1 that the trivial solution of (2.6) is stable on* $\mathbb{N}$.

2.2.2 Stability of non-linear systems

Consider the following non-linear system:

$$\begin{cases} \Delta x(k) = f(x(k)), \quad k \in \mathbb{N}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.7}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function, and suppose $f(0) = 0$, that is $x = 0$ is an equilibrium point for system (2.7).

Linearisation method

Theorem 2.2 *[31] Let J be the Jacobian matrix of f at 0.*
If all the eigenvalues $\lambda_j, 1 \leq j \leq n$, of J satisfies $|\lambda_j + 1| < 1$, then the trivial solution of (2.7) is asymptotically stable on $\mathbb{N}$.
Furthermore, if there is an eigenvalue λ of A withe $|\lambda + 1| > 1$, then the trivial solution of (2.7) is unstable on $\mathbb{N}$.

Example 2.4 *Consider the following non-linear system:*

$$\begin{cases} \Delta x_1(k) = \frac{2x_2(k)}{(1+x_1^2(k))} - x_1(k), \\ \Delta x_2(k) = \frac{x_1(k)}{(1+x_2^2(k))} - x_2(k). \end{cases} \tag{2.9}$$

Let $f = (f_1, f_2)^T$, where $f_1 = \frac{2x_2(k)}{(1+x_1^2(k))} - x_1(k)$ and $f_2 = \frac{x_1(k)}{(1+x_2^2(k))} - x_2(k)$, then the Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{\partial f_1(0)}{\partial x_1} & \frac{\partial f_1(0)}{\partial x_2} \\ \frac{\partial f_2(0)}{\partial x_1} & \frac{\partial f_2(0)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

the characteristic equation for $J = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ is $\lambda^2 - 2 = 0$ and hence the eigenvalues of J are $\lambda_{1,2} =^{+}_{-} \sqrt{2}$. Since

$$|\lambda_1 + 1| = 1 + \sqrt{2} > 1, \tag{2.10}$$

then, by Theorem 2.1 the trivial solution of (2.9) is unstable on $\mathbb{N}$.

Lyapunov direct method

Theorem 2.3 *[31] If there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$, (called **Lyapunov function**) which is continuous and such that:*

$$\begin{aligned} &V(0) = 0 \text{ and } V(x(k)) > 0, \ \forall x(k) \neq 0, \\ &\Delta V(x(k)) = V(x(k+1)) - V(x(k)) \leq 0, \ \forall k \in \mathbb{N}. \end{aligned} \tag{2.11}$$

Then the trivial solution of (2.7) is stable. Moreover if

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0, \ \forall k \in \mathbb{N}. \tag{2.12}$$

Then, the trivial solution of (2.7) is asymptotically stable.

Remark 2.2 *This theorem, helps to study stability not need to know the forme of solution, its dispointing fact is that there is no specific way to generate the Lyapunov functions.*

Example 2.5 *Consider the following non-linear system:*

$$\Delta x(k) = \frac{x(k)}{2 + x^2(k)} - x(k). \tag{2.13}$$

It has an equilibrium point at the origin. Our first choice of a Liapunov function will be $V(x) = x^2$, this is clearly continuous and positive definite on $\mathbb{R}$,

$$\Delta V(x(k)) = \left(\frac{x(k)}{2 + x^2(k)}\right)^2 - x^2(k) < 0.$$

Then, by Theorem 2.3 the trivial solution of (2.13) is asymptotically stable.

2.3 Stability of fractional order difference systems

2.3.1 Stability of linear systems

Now, we investigate the stability of the equilibrium point $x = 0$ of the $\alpha-$th order linear system of difference equations:

$$\begin{cases} {}^C\Delta_a^\alpha x(t) = Ax(t + \alpha - 1), & t \in \mathbb{N}_{a+1-\alpha}, \\ x(a) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.14}$$

where $0 < \alpha \leq 1$, $a \in \mathbb{R}$, and A is an $n \times n$ constant matrix.

Theorem 2.4 *[15] Suppose $0 < \alpha < 1$. Then the trivial solution of (2.14) is asymptotically stable if and only if the isolated zeroes, off the nonnegative real axis, of*

$$\det\left(I_n - z^{-1}(1 - z^{-1})^{-\alpha}A\right), \tag{2.15}$$

lie inside the unit disk.

Remark 2.3 *If $\alpha \to 1^-$, then the condition of Theorem 2.4 simplifies to the isolated zeroes, off the nonnegative real axis, of*

$$\det\left(I_n - \frac{1}{z-1}A\right) = \frac{1}{z-1}\det\left((z-1)I_n - A\right),$$

lie inside the unit disk, this means all the eigenvalues $\lambda = (z - 1)$ of A satisfy $|\lambda + 1| = |z| < 1$, it is the same result of Theorem 2.1.

Proof. By Theorem 1.15, the solution of (2.14) is given by

$$\begin{aligned} x(t) &= x(a) + \tfrac{1}{\Gamma(\alpha)} \sum_{s=a+1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \ {}^{C}\Delta_a^{\alpha} \ x(s) \\ &= x(a) + \tfrac{1}{\Gamma(\alpha)} A \sum_{s=a+1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} x(s+\alpha-1), \quad \forall t \in \mathbb{N}_{a+1}. \end{aligned} \tag{2.16}$$

Thus, for $t = a+k+1, k = 0, 1, 2, \cdots$, (2.16) simplifies to

$$x_k = x_{-1} + \sum_{s=0}^{k} B(k-s) x_{s-1}. \tag{2.17}$$

Here, the kernel $B(k)$ and x_k, is given by

$$B(k) = \frac{1}{(k)!} A \prod_{j=1}^{k} (\alpha + j - 1), \text{ and } x_k = x(a+k+1), \ k = -1, 0, 1, 2, \cdots. \tag{2.18}$$

Notice that (2.17) is a nonhomogeneous Volterra difference equation of the convolution type. Moreover, the aforementioned equation will be used heavily in the sequel.

We focus our attention on the scalar case with $A = \lambda$. Without loss of generality, we shall take $x_{-1} = 1$.

• If $\lambda > 0$. Since $x_k > 1$ for $k \geq 1$,

$$x_k > 1 + \lambda \sum_{s=0}^{k} \frac{1}{(k-s)!} \left(\prod_{j=1}^{k-s} (\alpha + j - 1) \right) = 1 + \lambda \sum_{s=0}^{k} \frac{1}{s!} \left(\prod_{j=1}^{s} (\alpha + j - 1) \right).$$

But

$$\frac{\prod_{j=1}^{s} (\alpha + j - 1)/s!}{1/(\alpha + s)} = \frac{\prod_{j=0}^{s} (\alpha + j)}{s!} = \alpha \frac{\prod_{j=1}^{s} (\alpha + j)}{s!} > \alpha,$$

this implies

$$x_k > 1 + \lambda \sum_{s=0}^{k} \frac{\alpha}{(\alpha + s)},$$

by the limit comparison test the solution $\{x_k\}$, of (2.14) diverges to infinity.

• If $\lambda = 0$, then $x_k = x_{-1}$ for $k = 0, 1, 2, \cdots$.

Hence, from now on, we assume that $\lambda < 0$, let $\tilde{x}(z) = Z(\{x_k\})$ be the unilateral Z-transform of the sequence $\{x_k\}$. Then

$$\tilde{x}(z) = (1 - z^{-1})^{-1} + \lambda (1 - z^{-1})^{-\alpha} (1 + z^{-1} \tilde{x}(z)), \quad |z| > R \geq 1.$$

Solving for $\tilde{x}$ yields

$$\tilde{x}(z) = \frac{(1 - z^{-1})^{-1} + \lambda (1 - z^{-1})^{-\alpha}}{1 - \lambda z^{-1} (1 - z^{-1})^{-\alpha}} = \frac{\frac{z}{z-1} + \lambda (\frac{z}{z-1})^{\alpha}}{1 - \lambda \frac{1}{z} (\frac{z}{z-1})^{\alpha}}.$$

That says

$$x_k = \frac{1}{2\pi i}\int_C z^{k-1}\tilde{x}(z)dz,$$

where C is any positively-oriented simple-closed contour in the analyticity region of $\tilde{x}$ that encircles all singular points of $\tilde{x}(z)$.

With that in mind, we consider the contour $C\rho$, depicted in Figure 2.1 below. The inner circles are of radius ρ which we take it small enough so that all isolated singularities of $\tilde{x}$ are inside the $C\rho$.

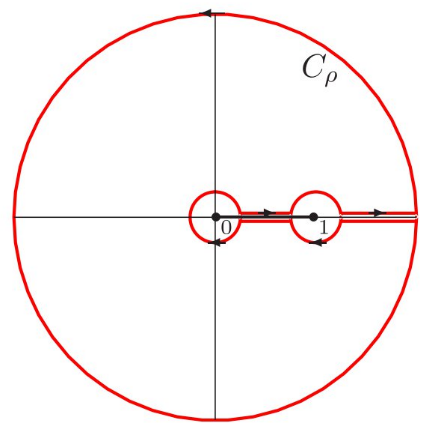


Figure 2.1: The contour $C\rho$

Since the line integrals of $z^{k-1}\tilde{x}(z)$ over the inner circles in Figure 2-1 tend to zero as ρ tends to zero, we have

$$\int_C z^{k-1}\tilde{x}(z)dz = \lim_{\rho\to 0}\int_{C_\rho} z^{k-1}\tilde{x}(z)dz.$$

But, by the Residue Theorem,

$$\tilde{x}_k = \frac{1}{2\pi i}\int_{C_\rho} z^{k-1}\tilde{x}(z)dz = \sum_i \operatorname{Re}s(z^{k-1}\tilde{x}(z), z_i) = \sum_i z_i^{k-1}\operatorname{Re}s(\tilde{x}(z), z_i),$$

where the z_i's are the isolated singularities $\tilde{x}(z)$. Indeed, the singularities of $\tilde{x}(z)$ are simple poles given by the zeros of

$$Q(z) = 1 - \lambda\frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha}.$$

To see this, suppose z_i is a zero of $Q(z)$. Then

$$\lambda\left(\frac{z_i}{z_i-1}\right)^{\alpha} = z_i,$$

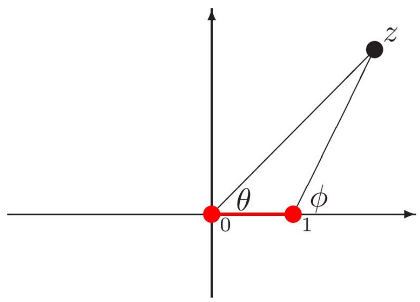
and so

$$Q'(z_i) = \frac{1-\alpha-z_i}{z_i(1-z_i)} \neq 0,$$

because $Q(1-\alpha) \neq 0$ if $0 < \alpha < 1$. Furthermore, with $P(z) = z/(z-1) + \lambda(z/(z-a))^{\alpha}$,

$$P(z_i) = \frac{z_i}{z_i - 1} + z_i \neq 0.$$

The "function" $(z/(z-1))^{\alpha} = z^{\alpha}/(z-1)^{\alpha}$ is multi-valued. As such, we introduce the branch cut shown in Figure 2-2.



$$|z| > 0, \ -\pi \leq \theta < \pi \text{ and } |z-1| > 0, \ -\pi \leq \phi < \pi$$

Figure 2.2: Branch cut of $(z/(z-1))^{\alpha}$

Given the branch depicted in Figure 2-2, let

$$z = |z|\, e^{i\theta} \quad \text{and} \quad z - 1 = |z-1|\, e^{i\phi},$$

where

$$|z|\,, |z-1| > 0 \quad \text{and} \quad -\pi \leq \theta, \phi < \pi.$$

That says,

$$\begin{aligned} Q(z) = 0 \quad &\Leftrightarrow \lambda \tfrac{|z|^{\alpha-1}}{|z-1|^{\alpha}} e^{-i[(1-\alpha)\theta + \alpha\phi]} = 1 \\ &\Leftrightarrow (1-\alpha)\theta + \alpha\phi = -\pi \quad \text{and} \quad \lambda \tfrac{|z|^{\alpha-1}}{|z-1|^{\alpha}} = -1. \end{aligned}$$

Furthermore, since $-\pi \leq \theta$, $\phi < \pi$, then $\theta = \phi = -\pi$ and, consequently, $|z-1| = |z| + 1$.
In view of the above argument, there are finitely many poles inside C_{ρ} and the residue of each is finite. Hence the result.
Following the same lines of reasoning employed in the scalar case and applying the Z-transform to (2.17), one obtains

$$\tilde{x}(z) = (1 - z^{-1})^{-1} x_{-1} + A(1 - z^{-1})^{-\alpha}(x_{-1} + z^{-1}\tilde{x}).$$

Re-arranging the terms and solving for $\tilde{x}(z)$, one gets

$$\tilde{x}(z) = \left(I_n - z^{-1}(1-z^{-1})A\right)^{-1}\left((1-z^{-1})^{-1} + (1-z^{-1})^{-\alpha}A\right)x_{-1},$$

where I_n is the identity matrix of order n.
Thus, like the above, we formulate the basic result. ■
An immediate consequence of Theorem 2.4 is the following corollary.

Corollaire 2.1 *[15] If $\alpha = 1/2$ and A is a triangular matrix with diagonal elements $\lambda_i, i = 1, \cdots, n$, then the zero solution of (2.14) is asymptotically stable if and only if*

$$-\sqrt{2} < \lambda_i < 0, \quad \forall i = 1, \cdots, n. \tag{2.19}$$

Example 2.6 *Consider the following $1/2$-order system of difference equations:*

$$\left({}^{C}\Delta^{1/2}_{a-1/2}\, x\right)(t) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x(t-1/2), \quad t \in \mathbb{N}_a,\ a \in \mathbb{R},\ \textit{and}\ \lambda_1, \lambda_2 \in \mathbb{R}, \tag{2.20}$$

subject to the initial condition

$$x(a-1/2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2.21}$$

Recall that Corollary 2.1 asserts that the zero solution of (2.20-2.21) is asymptotically stable if $-\sqrt{2} < \lambda_1, \lambda_2 < 0$. This, for different values of $\lambda_i, i = 1, 2$, is confirmed in Figure 2.3 below. The figure depicts the 2-norm of the solution $\{x_k\}$ for the first 100 iterates.

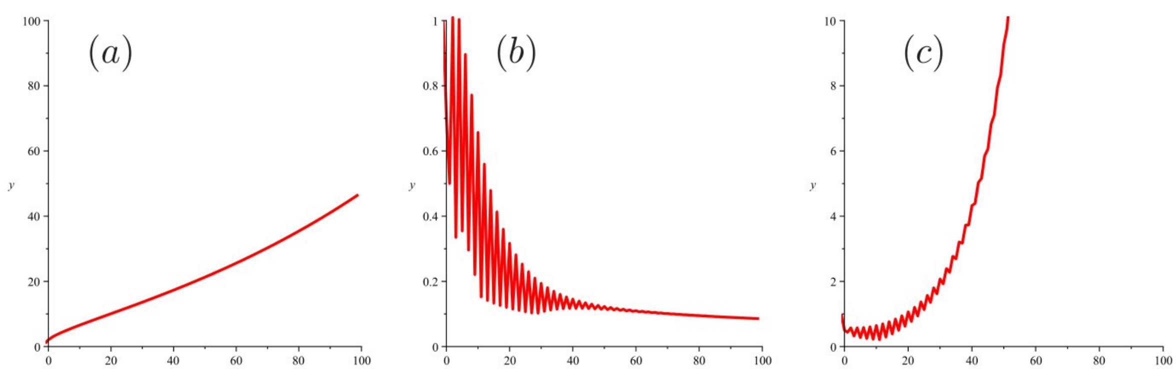


Figure 2.3: The first 100 iterates of 2-norm of the solution $\{x_k\}$ of Example 2.6, for (a) $\lambda_1 = 0.1$, $\lambda_2 = 0.01$, (b) $\lambda_1 = -1.3$, $\lambda_2 = -1$, and (c) $\lambda_1 = -1.5$, $\lambda_2 = -0.7$.

These results gave a condition for stability of (2.14), but they remain difficult to implement. Below we present a practical result, we introduce the set:

$$S^{\alpha}=\left\{z\in\mathbb{C}:|z|<\left(2\cos\frac{|\arg z|-\pi}{2-\alpha}\right)^{\alpha}\text{ and }|\arg z|>\frac{\alpha\pi}{2}\right\}. \tag{2.22}$$

Theorem 2.5 *[17] Let $\alpha\in(0,1)$ and A is an $n\times n$ constant matrix. If $\lambda\in S^{\alpha}$ for all the eigenvalues λ of A, then the trivial solution of (2.14) is asymptotically stable. In this case, the solutions of (2.14) decay towards zero algebraically (and not exponentially), more precisely*

$$\|x(t)\|=O(t^{-\alpha})\quad \text{as } t\to\infty,$$

for any solution x of (2.14).
Furthermore, if $\lambda\in\mathbb{C}\setminus cl(S^{\alpha})$ for an eigenvalue λ of A, the zero solution of (2.14) is not stable.

Proof. Without loss of generality, we shall take $a=0$, and use the variable change $t=k+1-\alpha$, system (2.14) become

$$({}^{C}\Delta_0^{\alpha}x)(k+1-\alpha)=Ax(k),\quad k=0,1,\cdots. \tag{2.23}$$

We have by definition

$$\begin{aligned}({}^{C}\Delta_0^{\alpha}x)(k+1-\alpha) &= \tfrac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}(k-\alpha-s)^{(-\alpha)}\Delta x(s)\\ &= \tfrac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}\tfrac{\Gamma(k-\alpha-s+1)}{\Gamma(k-s+1)}\Delta x(s)\\ &= \sum_{s=0}^{k}\tbinom{k-\alpha-s}{k-s}\Delta x(s)\\ &= \sum_{s=1}^{k}\left[\tbinom{k-\alpha-s+1}{k-s+1}-\tbinom{k-\alpha-s}{k-s}\right]x(s)-\tbinom{k-\alpha}{k}x(0)+\tbinom{-\alpha}{0}x(k+1).\end{aligned}$$

Using Pascal rule yields:

$$\begin{aligned}({}^{C}\Delta_0^{\alpha}x)(k+1-\alpha) &= \sum_{s=1}^{k}\tbinom{k-\alpha-s}{k-s+1}x(s)-\tbinom{k-\alpha}{k}x(0)+x(k+1)\\ &= \sum_{s=0}^{k}\tbinom{k-\alpha-s}{k-s+1}x(s)-\left[\tbinom{k-\alpha}{k}+\tbinom{k-\alpha}{k+1}\right]x(0)+x(k+1)\\ &= \sum_{s=0}^{k}(-1)^{k-s+1}\tbinom{\alpha}{k-s+1}x(s)-(-1)^{k+1}\tbinom{\alpha-1}{k+1}x(0)+x(k+1),\end{aligned}$$

so a sequence $x(k)$ is a solution of (2.14) if and only if it is a solution of

$$x(k+1)=Ax(k)+\sum_{s=0}^{k}B(k-s)x(s)+g(k),\quad k=0,1,\cdots, \tag{2.24}$$

where

$$B(k)=(-1)^{k}\binom{\alpha}{k+1}I_n\quad\text{and}\quad g(k)=(-1)^{k+1}\binom{\alpha-1}{k+1}x(0). \tag{2.25}$$

Comparing both the Volterra difference equations, the system (2.17) seems to be formally simpler and more suitable to analyze. However, the system (2.24) has an important advantage compared to (2.17), namely the property that the elements of its convolution kernel B given by (2.25) belong to the space $\ell^1(\mathbb{N}^{n\times n})$ of absolutely summable sequences, while the elements of kernel B given by (2.18) not. This will enable us to apply some relevant results of the theory of Volterra difference equations Theorem 1.6, where the assumption $B \in \ell^1(\mathbb{N}^{n\times n})$ is crucial.

(1) We consider (2.14) in its equivalent form (2.24) and first analyze its homogeneous part

$$x(k+1) = Ax(k) + \sum_{s=0}^{k} B(k-s)x(s), \quad k = 0, 1, \cdots . \tag{2.26}$$

To apply Theorem 1.6, we perform some necessary calculations related to the Z-transform of the kernel $b(k) = (-1)^k\binom{\alpha}{k+1}$. Using expansion into the binomial series (with the radius of convergence $R = 1$) we have

$$\tilde{b}(z) = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k+1}z^{-k} = z - z\left(1-\frac{1}{z}\right)^{\alpha}. \tag{2.27}$$

Then, based on Theorem 1.6, the homogeneous system (2.26) is asymptotically stable if all the zeros of the characteristic equation

$$\det\left(z\left(1-\frac{1}{z}\right)^{\alpha} I_n - A\right) = 0 \tag{2.28}$$

are located inside the unit circle. Obviously, z is a zero of this equation if and only if $z(1-z^{-1})^{\alpha}$ is an eigenvalue of A.

Now we consider the curve

$$\Gamma^{\alpha} = \left\{z\left(1-\frac{1}{z}\right)^{\alpha} : |z| = 1\right\},$$

defining the stability boundary for (2.14) and describe its structure. Let $z = e^{i\varphi}$ for $0 \le \varphi < 2\pi$ and let $1 - z^{-1} = re^{i\omega}$ for $r = r(\varphi) \ge 0$ and $\omega = \omega(\varphi), 0 \le \omega < 2\pi$. Then

$$1 - e^{-i\varphi} = re^{i\omega},$$

which, after equating real and imaginary parts, turns into

$$1 - \cos\varphi = r\cos\omega, \quad \sin\varphi = r\sin\omega.$$

If $\varphi = 0$, then $r = 0$. If $\varphi \ne 0$, then

$$\tan\omega = \frac{\sin\varphi}{1-\cos\varphi}.$$

Since
$$\frac{\sin\varphi}{1-\cos\varphi} = \frac{2\sin(\varphi/2)\cos(\varphi/2)}{2\sin^2(\varphi/2)} = \cot\frac{\varphi}{2} = \tan(\frac{\pi}{2}-\frac{\varphi}{2}),$$
we can write $\omega = \pi/2 - \varphi/2$. Further
$$r = 2\sin\frac{\varphi}{2}.$$

From here we get
$$\Gamma^\alpha = \left\{ \left(2\sin\frac{\varphi}{2}\right)^\alpha \exp(i\frac{2\varphi + (\pi-\varphi)\alpha}{2}) : 0 \le \varphi < 2\pi \right\}.$$

If we set $\theta = -\omega = \varphi/2 - \pi/2$, then
$$\Gamma^\alpha = \left\{ -(2\cos\theta)^\alpha \exp(i(2-\alpha)\theta) : -\frac{\pi}{2} \le \theta < \frac{\pi}{2} \right\}.$$

Using the polar form, $|z| = (2\cos\theta)^\alpha$, where
$$\theta = \begin{cases} \frac{\arg z - \pi}{2-\alpha} & \text{if } \arg z > 0, \\ \frac{\arg z + \pi}{2-\alpha} & \text{if } \arg z < 0. \end{cases}$$

This is equivalent to
$$\Gamma^\alpha = \left\{ z \in \mathbb{C} : |z| = \left(2\cos\frac{|\arg z| - \pi}{2-\alpha}\right)^\alpha \text{ and } |\arg z| \ge \frac{\alpha\pi}{2} \right\}. \tag{2.29}$$

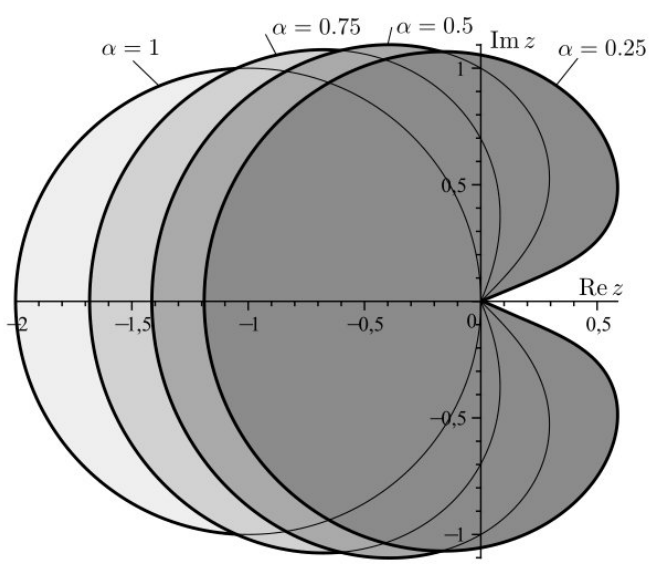


Figure 2.4: Asymptotic stability sets S^α for several values of α.

(2) As a next step, we show that $w_\alpha(z) = z(1-z^{-1})^\alpha$ maps the unit circle $D = \{z \in \mathbb{C} : |z| < 1\}$ onto S^α . First we consider the upper part of D in the form $D_u = \{z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im} z > 0\}$. Since w_α is holomorphic and nonconstant on D_u, it maps D_u (by the open mapping Theorem) to an open set. It means that for any $z_0 \in D_u$ there exists a neighborhood of $w_\alpha(z_0)$ contained in $w_\alpha(D_u)$. In other words, a point of D_u can not be mapped to the boundary of $w_\alpha(D_u)$. Further, the function w_α is continuously extendable on the closure $cl(D_u)$ (the singularity at the point $z = 0$ is removable as $\lim_{z\to 0} w_\alpha(z) = 0$) which is a compact set in $\mathbb{C}$ and thus $w_\alpha(D_u)$ is bounded. The same conclusion holds if we take the lower part of D in the form $D_\ell = \{z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im} z < 0\}$. Finally, the interval $(-1, 0)$ of the real axis (which is a part of the boundary ∂D_u as well as ∂D_ℓ) is mapped by w_α onto the interval $(-2^\alpha, 0)$ of the real axis, while the continuous extensions of w_α on $cl(D_u)$ and $cl(D_\ell)$ map the interval $[0, 1)$ onto abscissas lying in $w_\alpha(cl(D_u))$ and $w_\alpha(cl(D_\ell))$, respectively. It means that the interval $[-2^\alpha, 0]$ is a common part of the boundaries $\partial w_\alpha(D_\ell)$ and $\partial w_\alpha(D_u)$. In view of $w_\alpha(\partial D) = \Gamma^\alpha$ (see the previous step), the above arguments imply that $w_\alpha(D) = S^\alpha$.

(3) Let all the eigenvalues λ of A belong to S^α and let x be the solution of (2.24). Then by the variation of constants formula, we obtain

$$x(k) = R(k)x(0) + \sum_{s=0}^{k-1} R(s)g(k-s-1),$$

where $R(k)$ is the resolvent matrix of (2.26). So

$$\begin{aligned} x(k) &= R(k)x(0) + \sum_{s=0}^{k-1} R(s)g(k-s-1) \\ &= R(k)x(0) + \sum_{s=0}^{k-1} R(s)(-1)^{k-s}\tbinom{\alpha-1}{k-s}x(0) \\ &= \sum_{s=0}^{k} R(s)(-1)^{k-s}\tbinom{\alpha-1}{k-s}x(0) = \sum_{s=0}^{k} R(k-s)(-1)^{s}\tbinom{\alpha-1}{s}x(0). \end{aligned}$$

Using the asymptotic equivalence

$$(-1)^k \binom{\alpha-1}{k} \sim Ck^{-\alpha} \quad \text{as} \quad k \to \infty,$$

(with the constant C depending on α only) and taking the norms, we have

$$\begin{aligned} \|x(k)\| &= \left\| \sum_{s=0}^{k} R(k-s)(-1)^s \tbinom{\alpha-1}{s} x(0) \right\| \\ &\le \left\| \sum_{s=0}^{k} R(k-s)(-1)^s \tbinom{\alpha-1}{s} \right\| \|x(0)\| \\ &\le C_1 \sum_{s=0}^{k} \tfrac{1}{(s+1)^\alpha} \|R(k-s)\| \\ &= C_1 \left(\sum_{s=0}^{\lfloor k/2 \rfloor} \tfrac{1}{(s+1)^\alpha} \|R(k-s)\| + \sum_{s=\lfloor k/2 \rfloor+1}^{k} \tfrac{1}{(s+1)^\alpha} \|R(k-s)\| \right), \end{aligned}$$

where $C_1 > 0$ is a suitable real constant and the symbol $\lfloor . \rfloor$ stands for the floor function. Since each component of R belongs to $\ell^1(\mathbb{N}_0)$, we have $\|R(k)\| = O(k^{-1})$ as $k \to \infty$ and there exist $C_2, C_3 > 0$ such that

$$\sum_{s=0}^{\lfloor k/2 \rfloor} \frac{1}{(s+1)^\alpha} \|R(k-s)\| \leq \frac{C_2}{k+1} \sum_{s=0}^{\lfloor k/2 \rfloor} \frac{1}{(s+1)^\alpha} \leq \frac{C_3}{(k+1)^\alpha},$$

where we have used the inequality $\sum_{s=1}^{k} (s+1)^{-\alpha} \leq \int_0^k (x+1)^{-\alpha} dx$. Similarly, the second sum can be estimated as

$$\sum_{s=\lfloor k/2 \rfloor+1}^{k} \frac{1}{(s+1)^\alpha} \|R(k-s)\| \leq \frac{C_4}{(k+1)^\alpha} \sum_{s=\lfloor k/2 \rfloor+1}^{k} \|R(k-s)\| \leq \frac{C_5}{(k+1)^\alpha},$$

for suitable $C_4, C_5 > 0$. In summary, we have $\|x(k)\| \leq C_5(k+1)^{-\alpha}$, hence $\|x(k)\| = O(k^{-\alpha})$ as $k \to \infty$.

(4) It remains to show that if $\lambda \in \mathbb{C} \backslash cl(S^\alpha)$ for an eigenvalue λ of A, then the zero solution of (2.14) is not stable (equivalently, if there is a zero of $(\det\left(z\left(1-\frac{1}{z}\right)^\alpha I_n - A\right) = 0)$ with $|z| > 1$, then (2.14) is not stable).

If $|z| > 1$, then $\tilde{g}(z) = (1-z^{-1})^{\alpha-1} x(0)$, and the Z-transform of the solution x of (2.14) takes the form

$$\tilde{x}(z) = \left(z\left(1-\frac{1}{z}\right)^\alpha I_n - A\right)^{-1} z\left(1-\frac{1}{z}\right)^{\alpha-1} x(0). \tag{2.30}$$

A zero of $\det\left(z\left(1-\frac{1}{z}\right)^\alpha I_n - A\right)$ represents a singular point of $\tilde{x}$. It is known that if $\tilde{f}(z)$ is the Z-transform of a sequence $f : \mathbb{N} \to \mathbb{R}$, then its radius of convergence R is given by distance from origin to an outermost (non-removable) singular point. Hence, if there is a zero z_0 with $|z| > 1$, then also the radius of convergence of at least one component x_i of $\tilde{x}$ satisfies $R > 1$. Using the Cauchy-Hadamard theorem we have

$$r = \lim_{k\to\infty} \sup \sqrt[k]{|x_i(k)|} > 1$$

and, consequently, $\lim\limits_{k\to\infty} \sup |x_i(k)| = +\infty$ which proves that x is not bounded and thus (2.14) is not stable. ■

Remark 2.4 *The assertions of Theorem 2.4 and Theorem 2.5 describe the same stability region. But these analytical descriptions are different. In particular, the condition stated in Theorem 2.5 seems to be more convenient for practical purposes, due to the explicit form of S^α.*

Example 2.7 *Consider the following system:*

$${}^C\Delta_a^{\frac{1}{2}} x(t) = \begin{pmatrix} 0 & \frac{3}{2} \\ -\frac{2}{3} & -1 \end{pmatrix} x(t), \quad t \in \mathbb{N}_{a+1-\alpha}. \tag{2.31}$$

The characteristic equation for $A = \begin{pmatrix} 0 & \frac{3}{2} \\ -\frac{2}{3} & -1 \end{pmatrix}$ *is* $\lambda^2 + \lambda + 1 = 0$ *and hence the eigenvalues of* A *are* $\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ *and* $\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$*. Since*

$$|\lambda_2| = 1 > \left(2\cos\frac{|\arg\lambda_2| - \pi}{2 - \frac{1}{2}}\right)^{\frac{1}{2}} = \left(2\cos\frac{-4\pi}{9}\right)^{\frac{1}{2}} = 0.589318551 \Leftrightarrow \lambda_2 \notin S^{\frac{1}{2}}.$$

Then, by Theorem 2.5, the trivial solution of (2.31) is not stable.

Example 2.8 *Consider the following system:*

$$^{C}\Delta_a^{\frac{2}{3}} x(t) = \begin{pmatrix} -\frac{3}{8} & \frac{3}{2} \\ 0 & -\frac{17}{27} \end{pmatrix} x(t), \quad t \in \mathbb{N}_{a+1-\frac{2}{3}}. \tag{2.32}$$

The characteristic equation for $\begin{pmatrix} -\frac{3}{8} & \frac{3}{2} \\ 0 & -\frac{17}{27} \end{pmatrix}$ *is* $(\lambda + \frac{3}{8})(\lambda + \frac{17}{27}) = 0$ *and hence the eigenvalues of* A *are* $\lambda_1 = -\frac{3}{8}$ *and* $\lambda_2 = -\frac{17}{27}$*. Since*

$$|\lambda_1| = \frac{3}{8} < \left(2\cos\frac{|\arg\lambda_1| - \pi}{2 - \frac{2}{3}}\right)^{\frac{2}{3}} \simeq 0.6666666666, \quad \textit{and} \quad |\arg\lambda_1| = \pi > \frac{2\pi}{6} \Leftrightarrow \lambda_1 \in S^{\frac{2}{3}},$$

and

$$|\lambda_2| = \frac{17}{27} < \left(2\cos\frac{|\arg\lambda_2| - \pi}{2 - \frac{2}{3}}\right)^{\frac{2}{3}} \simeq 0.6666666666, \quad \textit{and} \quad |\arg\lambda_2| = \pi > \frac{2\pi}{6} \Leftrightarrow \lambda_2 \in S^{\frac{2}{3}}.$$

Then, by Theorem 2.5, the trivial solution of (2.32) is asymptotically stable.

2.3.2 Stability of non-linear systems

In this section, we extend the method of the Lyapunov functions to study the stability of solutions of the **Advanced time** and **Delay time** non-linear fractional order difference systems. First, we list some definitions that will be used in study the stability properties.

Definition 2.2 *[20] A function* $\phi(r)$ *is said to belong to the class* $\mathcal{K}$ *if and only if* $\phi \in C\left[[0,\rho), \mathbb{R}_+\right]$, $\phi(0) = 0$ *and* $\phi(r)$ *is strictly monotonically increasing in* r*. If* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\phi \in \mathcal{K}$, *and* $\lim_{r\to\infty} \phi(r) = \infty$, *then* ϕ *is said to belong to class* $\mathcal{KR}$.

Definition 2.3 *[20] A real valued function* $V(t,x)$ *defined on* $\mathbb{N}_a \times S_\rho$, *where* $S_\rho = \{x \in \mathbb{R} : \|x\| \le \rho\}$, *is said to be positive definite if and only if* $V(t,0) = 0$ *for all* $t \in \mathbb{N}_a$ *and there exists* $\phi(r) \in \mathcal{K}$ *such that* $\phi(r) \le V(t,x), \|x\| = r, (t,x) \in \mathbb{N}_a \times S_\rho$.

Definition 2.4 *[20] A real valued function $V(t,x)$ defined on $\mathbb{N}_a \times S_\rho$, where $S_\rho = \{x \in \mathbb{R} : \|x\| \leq \rho\}$, is said to be decrescent if and only if $V(t,0) = 0$ for all $t \in \mathbb{N}_a$ and there exists $\varphi(r) \in \mathcal{K}$ such that $V(t,x) \leq \varphi(r), \|x\| = r, (t,x) \in \mathbb{N}_a \times S_\rho$.*

Advanced time systems

Consider the following system:

$$\begin{cases} {}^C\Delta_{t_0}^{\alpha} x(t) = f(t+\alpha-1, x(t+\alpha-1)), \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.33}$$

where $t_0 = a + n_0 \in \mathbb{N}_a$ $(n_0 \in \mathbb{N})$, $t \in \mathbb{N}_{n_0}, a = \alpha - 1$, $f : \mathbb{N}_a \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $0 < \alpha \leq 1$. Let $f(t,0) = 0$, for all $t \in \mathbb{N}_a$ so that the system (2.33) admits the trivial solution.

Remark 2.5 *When $\alpha = 1$, the system (2.33) become*

$$\begin{cases} \Delta x(t) = f(t, x(t)), \quad t \in \mathbb{N}_{t_0}, \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n. \end{cases}$$

For the system (2.33) we have the following theorems.

Theorem 2.6 *[20] If there exists a positive definite and decrescent scalar function $V(t,x) \in C[\mathbb{N}_a \times S_\rho, \mathbb{R}_+]$ such that*

$$ {}^C\Delta_{t_0}^{\alpha} V(t, x(t)) \leq 0, \tag{2.34}$$

for all $t_0 \in \mathbb{N}_a$ and $(t,x) \in \mathbb{N} \times S_\rho$, then the trivial solution of (2.33) is uniformly stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be a solution of system (2.33). Since $V(t,x)$ is positive definite and decrescent, there exist $\varphi, \phi \in \mathcal{K}$ such that

$$\phi(\|x\|) \leq V(t,x) \leq \varphi(\|x\|),$$

for all $(t,x) \in \mathbb{N}_a \times S_\rho$.
For each $\epsilon > 0, 0 < \epsilon < \rho$ we choose a $\delta = \delta(\epsilon)$ such that

$$\varphi(\delta) < \phi(\epsilon).$$

For any solution $x(t)$ of (2.33) we have $\phi(\|x(t)\|) \leq V(t, x(t))$ with $\|x_0\| < \delta(\epsilon)$. Since ${}^C\Delta_{t_0}^{\alpha} V(t, x(t)) \leq 0$, by using Theorem 1.12 we have $V(t, x(t)) \leq V(t_0, x_0)$ for all $t \in \mathbb{N}_{t_0}$. Consequently,

$$\phi(\|x(t)\|) \leq V(t, x(t)) \leq V(t_0, x_0) \leq \varphi(\|x_0\|) < \varphi(\delta) < \phi(\epsilon),$$

and thus $\|x(t)\| < \epsilon$ for all $t \in \mathbb{N}_{t_0}$. ■

Theorem 2.7 [20] *If there exists a positive definite and decrescent scalar function $V(t,x) \in C\left[\mathbb{N}_a \times S_\rho, \mathbb{R}_+\right]$ such that*

$$^{C}\Delta_{t_0}^{\alpha} V(t, x(t)) \leq -\psi(\|x(t+\alpha-1\|), \quad \forall t_0 \in \mathbb{N}_a,\ (t,x) \in \mathbb{N} \times S_\rho, \tag{2.35}$$

where $\psi \in \mathcal{K}$, then the trivial solution of (2.33) is uniformly asymptotically stable.

Proof. Since all the conditions of Theorem 2.6 are satisfied, the trivial solution of the system (2.33) is stable. Let $0 < \epsilon < \rho$ and $\delta = \delta(\epsilon) > 0$ correspond to stability. Choose a fixed $\epsilon_0 < \rho$ and $\delta_0 = \delta(\epsilon_0) > 0$. Now, choose $\|x_0\| < \delta_0$ and $T(\epsilon)$ large enough such that $(T+a)^{(\alpha)} \geq (\varphi(\delta_0)/\psi(\delta(\epsilon)))\Gamma(\alpha+1)$. Such a large T can be chosen since $\lim_{T\to\infty}(\Gamma(T+\alpha)/\Gamma(T)) = \infty$. Now, we claim that $\|x(t,t_0,x_0)\| < \delta(\epsilon)$ for all $t \in [t_0, t_0+T] \cap \mathbb{N}_{t_0}$. If this is not true, due to (2.35) and Theorem 1.12, we get

$$\begin{aligned} V(t, x(t,t_0,x_0)) &\leq V(t_0,x_0) - \tfrac{1}{\Gamma(\alpha)} \sum_{s=t_0+1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \psi(\|x(s+\alpha-1)\|) \\ &\leq \varphi(\|x_0\|) - \tfrac{\psi(\delta)}{\Gamma(\alpha)} \sum_{s=n_0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \\ &\leq \varphi(\delta_0) - \tfrac{\psi(\delta)}{\Gamma(\alpha+1)} (t-n_0)^{(\alpha)}. \end{aligned}$$

Substituting $t = t_0 + T$, we get

$$0 < \phi(\delta(\epsilon)) \leq V(t_0+T, x(t_0+T,t_0,x_0)) \leq \varphi(\delta_0) - \frac{\psi(\delta)}{\Gamma(\alpha+1)}(T+t_0-n_0)^{(\alpha)} \leq 0,$$

which is a contradiction. Thus, there exists a $t \in [t_0, t_0+T]$ such that $\|x(t)\| < \delta(\epsilon)$. But in this case, since the trivial solution is uniformly stable and t is arbitrary, $\|x(t)\| < \epsilon$ for all $t \geq t_0 + T$ whenever $\|x_0\| < \delta_0$. ∎

Theorem 2.8 [20] *If there exists a function $V(t,x) \in C\left[\mathbb{N}_a \times \mathbb{R}^n, \mathbb{R}_+\right]$ such that*

$$\phi(\|x(t)\|) \leq V(t,x) \leq \varphi(\|x(t)\|) \quad \forall (t,x) \in \mathbb{N}_a \times \mathbb{R}^n,$$

$$^{C}\Delta_{t_0}^{\alpha} V(t, x(t)) \leq -\psi(\|x(t+\alpha-1)\|) \quad \forall t_0 \in \mathbb{N}_a,\ (t,x) \in \mathbb{N} \times \mathbb{R}^n,$$

where φ, ϕ, and $\psi \in \mathcal{KR}$ hold for all $(t,x) \in \mathbb{N}_a \times \mathbb{R}^n$, then the trivial solution of (2.33) is globally uniformly asymptotically stable.

Proof. Since the conditions of Theorem 2.7 are satisfied, the trivial solution of (2.33) is uniformly asymptotically stable. It remains to show that the domain of attraction of $x = 0$ is all of $\mathbb{R}^n$. Since $\lim_{r\to\infty} \phi(r) = \infty$, δ_0 in the proof of Theorem 2.7 may be chosen arbitrary large and ϵ can be chosen such that it satisfies $\varphi(\delta_0) < \phi(\epsilon)$. Thus, the globally uniformly asymptotic stability of $x = 0$ is concluded. ∎

Remark 2.6 *Theorem 2.8 gives a sufficient condition to analyze stability of (2.33). But it's not easy to construct ϕ, φ and ψ functions directly. Yet, so far there is no direct way to know the stability of system (2.33).*

Delay time systems

Now, we consider the following system:

$$\begin{cases} {}^C\Delta_a^\alpha x(t) = f(t+\alpha, x(t+\alpha)), & t \in \mathbb{N}_{a+1-\alpha}, \\ x(a) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.34}$$

where $f : \mathbb{N}_a \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $0 < \alpha \leq 1$. Let $f(t, 0) = 0$, for all $t \in \mathbb{N}_a$ so that the system (2.34) admits the trivial solution.

Remark 2.7 *when $\alpha = 1$, the system (2.34) become*

$$\begin{cases} \Delta x(t) = f(t+1, x(t+1)), & t \in \mathbb{N}_a, \\ x(a) = x_0, \quad x_0 \in \mathbb{R}^n. \end{cases}$$

The result of Theorem 2.8 remain correct in this case and is formulated as follows:

Theorem 2.9 *[21] If there exists a positive definite and decrescent scalar function $V(t, x)$, discrete class–$\mathcal{K}$ functions ϕ, φ and ψ such that*

$$\phi(\|x(t)\|) \leq V(t, x) \leq \varphi(\|x(t)\|), \quad t \in \mathbb{N}_a,$$

and

$${}^C\Delta_a^\alpha V(t, x(t)) \leq -\psi(\|x(t+\alpha)\|), \quad t \in \mathbb{N}_{a+1-\alpha},$$

then, the trivial solution of (2.34) is asymptotically stable.

Now, we present a sufficient condition for stability of (2.34). We first introduce the following Lemma. In the rest, we assume the initial point $a = 0$ for simplicity.

Lemma 2.1 *[21] For $0 < \alpha \leq 1$, and any discrete time $t \in \mathbb{N}_{a+1-\alpha}$, the following inequality holds:*

$${}^C\Delta_a^\alpha x^2(t) \leq 2x(t+\alpha)\, {}^C\Delta_a^\alpha x(t). \tag{2.35}$$

Proof. We need to equivalently prove

$${}^C\Delta_a^\alpha x^2(t) - 2x(t+\alpha)\, {}^C\Delta_a^\alpha x(t) \leq 0.$$

The left hand side can be rewritten explicitly as

$$\frac{1}{\Gamma(1-\alpha)}\sum_{s=a}^{t+\alpha-1}(t-s-1)^{(-\alpha)}\Delta_s\left(x^2(s)-2x(t+\alpha)x(s)\right).$$

We add $\Delta_s x^2(t+\alpha)$

$$^C\Delta_a^\alpha x^2(t)-2x(t+\alpha)^C\Delta_a^\alpha x(t)=\frac{1}{\Gamma(1-\alpha)}\sum_{s=a}^{t+\alpha-1}(t-s-1)^{(-\alpha)}\Delta_s\left(x(s)-x(t+\alpha)\right)^2.$$

Using the summation by parts. We obtain

$$\begin{aligned}^C\Delta_a^\alpha x^2(t)-2x(t+\alpha)^C\Delta_a^\alpha x(t) &= \tfrac{-1}{\Gamma(1-\alpha)}\sum_{s=a}^{t+\alpha-1}\left(x(s)-x(t+\alpha)\right)^2\Delta_s(t-s)^{(-\alpha)}\\ &\quad+\tfrac{(t-s)^{(-\alpha)}}{\Gamma(1-\alpha)}\left(x(s)-x(t+\alpha)\right)^2|_{s=a}^{t+\alpha}\\ &= \tfrac{-\alpha}{\Gamma(1-\alpha)}\sum_{s=a}^{t+\alpha-1}\left(x(s)-x(t+\alpha)\right)^2(t-s-1)^{(-\alpha-1)}\\ &\quad-\tfrac{(t-a)^{(-\alpha)}}{\Gamma(1-\alpha)}\left(x(a)-x(t+\alpha)\right)^2\leq 0.\end{aligned}$$

This completes the proof. ■

Theorem 2.10 *[21] If $x^T(t+\alpha)f(t+\alpha,x(t+\alpha))<0$ for any $t\in\mathbb{N}_{a+1-\alpha}$, then the trivial solution of (2.34) is asymptotically stable.*

Proof. Use a discrete Lyapunov candidate function $V(t,x(t))=\frac{1}{2}\sum_{i=1}^{n}x_i^2(t)$.
If $x^T(t+\alpha)f(t+\alpha,x(t+\alpha))<0$, then we have by Lemma 2.1

$$^C\Delta_a^\alpha V\leq\sum_{i=1}^{n}x_i(t+\alpha)^C\Delta_a^\alpha x_i(t)=x^T(t+\alpha)f(t+\alpha,x(t+\alpha))<0,$$

which means the fractional difference of $V(t,x(t))$ is negative definite.
Considering the Theorem 1.12, we can obtain

$$V(t,x(t))<V(0,x(0)),\quad\forall t\in\mathbb{N}_a,$$

or

$$\frac{1}{2}\sum_{i=1}^{n}x_i^2(t)<\frac{1}{2}\sum_{i=1}^{n}x_i^2(0).$$

According to the definition of the stability, we can determine the origin is stable.
On the other hand, the fractional difference of the V function results negative definite. The used Lyapunov function is positive definite. As a result, the equilibrium point is asymptotically stable from Theorem 2.9. This completes the proof. ■

Example 2.9 *Consider the linear discrete fractional equation:*

$$^{C}\Delta_a^{\alpha} x(t) = -x(t+\alpha), \quad x(0) = 0.1, \quad 0 < \alpha \leq 1, \quad t \in \mathbb{N}_{a+1-\alpha}.$$

We check that

$$x(t+\alpha)f(t+\alpha, x(t+\alpha)) = -x^2(t+\alpha) < 0, \;\; t \in \mathbb{N}_{a+1-\alpha}.$$

Hence, the trivial solution is asymptotically stable according to Theorem 2.10.

Example 2.10 *Consider a discrete time varying system of fractional order:*

$$\begin{cases} ^{C}\Delta_a^{\alpha} x_1(t) = -2x_1(t+\alpha) + tx_2(t+\alpha), & x_1(0) = 0.4, \quad 0 < \alpha \leq 1, \\ ^{C}\Delta_a^{\alpha} x_2(t) = -tx_1(t+\alpha) - x_2(t+\alpha), & x_2(0) = 0.8, \quad t \in \mathbb{N}_{a+1-\alpha}. \end{cases}$$

We can calculate

$$x^T(t+\alpha)f(t+\alpha, x(t+\alpha)) = -2x_1^2(t+\alpha) - x_2^2(t+\alpha) < 0, \quad \forall t \in \mathbb{N}_{a+1-\alpha}.$$

It is evident that the system is asymptotically stable from Theorem 2.10.

Chapter 3

New stability results for linear incommensurate fractional order difference systems

In the previous chapter, it mentioned some results for stability analysis of linear fractional-order difference systems (FoDS) with commensurate orders, in order to pave the way for introducing the main results of this chapter, which will be verified numerically, later on, in Section 2. Such results will clearly show the stability of the linear FoDS with incommensurate orders via some useful conditions formulated as theorems. First of all, consider the following linear incommensurate FoDS:

$$\begin{cases} {}^{C}\Delta_0^{\alpha_1} x_1(k+1-\alpha_1) = \sum\limits_{j=1}^{n} a_{1j} x_j(k), \\ {}^{C}\Delta_0^{\alpha_2} x_2(k+1-\alpha_2) = \sum\limits_{j=1}^{n} a_{2j} x_j(k), \\ \vdots \\ {}^{C}\Delta_0^{\alpha_n} x_n(k+1-\alpha_n) = \sum\limits_{j=1}^{n} a_{nj} x_j(k), \end{cases} \quad k = 0, 1, \cdots, \tag{3.1}$$

where ${}^{C}\Delta_0^{\alpha_i}$ is the Caputo fractional difference of order α_i, where $0 < \alpha_i \leq 1$, for $i = 1, 2, \cdots, n$, and $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$.

3.1 Main results

For the incommensurate fractional order linear difference system (3.1) we give the following results:

Theorem 3.1 *Consider system (3.1) subject to the initial vector condition* $X(0) = X_0 \in \mathbb{R}^n$. *Then,*

• *If all roots of the following characteristic equation:*

$$\det\left(diag\left(z(1-z^{-1})^{\alpha_1}, z(1-z^{-1})^{\alpha_2}, \cdots, z(1-z^{-1})^{\alpha_n}\right) - A\right) = 0, \tag{3.2}$$

lie inside the unit disk, then the zero solution of system (3.1) is asymptotically stable.

• *If there exists a zero, say* z^* *of (3.2) such that* $|z^*| > 1$, *then the zero solution of system (3.1) is not stable.*

Proof. As we observed in the proof of Theorem 2.5 that

$$({}^C\Delta_0^{\alpha_i} x_i)(k+1-\alpha) = \sum_{s=0}^{k}(-1)^{k-s+1}\binom{\alpha_i}{k-s+1}x(s) - (-1)^{k+1}\binom{\alpha_i-1}{k+1}x(0) + x(k+1), \quad \forall i = 1, 2, \cdots, n.$$

So we can rewrite system (3.1) as follows:

$$\begin{cases} x_1(k+1) &= \sum\limits_{s=0}^{k}(-1)^{k-s}\binom{\alpha_1}{k-s+1}x_1(s) + (-1)^{k+1}\binom{\alpha_1-1}{k+1}x_1(0) + \sum\limits_{j=1}^{n} a_{1j}x_j(k), \\ x_2(k+1) &= \sum\limits_{s=0}^{k}(-1)^{k-s}\binom{\alpha_2}{k-s+1}x_2(s) + (-1)^{k+1}\binom{\alpha_2-1}{k+1}x_2(0) + \sum\limits_{j=1}^{n} a_{2j}x_j(k), \\ & \vdots \\ x_n(k+1) &= \sum\limits_{s=0}^{k}(-1)^{k-s}\binom{\alpha_n}{k-s+1}x_n(s) + (-1)^{k+1}\binom{\alpha_n-1}{k+1}x_n(0) + \sum\limits_{j=1}^{n} a_{nj}x_j(k). \end{cases} \quad k = 0, 1, \cdots. \tag{3.3}$$

One might take the Z-transform to (3.3). This yields the following system:

$$\begin{cases} z\tilde{x}_1(z) - zx_1(0) &= (z - z(1-\frac{1}{z})^{\alpha_1})\tilde{x}_1(z) + (z(1-\frac{1}{z})^{\alpha_1-1} - z)x_1(0) + \sum\limits_{j=1}^{n} a_{1j}\tilde{x}_j(z), \\ z\tilde{x}_2(z) - zx_2(0) &= (z - z(1-\frac{1}{z})^{\alpha_2})\tilde{x}_2(z) + (z(1-\frac{1}{z})^{\alpha_2-1} - z)x_2(0) + \sum\limits_{j=1}^{n} a_{2j}\tilde{x}_j(z), \\ & \vdots \\ z\tilde{x}_n(z) - zx_n(0) &= (z - z(1-\frac{1}{z})^{\alpha_n})\tilde{x}_n(z) + (z(1-\frac{1}{z})^{\alpha_n-1} - z)x_n(0) + \sum\limits_{j=1}^{n} a_{nj}\tilde{x}_j(z), \end{cases} \tag{3.4}$$

where $\tilde{x}_i(z)$ indicates the Z-transform of $x_i(k)$, (i.e., $\tilde{x}_i(z) = Z[x_i(k)], 1 \le i \le n$). Consequently, we can rewrite system (3.4) as follows:

$$M(z).\begin{pmatrix} \tilde{x}_1(z) \\ \tilde{x}_2(z) \\ \vdots \\ \tilde{x}_n(z) \end{pmatrix} = \begin{pmatrix} z(1-\frac{1}{z})^{\alpha_1-1}x_1(0) \\ z(1-\frac{1}{z})^{\alpha_2-1}x_2(0) \\ \vdots \\ z(1-\frac{1}{z})^{\alpha_n-1}x_n(0) \end{pmatrix}, \tag{3.5}$$

in which

$$M(z)=\begin{pmatrix} z(1-\frac{1}{z})^{\alpha_1}-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & z(1-\frac{1}{z})^{\alpha_2}-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & z(1-\frac{1}{z})^{\alpha_n}-a_{nn} \end{pmatrix}. \tag{3.6}$$

Multiplying both sides of (3.5) by $(z-1)$ gives:

$$M(z).\begin{pmatrix} (z-1)\tilde{x}_1(z) \\ (z-1)\tilde{x}_2(z) \\ \vdots \\ (z-1)\tilde{x}_n(z) \end{pmatrix}=\begin{pmatrix} z^2(1-\frac{1}{z})^{\alpha_1}x_1(0) \\ z^2(1-\frac{1}{z})^{\alpha_2}x_2(0) \\ \vdots \\ z^2(1-\frac{1}{z})^{\alpha_n}x_n(0) \end{pmatrix}. \tag{3.7}$$

Now, we should note that if all roots of $\det M(z)=0$, lie inside the unit disk, then system (3.7) will be considered such that z satisfies $|z|\geq R$, withe $R\leq 1$ (R is the radius of convergence of $\tilde{X}(z)$). Actually, system (3.7) has a unique solution in this limited area represented by $((z-1)\tilde{x}_1(z),(z-1)\tilde{x}_2(z),\cdots,(z-1)\tilde{x}_n(z))$. Accordingly, we have:

$$\lim_{z\to 1}(z-1)\tilde{x}_i(z)=0, \quad i=1,2,\cdots,n. \tag{3.8}$$

Based on the assumption stated in first part of this theorem, and based also on the Final-Value Theorem associated with Z-transform, we obtain:

$$\lim_{k\to\infty}x_i(k)=\lim_{z\to 1}(z-1)\tilde{x}_i(z)=0, \quad i=1,2,\cdots,n. \tag{3.9}$$

On the other hand, considering the second part of this theorem implies that the convergence radius R of the series:

$$\sum_{k=0}^{\infty}X(k)z^{-k}=\tilde{X}(z), \tag{3.10}$$

is greater than 1 (i.e., $R>1$). Therefore, there exist i_0, where $1\leq i_0\leq n$, which makes the convergence radius R_{i_0}, of the series:

$$\sum_{k=0}^{\infty}x_{i_0}(k)z^{-k}=\tilde{x}_{i_0}(z), \tag{3.11}$$

also be greater than 1 (i.e., $R_{i_0}>1$). Thus, by using the Cauchy-Hadamard Theorem, we obtain:

$$R_{i_0}=\lim_{k\to\infty}\sup\sqrt[k]{|x_{i_0}(k)|}>1. \tag{3.12}$$

Consequently, $\lim_{k\to\infty}\sup|x_{i_0}(k)|=\infty$. This, however, implies that x will be never bounded and hence (3.1) is not stable. ■

Remark 3.1 *If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, then Theorem 3.1 becomes similar to Theorem 2.4 which is equivalent, as it is known, to Theorem 2.5.*

Corollaire 3.1 *Suppose that A is a triangular matrix with diagonal elements $\lambda_i, i = 1, \cdots, n$. If $-2^{\alpha_i} < \lambda_i < 0, \forall i$, then the zero solution of system (3.1) is asymptotically stable. Furthermore, such solution is not stable if either $\lambda_i > 0$ or $\lambda_i < -2^{\alpha_i}$, for some i.*

Proof. Consider the first part of Theorem 3.1 and A as assumed here. This will turn (3.2) into the form:

$$\prod_{i=1}^{n} \left(z(1 - z^{-1})^{\alpha_i} - \lambda_i \right) = 0. \tag{3.13}$$

It means that there exists i, where $0 \leq i \leq n$ such that:

$$\left(z(1 - z^{-1})^{\alpha_i} - \lambda_i \right) = 0. \tag{3.14}$$

Now, according to the assumption that supposes all the roots of (3.2) lie inside the unit disk, we deduce that all λ_i's belong to the set $\left\{ z(1 - z^{-1})^{\alpha_i},\ z \in \mathbb{C} \text{ and } |z| < 1 \right\}$, for $0 \leq i \leq n$. Based on the proof of Theorem 2.5, we also deduce:

$$\left\{ z(1 - z^{-1})^{\alpha_i},\ z \in \mathbb{C} \text{ and } |z| < 1 \right\} = \left\{ z \in \mathbb{C} : |z| < \left(2 \cos \frac{|\arg z| - \pi}{2 - \alpha} \right)^{\alpha_i} \text{ and } |\arg z| > \frac{\alpha_i \pi}{2} \right\},$$

where $\lambda_i \in \mathbb{R}, 0 \leq i \leq n$, This means that $-2^{\alpha_i} < \lambda_i < 0$, as desired. ∎

Example 3.1 *Consider the vector difference equation*

$$\begin{pmatrix} {}^C\Delta_0^{\sqrt{2}} x_1(k + 1 - \sqrt{2}) \\ {}^C\Delta_0^{\pi} x_2(k + 1 - \pi) \\ {}^C\Delta_0^{e} x_3(k + 1 - e) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 5 & 6 \\ 0 & \lambda_2 & 8 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix}, \quad k = 0, 1, 2, \cdots, \tag{3.15}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, by Corollary 3.1 the zero solution of (3.15) is asymptotically stable if

$$-2^{\sqrt{2}} < \lambda_1 < 0,$$

$$-2^{\pi} < \lambda_2 < 0,$$

and

$$-2^{e} < \lambda_3 < 0.$$

If either $\lambda_i > 0$ or $\lambda_i < -2^{\alpha_i}$, for some $i = 1, 2, 3$. Then, the zero solution of (3.15) is not stable.

As a matter of fact, the above result, represented by Corollary 3.1, is deemed one of the main significant results in this chapter. It can be easily implemented for exploring the stability of some linear incommensurate FoDSs which involve just a triangular matrix A in their forms. On the contrary, one can find it extremely hard to verify condition (3.2) in the proposed result represented by Theorem 3.1. Actually, this condition concerns with the full matrix that might be found in the linear incommensurate FoDSs. To deal with this problem, we present below another more practical result which is equivalent to Theorem 3.1.

Theorem 3.2 *Consider $0 < \alpha_i < 1$, for $i = 1, 2, \cdots, n$, and M is the lowest common multiple of the denominators u_i of α_i's, in which $\alpha_i = \frac{v_i}{u_i}$ and $(u_i, v_i) = 1$, where $u_i, v_i \in \mathbb{Z}_+, \forall i$. Then, the zero solution of system (3.1), subject to the initial vector condition $X(0) = X_0 \in \mathbb{R}^n$, is:*
- *Asymptotically stable if any zero solution of the polynomial:*

$$\det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \cdots, \lambda^{M\alpha_n}\right) - (1-\lambda^M)A\right), \tag{3.16}$$

lie inside the set

$$\mathbb{C}\backslash K^\gamma,$$

where $\gamma = \frac{1}{M}$ and where

$$K^\gamma = \left\{ z \in \mathbb{C} : |z| \leq \left(2\cos\frac{|\arg z|}{\gamma}\right)^\gamma \text{ and } |\arg z| \leq \frac{\gamma\pi}{2} \right\}. \tag{3.17}$$

- *Not stable, furthermore, if there is a zero, say λ, of (3.16) such that $\lambda \in IntK^\gamma$.*

Proof. In accordance with Theorem 3.1, system (3.1) is asymptotically stable if all the zeros of the following characteristic equation:

$$\det\left(diag\left(z(1-z^{-1})^{\alpha_1}, z(1-z^{-1})^{\alpha_2}, \cdots, z(1-z^{-1})^{\alpha_n}\right) - A\right) = 0,$$

are located inside the unit circle. However, setting

$$1 - \frac{1}{z} = \lambda^M \Leftrightarrow z = \frac{1}{1-\lambda^M}, \lambda^M \neq 1,$$

will turn the above characteristic equation to be in the form:

$$\det\left(diag\left(\frac{\lambda^{M\alpha_1}}{1-\lambda^M}, \frac{\lambda^{M\alpha_2}}{1-\lambda^M}, \cdots, \frac{\lambda^{M\alpha_n}}{1-\lambda^M}\right) - A\right) = 0. \tag{3.18}$$

Multiplying both sides of (3.18) by $(1-\lambda^M)^n$ yields:

$$\det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \cdots, \lambda^{M\alpha_n}\right) - (1-\lambda^M)A\right) = 0. \tag{3.19}$$

Now, one finds that it is necessary to prove the two assertions $z \in \{z \in \mathbb{C} : 0 < |z| < 1\} \Leftrightarrow \lambda \in \mathbb{C} \backslash K^\gamma$ and $z \in \{z \in \mathbb{C} : |z| > 1\} \Leftrightarrow \lambda \in IntK^\gamma$. For achieving those goals, consider the following steps:

• Step 1: (Defining the stability boundary). Consider the following curve:

$$L^\gamma = \left\{ \left(1 - \frac{1}{z}\right)^\gamma : |z| = 1 \right\}, \tag{3.20}$$

which defines the stability boundary for system (3.1) and also describes its structure. Suppose $z = e^{i\varphi}$ and $1 - z^{-1} = re^{i\omega}$ for $0 \leq \varphi < 2\pi$ and $r = r(\varphi) \geq 0$, and also suppose $\omega = \omega(\varphi)$, where $\omega \in [0, 2\pi)$. Then,

$$1 - e^{-i\varphi} = re^{i\omega}.$$

This equation, after the imaginary and real parts are equated, will be turned into the following two components:

$$\sin\varphi = r\sin\omega, \quad 1 - \cos\varphi = r\cos\omega.$$

Observe that, when $\varphi = 0$, then $r = 0$. Otherwise, we have:

$$\tan\omega = \frac{\sin\varphi}{1 - \cos\varphi}.$$

In view of the fact that:

$$\frac{\sin\varphi}{1 - \cos\varphi} = \frac{2\sin(\varphi/2)\cos(\varphi/2)}{2\sin^2(\varphi/2)} = \cot\frac{\varphi}{2} = \tan(\frac{\pi}{2} - \frac{\varphi}{2}),$$

we can write r and ω as $r = 2\sin\frac{\varphi}{2}$ and $\omega = \pi/2 - \varphi/2$, respectively. From here, we obtain:

$$L^\gamma = \left\{ \left(2\sin\frac{\varphi}{2}\right)^\gamma \exp(i\frac{(\pi - \varphi)\gamma}{2}) : 0 \leq \varphi < 2\pi \right\}.$$

Observe that setting $\theta = -\omega = \varphi/2 - \pi/2$, will turn L^γ to be as follows:

$$L^\gamma = \left\{ (2\cos\theta)^\gamma \exp(-i\gamma\theta) : -\frac{\pi}{2} \leq \theta < \frac{\pi}{2} \right\}.$$

One can use the polar form represented by ($|z| = (2\cos\theta)^\gamma$), where $\arg z = -\gamma\theta$, to obtain:

$$L^\gamma = \left\{ z \in \mathbb{C} : |z| = \left(2\cos\frac{|\arg z|}{\gamma}\right)^\gamma \text{ and } |\arg z| \leq \frac{\gamma\pi}{2} \right\}. \tag{3.21}$$

• Step 2: (Showing that $w_\gamma(z) = (1 - \frac{1}{z})^\gamma$ maps the set $D_E = \{z \in \mathbb{C} : |z| > 1\}$ onto $IntK^\gamma$, with noting that $w_\gamma(2) = 2^{-\gamma} \in IntK^\gamma$). In view of the Open mapping theorem, and since w_γ is nonconstant holomorphic on D_E, then it maps D_E to an open set. In other words, we have a neighborhood of $w_\gamma(z^*)$ included in $w_\gamma(D_E), \forall z^* \in D_E$. This implies that the boundary of $w_\gamma(D_E)$ can not be mapped by any point of D_E. This means that $w_\gamma(D_E) \subset IntK^\gamma$. Similarly, one can prove that $w_\gamma(D_I) \subset \mathbb{C}\backslash K^\gamma$, where $D_I = \{z \in \mathbb{C} : 0 < |z| < 1\}$. In view of $w_\gamma(\{z \in \mathbb{C} : |z| = 1\}) = L^\gamma$ (see the previous step), and also in view of the continuity of w_γ, the above arguments imply that $w_\gamma(D_E) = IntK^\gamma \backslash \{1\}$.

• Step 3: For the purpose of showing the other part this theorem, we first assume that there is a solution λ of (3.16) with $\lambda \in IntK^\gamma$. This implies that $z = \frac{1}{1-\lambda^M}$ is a solution of (3.2) with $|z| > 1$. Thus, we can deduce, in view of Theorem 3.1, that there is an instability of the zero solution of system (3.1). On the other hand, if each solution of (3.16) belongs to $\mathbb{C}\backslash K^\gamma$, then all solutions of (3.1) will belong to $\{z \in \mathbb{C} : |z| < 1\}$, which makes the zero solution of system (3.1), via Theorem 3.1, asymptotically stable, as required. ■

3.2 Numerical simulations

To highlight the primary outcomes of this chapter in exploring the stability of the linear incommensurate FoDSs, Theorem 3.2 will be utilized to investigate two examples that explore such stability when these systems involve full matrices in their forms.

Example 3.2 *Consider the following linear incommensurate FoDS:*

$$\begin{pmatrix} {}^C\Delta_0^{\frac{1}{2}} x_1(k+1-\frac{1}{2}) \\ {}^C\Delta_0^{\frac{1}{4}} x_2(k+1-\frac{1}{4}) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -\frac{9}{16} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}, \quad k = 0, 1, 2, \cdots. \tag{3.22}$$

One can obtain M to be equal 4. This, however, implies:

$$\det\left(\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix} - (1-\lambda^4)\begin{pmatrix} -1 & 1 \\ -\frac{9}{16} & \frac{1}{2} \end{pmatrix}\right) = 0$$

$\Leftrightarrow$

$$\det\begin{pmatrix} \lambda^2 + (1-\lambda^4) & -(1-\lambda^4) \\ \frac{9}{16}(1-\lambda^4) & \lambda - \frac{1}{2}(1-\lambda^4) \end{pmatrix} = 0$$

$\Leftrightarrow$

$$\frac{1}{16}\lambda^8 + \frac{1}{2}\lambda^6 - \lambda^5 - \frac{1}{8}\lambda^4 + \lambda^3 - \frac{1}{2}\lambda^2 + \lambda + \frac{1}{16} = 0. \tag{3.23}$$

Accordingly, the solution of (3.23) will be in the following form:

$$\begin{aligned}
\lambda_1 &= -1.1634, \\
\lambda_2 &= -6.0451 \times 10^{-2}, \\
\lambda_3 &= -0.78732 + 3.1894i, \\
\lambda_4 &= -0.78732 - 3.1894i, \\
\lambda_5 &= 1.3269 - 0.4875i, \\
\lambda_6 &= 1.3269 + 0.4875i, \\
\lambda_7 &= 7.2415 \times 10^{-2} + 0.80874i, \\
\lambda_8 &= 7.2415 \times 10^{-2} - 0.80874i.
\end{aligned}$$

As per Theorem 3.2, and due to $\lambda_i \in \mathbb{C}\backslash K^{\frac{1}{4}}, \forall i = 1, 2, \cdots, 8,$ *then system (3.22) is asymptotically stable about its zero solution.*
In order to demonstrate the validity of the obtained outcomes, one can observe, from Figure 3.1, that the two states of system (3.22) converge to zero, and hence it is stable.

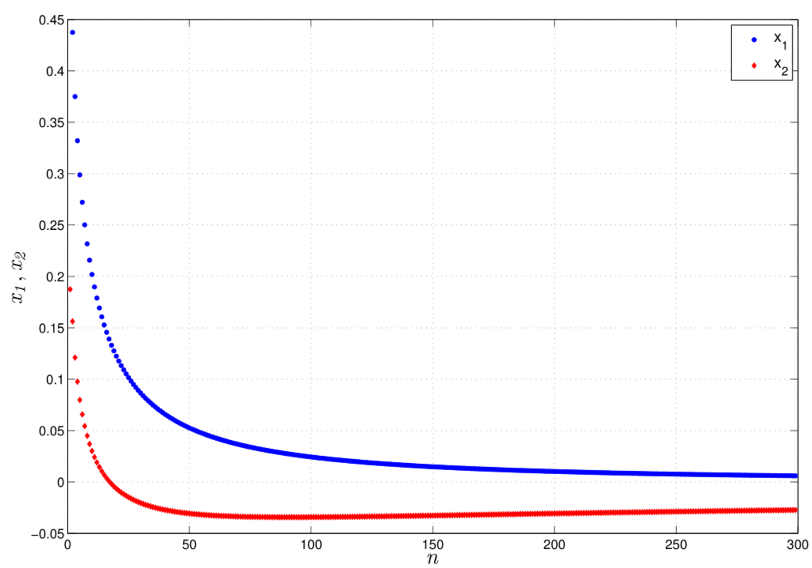


Figure 3.1: The stability of the zero solution of system (3.22).

Example 3.3 *Consider the following linear incommensurate FoDS:*

$$\begin{pmatrix} {}^C\Delta_0^{\frac{1}{2}} x_1(k+1-\frac{1}{2}) \\ {}^C\Delta_0^{\frac{1}{3}} x_2(k+1-\frac{1}{3}) \\ {}^C\Delta_0^{\frac{2}{3}} x_3(k+1-\frac{2}{3}) \end{pmatrix} = \begin{pmatrix} -1 & 0 & -0.2 \\ 3.4 & -1 & 0.2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix}, \quad k = 0, 1, 2, \cdots . \tag{3.24}$$

One can find $M = 6$, which leads to:

$$\det\left(\begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^4 \end{pmatrix} - (1-\lambda^6) \begin{pmatrix} -1 & 0 & -0.2 \\ 3.4 & -1 & 0.2 \\ 0 & 0 & -1 \end{pmatrix} \right) = 0$$

$\Leftrightarrow$

$$\begin{gathered} -\lambda^{18} + \lambda^{16} + \lambda^{15} + \lambda^{14} - \lambda^{13} + 2\lambda^{12} - \lambda^{11} - 2\lambda^{10} - \lambda^9 \\ -2\lambda^8 + \lambda^7 - 2\lambda^6 + \lambda^5 + \lambda^4 + \lambda^3 + \lambda^2 + 1 = 0. \end{gathered} \tag{3.25}$$

Consequently, the solution of (3.25) will be in the form:

$$\begin{pmatrix} -0.32131 - 0.87498i \\ -0.32131 + 0.87498i \\ 0.32131 - 0.87498i \\ 0.32131 + 0.87498i \\ -0.85180 \\ -0.54463 + 0.7276i \\ -0.54463 - 0.7276i \\ 0.54463 + 0.7276i \\ 0.54463 - 0.7276i \\ 0.42590 - 0.73768i \\ 0.42590 + 0.73768i \\ -1.1510 \\ 1.1510 \\ -1.2106 \\ 1.2106 \\ -0.58699 - 1.0167i \\ -0.58699 + 1.0167i \\ 1.1740 \end{pmatrix}$$

where

$$K^{\frac{1}{6}} = \left\{ z \in \mathbb{C} : |z| \le (2\cos 6\,|\arg z|)^{\frac{1}{6}} \text{ and } |\arg z| \le \frac{\pi}{12} \right\}.$$

Hence, in view of Theorem 3.2, system (3.24) is also asymptotically stable about its zero solution. To confirm the final inference of Example 3.2, Figure 3.2 illustrates such stability by exhibiting the convergence of all system's states to zero, which shows the validity of the proposed results.

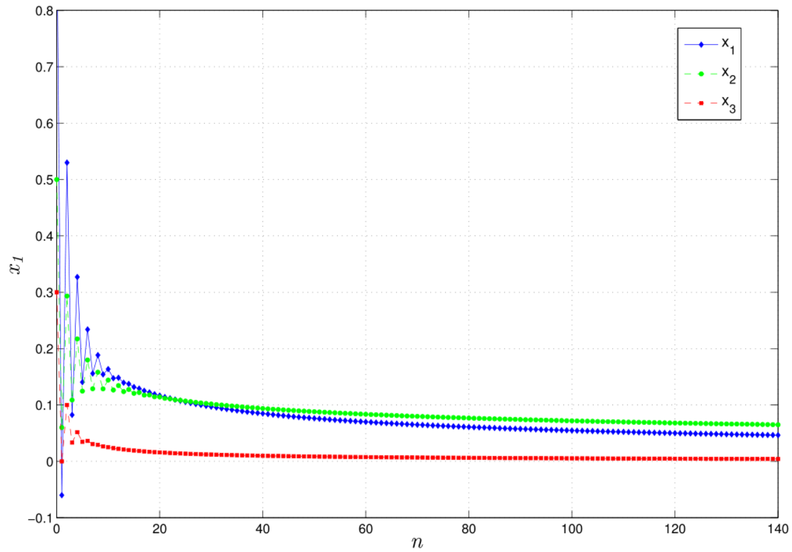


Figure 3.2: The stability of the zero solution of system (3.24).

Chapter 4

Synchronization of incommensurate fractional order difference systems

In this chapter, we study the synchronization of incommensurate fractional order discrete systems. The considered synchronization scheme can be tailored to encompass several types of classical synchronization. Numerical examples are presented to test the findings of the study.

4.1 Synchronization

4.1.1 Master-slave system

We consider a master system represented by:

$$\begin{cases} {}^C\Delta_0^{\alpha_1} x_1(k+1-\alpha_1) = f_1(X(k)), \\ {}^C\Delta_0^{\alpha_2} x_2(k+1-\alpha_2) = f_2(X(k)), \\ \vdots \\ {}^C\Delta_0^{\alpha_n} x_n(k+1-\alpha_n) = f_n(X(k)), \end{cases} \quad k = 0, 1, \cdots, \tag{4.1}$$

where ${}^C\Delta_0^{\alpha_i}$ is the Caputo fractional difference of order $\alpha_i, 0 < \alpha_i \leq 1$, for $i = 1, 2, \cdots, n$, $X(k) = (x_1(k), x_2(k), \cdots, x_n(k))^T \in \mathbb{R}^n$ is the state of the system (4.1) and $(f_1, f_2, \cdots, f_n)^T : \mathbb{R}^n \to \mathbb{R}^n$. And a slave system given by:

$$\begin{cases} {}^C\Delta_0^{\alpha_1} y_1(k+1-\alpha_1) = g_1(X(k)) + U_1, \\ {}^C\Delta_0^{\alpha_2} y_2(k+1-\alpha_2) = g_2(X(k)) + U_2, \\ \vdots \\ {}^C\Delta_0^{\alpha_n} y_n(k+1-\alpha_n) = g_n(X(k)) + U_n, \end{cases} \quad k = 0, 1, \cdots, \tag{4.2}$$

where $Y(k) = (y_1(k), y_2(k), \cdots, y_n(k))^T \in \mathbb{R}^n$ is the state of the system (4.2), $(g_1, g_2, \cdots, g_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ and $U = (U_1, U_2, \cdots, U_n)^T \in \mathbb{R}^n$ is a control vector to be determined.

4.1.2 Synchronization types

Complete synchronization (C.S)

Definition 4.1 *[37] The problem of complete synchronization is to determine the control U so that*

$$\lim_{k\to\infty} \|Y(k) - X(k)\| = 0, \tag{4.3}$$

where $\|.\|$ is the euclidean norm.

Remark 4.1 *If $(f_1, f_2, \cdots, f_n) = (g_1, g_2, \cdots, g_n)$, the relationship becomes identical complete synchronization.*
If $(f_1, f_2, \cdots, f_n) \neq (g_1, g_2, \cdots, g_n)$, it is a non-identical complete synchronization.

Anti-synchronization

Definition 4.2 *[37] The problem of anti-synchronization is to determine the control U so that*

$$\lim_{k\to\infty} \|Y(k) + X(k)\| = 0. \tag{4.4}$$

Projective synchronization

Definition 4.3 *[38] We say that we have a projective synchronization between the systems (4.1) and (4.2), if there exists a diagonal matrix $H = diag(h_1, \cdots, h_n)$, such as:*

$$\lim_{k\to\infty} \|Y(k) - H \times X(k)\| = 0. \tag{4.5}$$

Remark 4.2 *The case where all h_i are equal to 1 represents a case of complete synchronization.*
The case where all h_i are equal to -1 represents a case of anti synchronization.

FSHP synchronization (FSHP.S)

Definition 4.4 *[39] We say that we have a FSHP synchronization (full state hybrid projective synchronization) between the master system (4.1) and the slave system (4.2), if there exists a controls $U_i, 1 \le i \le n$, and a constants $(\gamma_{ij})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$, such as:*

$$\lim_{k\to\infty} \left| y_i(k) - \sum_{j=1}^{n} \gamma_{ij} x_j(k) \right| = 0, \quad i = 1, \cdots, n. \tag{4.6}$$

Inverse FSHP synchronization (I.FSHP.S)

Definition 4.5 *[40] We say that we have inverse FSHP synchronization between the master system (4.1) and the slave system (4.2), if there exists a controls* $U_i, 1 \leq i \leq n$, *and a constants* $(\beta_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$, *so that synchronization errors*

$$\lim_{k\to\infty} \left| x_i(k) - \sum_{j=1}^{n} \beta_{ij} y_j(k) \right| = 0, \quad i = 1, \cdots, n. \tag{4.7}$$

Generalized synchronization (G.S)

Definition 4.6 *[41] If there exists a controller* U *and a function* $\phi : \mathbb{R}^n \to \mathbb{R}^n$, *check*

$$\lim_{k\to\infty} \|Y(k) - \phi(X(k))\| = 0, \tag{4.8}$$

then, systems (4.1) and (4.2) synchronize in the generalized sense with respect to the function ϕ.

Remark 4.3 *Generalized synchronization is considered to be a generalization of complete synchronization, anti-synchronization, projective synchronization and FSHP synchronization.*

Inverse generalized synchronization (I.G.S)

Definition 4.7 *[42] If there exists a controller* U *and a function* $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, *check*

$$\lim_{k\to\infty} \|X(k) - \varphi(Y(k))\| = 0, \tag{4.9}$$

then, systems (4.1) and (4.2) synchronize in the inverse generalized sense with respect to the function φ.

Remark 4.4 *If the function* ϕ *is defined by* $\phi(Y(k)) = BY(k)$ *where* $B = (\beta_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$, *we say that we have a inverse full-state hybrid projective synchronization.*

Synchronization Q-S

Definition 4.8 *[43] We say that systems (4.1) and (4.2) are in* $Q - S$ *synchronization in dimension* d, *if there is a controller* U *and two functions* $Q : \mathbb{R}^n \to \mathbb{R}^d, S : \mathbb{R}^n \to \mathbb{R}^d$ *such that*

$$\lim_{t\to\infty} \|Q(X(k)) - S(Y(k))\| = 0. \tag{4.10}$$

Remark 4.5 $Q - S$ *synchronization is considered to be a generalization of all types of previous synchronizations.*

4.2 Analytical results

Let us consider the master system given by:

$$\begin{cases} {}^C\Delta_0^{\alpha_1} x_1(k+1-\alpha_1) = \sum_{j=1}^{n} a_{1j}x_j(k) + f_1(X(k)), \\ {}^C\Delta_0^{\alpha_2} x_2(k+1-\alpha_2) = \sum_{j=1}^{n} a_{2j}x_j(k) + f_2(X(k)), \\ \vdots \\ {}^C\Delta_0^{\alpha_n} x_n(k+1-\alpha_n) = \sum_{j=1}^{n} a_{nj}x_j(k) + f_n(X(k)), \end{cases} \quad k = 0, 1, \cdots, \tag{4.11}$$

with $X(k) = (x_1(k), x_2(k), \cdots, x_n(k))^T$ denoting its state vector, $A = (a_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$, and $(f_1, f_2, \cdots, f_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ being a nonlinear function. Let us also consider as slave the system:

$$\begin{cases} {}^C\Delta_0^{\alpha_1} y_1(k+1-\alpha_1) = \sum_{j=1}^{n} b_{1j}y_j(k) + g_1(Y(k)) + U_1, \\ {}^C\Delta_0^{\alpha_2} y_2(k+1-\alpha_2) = \sum_{j=1}^{n} b_{2j}y_j(k) + g_2(Y(k)) + U_2, \\ \vdots \\ {}^C\Delta_0^{\alpha_n} y_n(k+1-\alpha_n) = \sum_{j=1}^{n} b_{nj}y_j(k) + g_n(Y(k)) + U_n, \end{cases} \quad k = 0, 1, \cdots, \tag{4.12}$$

where, again, $Y(k) = (y_1(k), y_2(k), \cdots, y_n(k))^T$ are the states, $B = (b_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$, and $(g_1, g_2, \cdots, g_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function. The vector $U = (U_1, U_2, \cdots, U_n)^T$ is a controller used to drive the states of the slave system towards a specific trajectory to synchronize with the master.

To achieve synchronization between systems (4.11) and (4.12). The following theorem presents the control laws.

Theorem 4.1 *The master–slave pair (4.11)–(4.12) is globally complete synchronized by means of the control law:*

$$U_i = -\left(\sum_{j=1}^{n} (c_{ij} - b_{ij})e_j(k) + \sum_{j=1}^{n} b_{ij}y_j(k) + g_i(Y(k)) - \sum_{j=1}^{n} a_{ij}x_j(k) - f_i(X(k)) \right), \quad \forall i = 1, 2, \cdots, n, \tag{4.13}$$

subject to the control matrix C being selected as follow:

$$c_{ij} = b_{ij}, \text{ if } i \neq j \text{ and } -2^{\alpha_i} < b_{ii} - c_{ii} < 0, \quad 1 \leq i, j \leq n. \tag{4.14}$$

Remark 4.6 *The matrix C can also be selected so that:*

$$-2^{\alpha_i} < b_{ii} - c_{ii} < 0 \quad 1 \leq i \leq n, \text{ and } B - C \text{ is a triangular matrix.} \tag{4.15}$$

Proof. The Caputo fractional difference of the error system:

$$e_i(k) = y_i(k) - x_i(k), \quad i = 1, 2, \cdots, n,$$

is

$$\begin{aligned} {}^C\Delta_0^{\alpha_i} e_i(k+1-\alpha_i) &= {}^C\Delta_0^{\alpha_i} y_i(k+1-\alpha_i) - {}^C\Delta_0^{\alpha_i}\ x_i(k+1-\alpha_i) \\ &= \sum_{j=1}^{n} b_{ij} y_j(k) + g_i(Y(k)) + U_i - \sum_{j=1}^{n} a_{ij} x_j(k) - f_i(X(k)), \quad i = 1, 2, \cdots, n. \end{aligned} \tag{4.16}$$

By defining a control matrix $C = (c_{ij})_{1 \le i,j \le n} \in \mathbb{R}^{n \times n}$ and

$$R_i = \sum_{j=1}^{n} (c_{ij} - b_{ij}) e_j(k) + \sum_{j=1}^{n} b_{ij} y_j(k) + g_i(Y(k)) - \sum_{j=1}^{n} a_{ij} x_j(k) - f_i(X(k)), \quad i = 1, 2, \cdots, n, \tag{4.17}$$

we can rewrite (4.16) in the form

$${}^C\Delta_0^{\alpha_i} e_i(k+\alpha_i-1) = \sum_{j=1}^{n} (b_{ij} - c_{ij}) e_j(k) + R_i + U_i(k), \quad i = 1, 2, \cdots, n. \tag{4.18}$$

Substituting (4.13) and (4.14) into (4.18), the fractional error system can be described as

$${}^C\Delta_a^{\alpha_i} e_i(k+\alpha_i-1) = (b_{ii} - c_{ii}) e_i(k), \quad i = 1, 2, \cdots, n. \tag{4.19}$$

It is easy to see that the matrix $B - C$ is a diagonal matrix with diagonal elements $\lambda_i = b_{ii} - c_{ii}, i = 1, 2, \cdots, n$, satisfy the conditions

$$-2^{\alpha_i} < \lambda_i < 0. \tag{4.20}$$

It, therefore, follows immediately from Corollary 3.1 that the zero solution of (4.19) is asymptotically stable and thus the master (4.11) and slave (4.12) are globally complete synchronized.

■

4.3 Numerical results

In order to put the control laws proposed in Theorems 4.1, we will present two numerical examples for a pair of fractional chaotic systems with different dimensions.

4.3.1 Synchronization in 2D

We consider as master the 2D fractional lorenz map [44] of the form:

$$\begin{cases} {}^C\Delta_a^{\alpha_1} x_1(k+1-\alpha_1) = \sum_{j=1}^{2} a_{1j} x_j(k) + f_1(x_1(k), x_2(k)), \\ {}^C\Delta_a^{\alpha_2} x_2(k+1-\alpha_2) = \sum_{j=1}^{2} a_{2j} x_j(k) + f_2(x_1(k), x_2(k)), \end{cases} \quad k = 0, 1, \cdots, \tag{4.21}$$

which exhibits a chaotic attractor, for instance, when $(a_{11}, a_{12}, a_{21}, a_{22}) = (0.9375.1, 0, 0, -0.75)$,

$$f_1(x_1(k), x_2(k)) = -0.75x_1(k)x_2(k),$$

$$f_2(x_1(k), x_2(k)) = 0.75x_1^2(k),$$

$\alpha_1 \neq \alpha_2, a = 0$ and initial conditions are $x_1(0) = 0.1, x_2(0) = 0$. The resulting chaotic attractor is shown in Figure 4.1 and its general shape is similar to that of the integer order one.

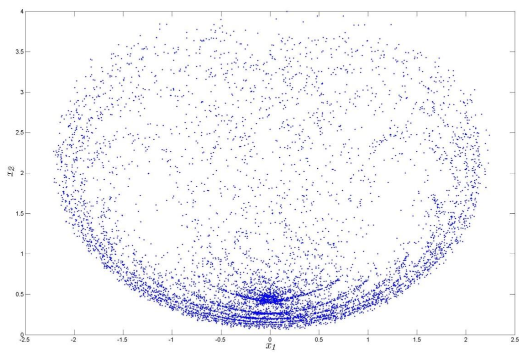


Figure 4.1: Chaotic attractor of the fractional order Lorenz map for $\alpha_1 = 0.98, \alpha_2 = 0.8$.

As for the slave, we select the fractional flow map proposed in [44]. The slave system is described as:

$$\begin{cases} {}^C\Delta_a^{\alpha_1} y_1(k+1-\alpha_1) = \sum\limits_{j=1}^{2} b_{1j}y_j(k) + g_1(y_1(k), y_2(k)) + U_1, \\ {}^C\Delta_a^{\alpha_2} y_2(k+1-\alpha_2) = \sum\limits_{j=1}^{2} b_{2j}y_j(k) + g_2(y_1(k), y_2(k)) + U_2, \end{cases} \quad k = 0, 1, \cdots, \tag{4.22}$$

where y_1, y_2 are states of the slave systems, respectively, and $(b_{11}, b_{12}, b_{21}, b_{22}) = (-1.1, 1, 0, -1)$,

$$g_1(y_1(k), y_2(k)) = 0,$$

$$g_2(y_1(k), y_2(k)) = y_1^2(k) - 1.7,$$

$a = 0$, $\alpha_1 \neq \alpha_2$ and $U_i(k), i = 1, 2$, are controllers, the uncontrolled map (4.22) with $U_1 = U_2 = 0$ is chaotic as shown in Figure 4.2.

It is easy to see that the linear part of the slave system (4.22) is given by:

$$B = \begin{pmatrix} -1.1 & 1 \\ 0 & -1 \end{pmatrix}.$$

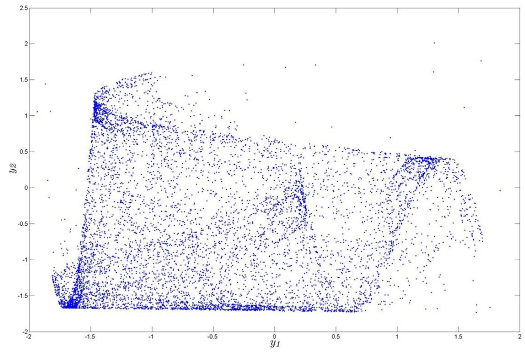


Figure 4.2: Chaotic attractor of the fractional order Flow map for $\alpha_1 = 0.98, \alpha_2 = 0.8$.

According to Theorem 4.1, there exists a control matrix C such that $B - C$ satisfies condition (4.15). One can, for instance, choose the case

$$C = \begin{pmatrix} -0.1 & 0 \\ 0 & 0 \end{pmatrix},$$

which clearly satisfies the condition, and by extension systems (4.21) and (4.22) are synchronized in 2D. Now, it is rather easy to construct the control law according to Theorem 4.1. The resulting error system is of the form:

$$\begin{cases} {}^{C}\Delta_a^{\alpha_1} e_1(k+1-\alpha_1) = -e_1(k) + e_2(k), \\ {}^{C}\Delta_a^{\alpha_2} e_2(k+1-\alpha_2) = -e_2(k), \end{cases} \quad k = 0, 1, \cdots .$$

The time evolution of the errors is depicted in Figure 4.3. Clearly, synchronization is achieved as the errors converge to zero in sufficient time.

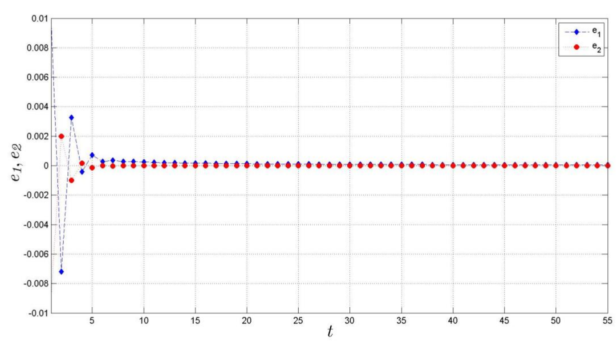


Figure 4.3: Evolution of states of the error system for $\alpha_1 = 0.98, \alpha_2 = 0.8$.

4.3.2 Synchronization in 3D

We consider as master the 3D fractional Stefanski map [45] of the form

$$\begin{cases} {}^{C}\Delta_a^{\alpha_1} x_1(k+1-\alpha_1) = \sum_{j=1}^{3} a_{1j}x_j(k) + f_1(x_1(k), x_2(k), x_3(k)), \\ {}^{C}\Delta_a^{\alpha_2} x_2(k+1-\alpha_2) = \sum_{j=1}^{3} a_{2j}x_j(k) + f_2(x_1(k), x_2(k), x_3(k)), \quad k = 0, 1, \cdots, \\ {}^{C}\Delta_a^{\alpha_3} x_3(k+1-\alpha_3) = \sum_{j=1}^{3} a_{3j}x_j(k) + f_3(x_1(k), x_2(k), x_3(k)), \end{cases} \tag{4.23}$$

which exhibits a chaotic attractor, for instance, when $(a_{11}, a_{12}, a_{13}, a_{21}, a_{23}, a_{32}, a_{33}) = (-1, 0, 1, 0, 0, 0, -1)$, $-1 < a_{22} < 0$, $a_{31} = a_{22} + 1$,

$$f_1(x_1(k), x_2(k), x_3(k)) = 1 - \alpha x_2^2(k),$$

$$f_2(x_1(k), x_2(k), x_3(k)) = 1 - \alpha x_1^2(k),$$

$$f_3(x_1(k), x_2(k), x_3(k)) = 0,$$

$a = 0$, $\alpha > 0$ and $\alpha_1 \neq \alpha_2 \neq \alpha_3$. The resulting chaotic attractor is shown in Figure 4.4 and its general shape is similar to that of the integer order one.

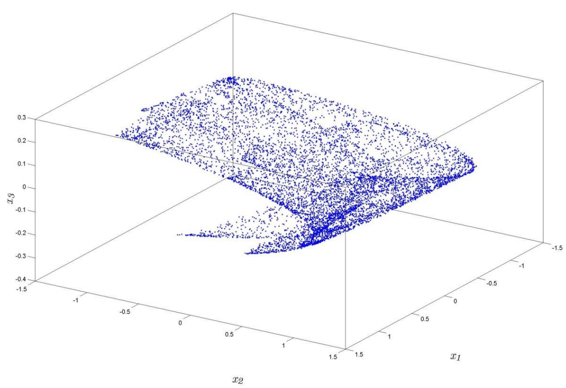


Figure 4.4: Chaotic attractor of the fractional order Stefansky map for $\alpha_1 = 0.97, \alpha_2 = 0.969, \alpha_3 = 0.975$.

As for the slave, we select the more general $3-$component fractional system proposed in [44].

The system is described as

$$\begin{cases} {}^C\Delta_a^{\alpha_1} y_1(k+1-\alpha_1) = \sum_{j=1}^{3} b_{1j}y_j(k) + g_1(y_1(k), y_2(k), y_3(k)) + U_1, \\ {}^C\Delta_a^{\alpha_2} y_2(k+1-\alpha_2) = \sum_{j=1}^{3} b_{2j}y_j(k) + g_2(y_1(k), y_2(k), y_3(k)) + U_2, \quad k = 0, 1, \cdots, \\ {}^C\Delta_a^{\alpha_3} y_3(k+1-\alpha_3) = \sum_{j=1}^{3} b_{3j}y_j(k) + g_3(y_1(k), y_2(k), y_3(k)) + U_3, \end{cases} \tag{2.24}$$

where $(b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{23}, b_{32}, b_{33}) = (-1.9, 0.5, 0, -1.9, 0, 0.2, 0, 0, -1.9)$,

$$g_1(y_1(k), y_2(k), y_3(k)) = 0,$$

$$g_2(y_1(k), y_2(k), y_3(k)) = 0,$$

$$g_3(y_1(k), y_2(k), y_3(k)) = 2 - 0.6y_2(k)y_3(k),$$

U_1, U_2 and U_3 are controllers, $a = 0$ and $\alpha_1 \neq \alpha_2 \neq \alpha_3$, the uncontrolled map (2.24) with $U_1 = U_2 = U_3 = 0$ is chaotic as shown in Figure 4.5.

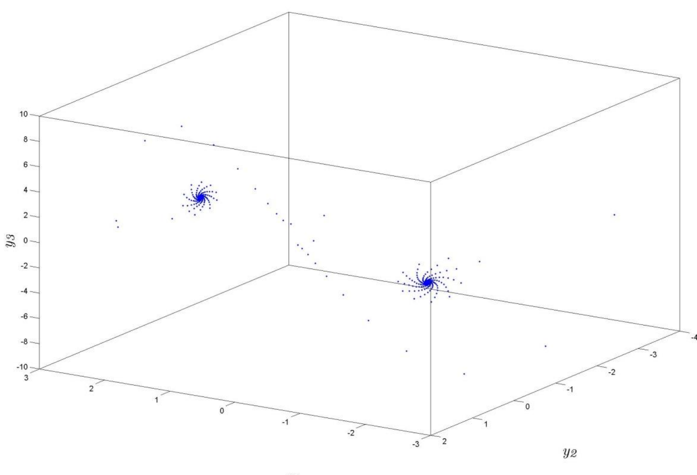


Figure 4.5: Chaotic attractor of the fractional order Wang map for $\alpha_1 = 0.97, \alpha_2 = 0.969, \alpha_3 = 0.975$.

The linear part of the slave system is

$$B = \begin{pmatrix} -1.9 & 0.5 & 0 \\ -1.9 & 0 & 0.2 \\ 0 & 0 & -1.9 \end{pmatrix}.$$

According to Theorem 4.1, there exists a control matrix C such that $B - C$ satisfies condition (4.15). One can, for instance, choose the case

$$C = \begin{pmatrix} -1.6 & 0 & 0 \\ -1.9 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which clearly satisfies the conditions. As a result, systems (4.23) and (4.24) are synchronized in three dimensions. Figure 4.6 depicts the convergence of the errors, which belong to the fractional system

$$\begin{cases} {}^C\Delta_a^{\alpha_1} e_1(k+1-\alpha_1) = -0.3e_1(k) + 0.5e_2(k), \\ {}^C\Delta_a^{\alpha_2} e_2(k+1-\alpha_2) = -e_2(k) + 0.2e_3(k), \qquad k = 0, 1, \cdots, \\ {}^C\Delta_a^{\alpha_3} e_3(k+1-\alpha_3) = -0.9e_3(k), \end{cases}$$

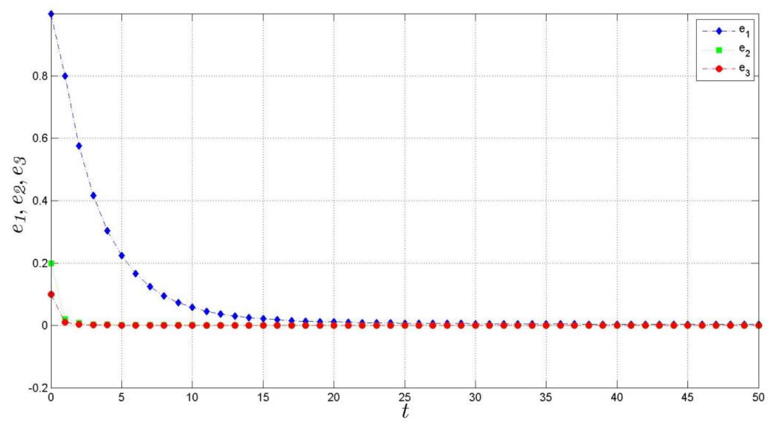


Figure 4.6: Evolution of states of the error system for $\alpha_1 = 0.97, \alpha_2 = 0.969, \alpha_3 = 0.975$.

General conclusion and perspectives

In this thesis, some simple applicable conditions for judging the stability of the linear incommensurate fractional-order difference systems with have been reported as novel results. These results have really been verified numerically by illustrating the stability of the solutions of such systems via several examples. To achieve the intended goals, we began by presenting preliminary chapters on the stability of integer-order difference systems and commensurate fractional-order difference systems. All these results are applicable to be implemented in lots of difference systems, like e.g., the Duffing oscillator system which has been successfully employed to model a set of physical schemes such as beam buckling, ionization waves in plasmas, nonlinear electronic circuits, stiffening springs, and superconducting Josephson parametric amplifiers. Such investigation together with studying the dynamics of the linear incommensurate FoDSs will be some of several targets that left for future consideration. These results have been applied in synchronization. More specifically in synchronization of incommensurate fractional order dynamical systems.

We hope in our future work to solve the problem of stability of non linear incommensurate fractional order difference systems, which will help to study synchronization of non linear incommensurate fractional order difference systems more broadly.

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