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**DOCTORAL THESIS**

**Field**: Mathematics and Computer Science

**Option**:Mathematics

**Specialty**: Applied Mathematics

Theme:

# Dynamics of a Turing-type reaction-diffusion model

**Presented by:**

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بسم الله الرحمن الرحيم

# Dedicace

To my lovely parents..
To my dear brothers..
To my dear husband..
To all of my family, especially those closest to my heart, my grandmother and grandfather, my uncle and my aunts..

# Acknowledgements

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Abstract

The aim of this thesis is to study the dynamics of a Turing-type reaction-diffusion model, i.e. study a reaction-diffusion model that exhibits Turing properties, whereas the reaction-diffusion defines the mechanism in which several interacting chemicals or agents react together while diffusing or spreading across a liquid or gaseous medium simultaneously, usually, these processes are studied for their ability to produce nontrivial patterns that evolve over time. Such patterns are driven by diffusion, also referred to as Turing structures or Turing patterns. The Turing patterns are measured in the presence of diffusion, but are not present in the absence of diffusion, this mechanisme called "diffusion-driven instability" or "Turing instability". In this work, the study was related "the well-known Degn-Harrison reaction diffusion model and its generalization". Our thechnique to prove the asymptotic stability of the steady state solution is based on the eigen-analysis, the Poincaré–Bendixson theorem and the direct Lyapunov method.
**keywords:**

Reaction-Diffusion Systems, Turing Instability, Global Existence, Lyapunov Functional, Stability Analysis.

# Résumé

Le but de cette thèse est d'étudier la dynamique d'un modèle de réaction-diffusion de type Turing, i.e. d'étudier un modèle de réaction-diffusion qui présente les propriétés de Turing, telle que le réaction-diffusion définit le mécanisme par lequel plusieurs produits chimiques ou agents interagissants réagissent ensemble tout en diffusant ou en diffusant simultanément sur un milieu liquide ou gazeux, habituellement, ces processus sont étudiés pour leur capacité à produire des modèles non triviaux qui évoluent au fil du temps. Ces modèles sont entraînés par la diffusion, également appelée structures de Turing ou modèles de Turing. Les modèles de Turing sont mesurés en la présence de diffusion, mais ne sont pas présents en l'absence de diffusion, ce mécanisme appelé "instabilité de diffusion" ou "instabilité de Turing". Dans ce travail, l'étude a été liée "le modèle bien connu de réaction-diffusion Degn-Harrison et sa généralisation". Notre technique pour prouver la stabilité asymptotique de la solution à l'état d'équilibre est basée sur l'analyse du valeurs propres, le théorème de Poincaré–Bendixson et la méthode directe de Lyapunov.

## Mots clés:

Système de Réaction-Diffusion, Instabilité de Turing, Existence Globale, Fonction de Lyapunov, Analyse de Stabilité.

# ملخص

الهدف من هذه الأطروحة هو دراسة ديناميكيات أنظمة التفاعل-الانتشار من النوع نموذج تورينج ، أي دراسة نموذج التفاعل-الانتشار الذي يعرض خصائص تورينج، حيث يعرف التفاعل-الانتشار الآلية التي يتفاعل فيها العديد من المواد الكيميائية أو تتفاعل العوامل معا أثناء الانتشار أو الانتشار عبر وسيط سائل أو غازي في نفس الوقت، تُدرس هذه العمليات عادة لقدرتها على إنتاج أنماط غير بديهية تتطور مع مرور الوقت وهذه الأنماط يحركها الانتشار، ويشار إليها أيضاً باسم"هياكل تورينج " أو "أنماط تورينج" وتُقاس أنماط تورينج بوجود الانتشار، ولكن ليس في غيابه، هذه الآلية تسمى "عدم الاستقرار المدفوع بالانتشار" أو "عدم استقرار تورينج". في هذا العمل ارتبطت الدراسة بنموذج التفاعل-الانتشار المعروف ديجن-هاريسون والتعميم الخاص به، حيث اعتمدنا لإثبات الاستقرار المقارب لحل الحالة المستقرة على تحليل القيم الذاتية، نظرية بوانكاري-بنديكسو وطريقة ليابونوف المباشرة.

## الكلمات المفتاحية:

أنظمة التفاعل-الانتشار، عدم استقرار تورينج، الوجود الكلي، دالة ليابونوف، تحليل الاستقرار.

# CONTENTS

# General Introduction

Nature refers to everything found in the universe from various remove commas, pattern formation which is a complex process. These patterns occur in different contexts and can sometimes be modelled mathematically. Natural patterns include symmetries (animal coats, snowflakes, flowers, echinoderms, crystals), trees, spirals, meanders (sinuous bends in rivers), waves (dunes, wind waves, sea waves), foams, tessellations(honeycomb, bony fish, reptiles), cracks, stripes (angelfish, zebras) and spots (leopards, ladybirds). There even exist microscopic patterns in nature such as the patterns in the connectivity of neurons in the brain's visual cortex. Despite their complexity and wide variety, the abundance of patterns in nature suggests that there may be a set of simple principles governing pattern formation in general.

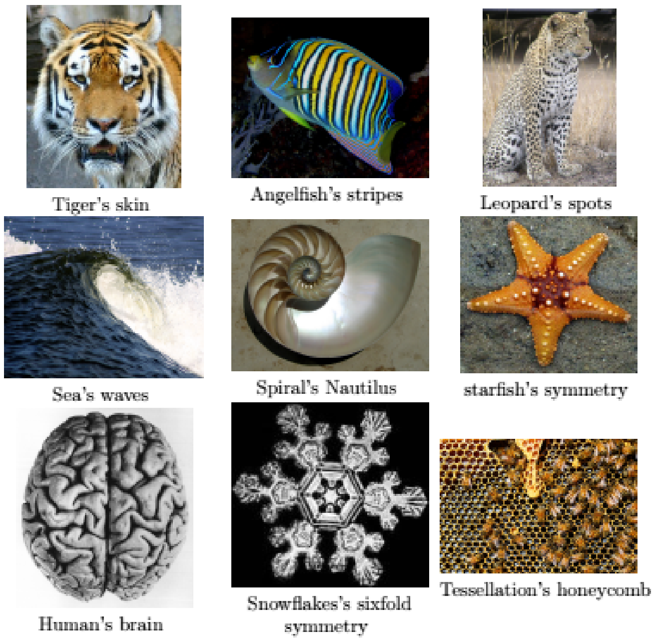


Figure 1: Pattern formation examples

Scientists have become more and more interested in understanding the processes of biological pattern formation over the past fifty years, and this area has become a fertile ground for collaborations between experimental groups and applied mathematicians.

In 1952, the British mathematician, logician, cryptanalyst and theoretical biologist, Alan Turing, proposed that pattern formation could be understood using a simple system of reaction-diffusion equations representing interacting chemicals. More importantly, he suggested that patterns could originate due to the interactions of otherwise stabilizing processes. In his seminal paper entitled **"The Chemical Basis of Morphogenesis"**[50], he predicted the striking idea of "diffusion-driven instability", which states that instead of acting to equalize concentration differences in space, diffusion can be coupled to suitable reaction–diffusion systems to destabilize a stable homogeneous steady state and generate stable and time-independent concentration patterns.

Over years, the concept of Turing instability attracted the interest of a large number of researchers and its theoretical aspects were successfully analyzed. Not only has it been studied in the biological and chemical fields, some investigations extend as far as economics, the physics of semiconductors and star formation [38].

In 1990, nearly 40 years after Turing's paper was written, De Kepper et al. ([19, 15]) introduced the first experimental evidence of Turing pattern through the chlorite-iodide-malonic acid and starch (CIMA) reaction in an open unstirred gel reactor. This CIMA model showed that under certain conditions the Laplacian (diffusion) driven instability of the model gives rise to oscillatory solutions and, therefore, pattern formation. The fact that there are five reactants involved in the CIMA reaction makes the mathematical description very complicated. However, observing that three of the five reactants remain nearly constants in the CIMA reaction, Lengyel and Epstein ([35, 37]) were able to reduce it to a $2 \times 2$ system.

In our work, we focus on the Degn-Harrison model, which is another Turing-type system. This model was first proposed as early as 1969 by Degn and Harrison [18] to describes the respiratory behavior of the Klebsiella Aerogenes bacterial culture, which is shown in Figure 2. The following is a brief description of the main contents and contribution made in this thesis.

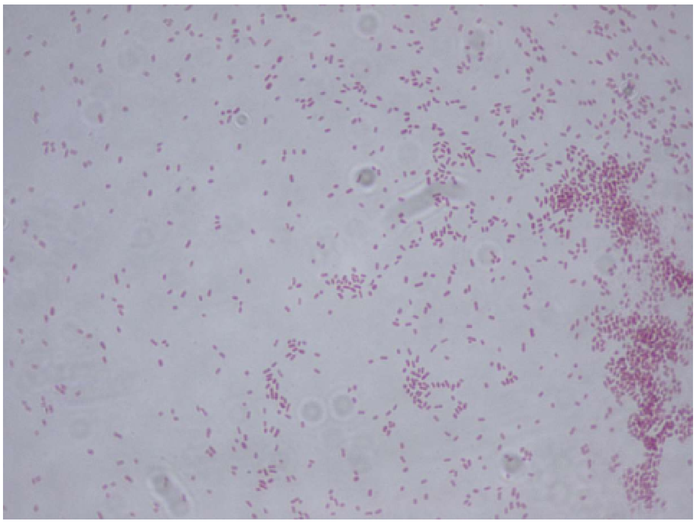

Figure 2: Klebsiella Aerogenes bacterial culture

# Outline of the thesis

The thesis is organized as follows:

The first Chapter is devided into three sections which are:

- Nomenclature.
- General notions: presents the basic definitions, theorems, formulas, and inequalities that are used as analytical tools throughout the thesis.
- Numerical methods: gives a description of the numerical finite difference method, a Matlab implementation of which will be used to validate the theoretical results presented throughout this thesis.

The second chapter presents the theoretical background based on which the present work stands. It is divided into two sections: reaction diffusion systems and Turing instability. In the first section, we give a general introduction to reaction diffusion systems, in which we define the general form of a two-dimensional reaction-diffusion system and simplify it by means of the nondimensionalization of variables. Next, we establish the equilibruim solution and the linearization of the system. Finally, we discus the stability analysis and the local stability in the ODE and the PDE senses. In the second section, we introduce the Turing instability and its conditions. Then, we describe for the activator–inhibitor of a system. Also, we complete the stability analysis from the previous section. We mention some methods to obtain the global asymptotic stability. In the end, we introduce the Degn Harrison model, give a brief history, and mention the most important works and research related it. Also, we talk about its generalization.

The third Chapter presents the first main contribution of this thesis. First, we study the well-known Degn-Harrison reaction diffusion model. In Section 2, we prove the asymptotic stability of the system, both in the local and global senses. Also, weaker conditions than those of previous studies are derived. In Section 3, our results are validated using Matlab computer simulations.

The fourth Chapter constitutes the second main part of this thesis, in which we study the Degn-Harrison system with a generalized reaction term. In Section 2, once an invariant rectangle is identified for the system, we prove the existence of a unique solution for all $t > 0$ and establish its boundedness. In Section 3, the eigenfunction expansion method is used to settle the local asymptotic stability of the steady state solution. Then, the direct Lyapunov method is employed to obtain the conditions, assuring the global convergence to the homogeneous equilibrium solution. In Section 4, we discuss the elliptic boundary value problem obtaining a priori estimates for the nonconstant steady state solutions.

Moreover, the nonexistence of non-constant positive solutions is be proved. Finally, in Section 5, numerical simulations are performed in order to corroborate the analytical findings of Section 3.

# CHAPTER 1

## Preliminaries

In this chapter, we present some of the necessary nomenclature and notions used throughout the thesis. Then, we give a brief description of the finite difference-based numerical analysis method used to validate the theoretical results.

## 1.1 Nomenclature

- $\mathbb{R}$ : Set of real numbers.
- $\mathbb{R}^+$ : Set of all nonnegative real numbers.
- $\mathbb{R}^N$ : Set of all $N$-tuples $x = (x_1, x_2, ..., x_N)$.
- $\mathbb{C}(\Omega)$ : Space of continuous functions on $\Omega$.
- $\mathbb{C}^K(\Omega), k = 1, 2$ : Set of $k$-times continuously differentiable functions in $\Omega$.
- $\mathbb{L}^2(\Omega)$ : Set of square-integrable functions on $\Omega$.
- $\mathbb{L}^q(\Omega)$ : Space of measurable functions on $\Omega$ for which the $q^{th}$ power of the absolute value is Lebesgue integrable .
- $\mathbb{L}^\infty(\Omega)$ : Space of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$, for which there is a constant $M \geq 0$ such that $|u(x)| \leq M$ a.e $x \in \mathbb{R}^N$.
- $\mathbb{H}^1, \mathbb{W}_0^{1,q}(\Omega)$ : Sobolev space.
- $\mathbb{H}^2$ : Hilbert space.
- $\Omega$ : Bounded domain of $\mathbb{R}^N$.
- $\partial\Omega$ : Boundary of domain $\Omega$.
- $\overline{\Omega}$ : Closer of domain $\Omega$.
- $|\Omega|$ : Volume of domain $\Omega$.
- $\det(J)$ : Determinant of the matrix $J$.
- $tr(J)$ : Trace of the matrix $J$.
- $\mathrm{Re}(\lambda)$ : Real part of the complex number $\lambda$.
- $\dot{V}(u)$ : The derivative of $V$ i.e. $\dot{V}(u) = \frac{d}{dt}V(u)$.
- $\dfrac{\partial u}{\partial t}, \partial_t u$ : Partial derivative with respect to $t$.

- $\frac{\partial u}{\partial \upsilon}$ : Normal derivative of $u$ outside of $\partial\Omega$.

- $\upsilon$ : Outward unit normal vector of the boundary $\partial\Omega$

- $U_x(x_0)$ : Derivative of $U$ with respect to $x$ evaluated at $x = x_0$.

- $O(h^n)$ : Unkown error term.

- $\Delta u, \nabla^2 u$ : Laplacian operator of $u$ defined by $\Delta u = \nabla^2 u = \sum_{i=1}^{i=N} \frac{\partial^2 u}{\partial x_i^2}$.

- $\nabla u$ : Gradient of $u$ defined by $\nabla u = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_N} \right)$.

- $\max u(x), \min u(x)$ : Maximum of $u(x)$, minimum of $u(x)$.

- $\sup u(x), \inf u(x)$ : Superior $u(x)$, inferior $u(x)$.

- $\langle ., . \rangle$ : Inner product.

- $\|u\|$ : Norm.

## 1.2 General Notions

In this section, we present the basic notions used in this thesis. Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with reasonably smooth boundary $\partial\Omega$. For $(x,t) \in \Omega \times \mathbb{R}^+$, we consider the following system of reaction-diffusion equations

$$U_t - D\Delta U = H(U), \tag{1.1}$$

where $U = (u_1, u_2, \ldots, u_N)$, $N \geq 1$, and $D$ is a constant positive definite matrix. Together with (1.1), we assume that $U$ satisfies the initial condition

$$U(x,0) = U_0(x), x \in \Omega \tag{1.2}$$

and the Neumann boundary condition

$$\frac{\partial U}{\partial \upsilon} = 0 \qquad \text{on } \partial\Omega, \tag{1.3}$$

where $\upsilon$ is the outward unit normal vector of the boundary $\partial\Omega$.

### 1.2.1 Basic Theorems and Definitions

If $U = (u, v)$, system (2.1) can be reduced to

$$\begin{cases} u_t - d_1 \Delta u = F(u, v), \\ v_t - d_2 \Delta v = G(u, v), \end{cases} \tag{1.4}$$

where $u, v$ are the chemical species and $d_1, d_2$ are the specific diffusion coeffecients.

**Definition 1.1** ***(Invariant set)*** *[32]*

*Let $\Re$ be a domain enclosed by a simple curve $\partial\Re$ (in the phase plane). $\Re$ is said to be an invariant set for the ODE of system (1.4) if any solution with initial conditions in $\Re$ remains inside $\Re$ for all $t > 0$.*

**Definition 1.2** *[54]*

*A rectangle $\Re = (0, r_1) \times (0, r_2)$ is called an invariant rectangle if the vector field $(F, G)$ on the boundary $\partial\Re$ points inside. That is*

$$\begin{cases} F(0, v) \geq 0 \text{ and } F(r_1, v) \leq 0 \text{ for } 0 < v < r_2, \\ G(u, 0) \geq 0 \text{ and } G(u, r_2) \leq 0 \text{ for } 0 < u < r_1. \end{cases}$$

**Definition 1.3** *[41]*

*The rectangle $\Re = [\delta_1, \delta_2] \times [\gamma_1, \gamma_2]$ is an invariant rectangle if the vector field $(F, G)$ defined on boundary $\partial\Re$ points inside. That is*

$$\begin{cases} F(\delta_1, v) > 0 \text{ and } F(\delta_2, v) < 0 \text{ for } \gamma_1 < v < \gamma_2, \\ G(u, \gamma_1) > 0 \text{ and } G(u, \gamma_2) < 0 \text{ for } \delta_1 < u < \delta_2. \end{cases}$$

***Theorem 1.4 (Global existence)*** *[32]*

*If there exists an invariant rectangle for system (1.1), then system (1.1) with initial conditions and boundary conditions in $\Re$ has a unique global solution.*

**Definition 1.5** ***(Limit cycles)***

*A limit cycle is an isolated closed trajectory ("isolated" means that neighbouring trajectories are not closed), which only occur in nonlinear systems.*

**Proposition 1.6** *[42]*

*Given the functions $g \in \mathbb{C}\left(\overline{\Omega} \times \mathbb{R}\right)$ and $U \in \mathbb{C}^2(\Omega) \cap \mathbb{C}^1\left(\overline{\Omega}\right)$, it follows that*

**(i)** *if*

$$\Delta U(x) + g(x, U(x)) \geq 0 \quad \text{in} \quad \Omega,$$

*with* $\frac{\partial U}{\partial \upsilon} \leq 0$ *on* $\partial\Omega$ *and* $U(x_0) = \max_{\overline{\Omega}}(U(x))$, *then*

$$g(x_0, U(x_0)) \geq 0.$$

**(ii)** *Alternatively, if*

$$\Delta U(x) + g(x, U(x)) \leq 0 \quad \text{in } \Omega,$$

*with* $\frac{\partial U}{\partial \upsilon} \geq 0$ *on* $\partial\Omega$ *and* $U(x_0) = \min_{\overline{\Omega}}(U(x))$, *then*

$$g(x_0, U(x_0)) \leq 0.$$

**Theorem 1.7** ***(Weak maximum principle)*** *[32]*

*Let* $U \in \mathbb{C}^2(\Omega \times (0,T)) \cap \mathbb{C}(\Omega \times (0,T)), c(x,t) \geq c_{\min}$ *and*

$$D\Delta U - cU - U_t \geq 0, \text{ in } \Omega \times (0,T).$$

*Furthermore, let* $U \leq 0$ *in* $\Omega \times \{0\}$ *(i.e. for the initial condition) and in* $\partial\Omega \times (0,T)$ *(i.e. on the boundaries). Then,*

$$U(x,t) \leq 0, \forall (x,t) \in \Omega \times (0,T).$$

### 1.2.2 Basic Formulas

**Green's Formula [7]**

**Theorem 1.8** *Let* $u, v$ *are functions of Sobolev space* $\mathbb{H}^1(\Omega)$ *and* $\partial\Omega$ *be smooth, we have*

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = -\int_\Omega \frac{\partial v}{\partial x_i} u dx + \int_{\partial\Omega} uv\eta_i d\sigma, \qquad 1 \leq i \leq n.$$

*We design by* $\eta_i$ *the* $i^{th}$ *consinus director of normal* $\eta$ *in* $\partial\Omega$ *directed towards the outside of* $\Omega$ *and we write* $\eta_i = (\overrightarrow{\eta}.\overrightarrow{e_i})d\sigma$ *the superficial measure on* $\partial\Omega$.

**Corollary 1.9** *For all functions* $(u, v)$ *of Sobolev space* $\mathbb{H}^1(\Omega)$, *we have the Green formula*

$$\int_\Omega (\Delta u) v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma - \int_\Omega \nabla u \nabla v dx.$$

***Taylor's Formula*** *[22]*

For $f:\mathbb{R}\to\mathbb{R}$, the Taylor series is defined as

$$f(u)=f(u^*)+\frac{\partial f}{\partial u}(u-u^*)+\frac{1}{2!}\frac{\partial^2 f}{\partial u^2}(u-u^*)^2+\cdots+\frac{1}{n!}\frac{\partial^n f}{\partial u^n}(u-u^*)^n+O(u,u^*).$$

And for $f:\mathbb{R}^2\to\mathbb{R}$, we have

$$\begin{aligned} f(u,v) &= f(u^*,v^*)+\frac{\partial f}{\partial u}(u-u^*)+\frac{\partial f}{\partial v}(v-v^*)+\frac{1}{2!}\frac{\partial^2 f}{\partial u^2}(u-u^*)^2+\frac{1}{2!}\frac{\partial^2 f}{\partial v^2}(v-v^*)^2 \\ &\quad +\frac{\partial^2 f}{\partial u\partial v}(u-u^*)(v-v^*)+O\left((u-u^*)^2+(v-v^*)^2\right). \end{aligned}$$

### 1.2.3 Basic Inequalities

The following inequalities are avaible at [22].

**Cauchy-Shwarz's Inequality**

For all $(u,v)\in\mathbb{L}^2(\Omega)$,

$$\left|\int_\Omega u(x)\,v(x)\,dx\right|\le\int_\Omega|u(x)\,v(x)|\,dx\le\left(\int_\Omega|u(x)|^2\,dx\right)^{1/2}\left(\int_\Omega|v(x)|^2\,dx\right)^{1/2}.$$

**Cauchy's Inequality with $\epsilon$ ($\epsilon$-Inequality)**

For all $\epsilon>0$ and $(u,v)\in\mathbb{R}^2$,

$$|uv|\le\frac{\epsilon}{2}|u|^2+\frac{1}{2\epsilon}|v|^2.$$

**Young's Inequality**

For all $(u,v)\in\mathbb{R}^2$,

$$|uv|\le\frac{1}{p}|u|^p+\frac{1}{q}|v|^q,$$

where $p,q$ strictly positive real numbers linked by the relation $(\frac{1}{p}+\frac{1}{q}=1)$.

**$\epsilon-$Young's Inequality**

For all $\epsilon>0$ and $(u,v)\in\mathbb{R}^2$,

$$|uv|\le\epsilon|u|^p+c(\epsilon)|v|^q,$$

where $p,q$ strictly positive real numbers linked by the relation $(\frac{1}{p}+\frac{1}{q}=1)$.

**Poincaré's Inequality**

Let $\Omega$ be a bounded, connected, open subset of $\mathbb{R}^N$,with a $\mathbb{C}^1$ boundary $\partial\Omega$. Assume

$1 \leq q < \infty$. Then there exists a constant $C$, depending only on $N, q$ and $\Omega$, such that

$$\|u\|_{\mathbb{L}^q(\Omega)} \leq C \|\nabla u\|_{\mathbb{L}^q(\Omega)}, \text{ for all } u \in W_0^{1,q}(\Omega).$$

## 1.3 Numerical Analysis

To go from an exact continuous problem governed by PDE to the discrete approximate problem, there are three main of methods: **finite differences, finite volumes, finite elements.**

In our work, we focus on the finite differences method.

### 1.3.1 The Finite Differences Method

In numerical analysis, finite-difference methods (FDM) are discretizations used for solving differential equations by approximating them with difference equations that approximate the derivatives.

FDMs convert linear ordinary differential equations **(ODE)** or non-linear partial differential equations **(PDE)** into a system of equations that can be solved by matrix algebra techniques. The reduction of the differential equation to a system of algebraic equations makes the problem of finding the solution to a given ODE/PDE ideally suited to modern computers, hence the widespread use of FDMs in modern numerical analysis. Today, FDMs are the dominant approach to numerical solutions of PDEs.

The following information is available at [16].

#### Taylor's Theorem

First, assuming the function whose derivatives are to be approximated is properly-behaved, **Taylor's theorem** states the following.

**Theorem 1.10** *Let $U(x)$ have $n$ continuous derivatives over the interval $(a, b)$. Then, for $a < x_0, x_0 + h < b$,*

$$U(x_0 + h) = U(x_0) + \frac{U_x(x_0)}{1!}h + \frac{U_{xx}(x_0)}{2!}h^2 + \ldots + \frac{U_{x(n-1)}(x_0)}{(n-1)!}h^{n-1} + O(h^n), \quad (1.5)$$

*where*

- $U_x = \frac{dU}{dx},\ U_{xx} = \frac{d^2U}{dx^2}, \ldots, \frac{d^{n-1}U}{dx^{n-1}}$.
- $U_x(x_0)$ *is the derivative of $U$ with respect to $x$ evaluated at $x = x_0$.*
- $O(h^n)$ *is an unkown error term.*

The usual interpretation of Taylor's theorem says that if we know the value of $U$ and the value of its derivatives at a point $x_0$, then we can write down equation (1.5) for its value at the (nearby) point $x_0 + h$.

This expression contains an unkown quantity which is written as $O(h^n)$ and pronounced 'order $h$ to the $n$'. If we discared the term $O(h^n)$ in (1.5) (i.e truncate the right hand side of (1.5)) we get an approximation to $U(x_0 + h)$ . The error in this approximation is $O(h^n)$ .

#### Taylor's Theorem Applied to the Finite Difference Method (FDM)

In the FDM we know the $U$ values at the grid points and we want to replace the partial derivatives of the PDE we are solving by their approximates at these grid points. We do this by interpreting (1.5) in another way. In the FDM both $x_0$ and $x_0 + h$ are grid points and $U(x_0)$ and $U(x_0 + h)$ are known. This allows us to rearrange equation (1.5) to get the so called Finite Difference (FD) approximations to derivatives which have $O(h^n)$ errors.

#### Simple Finite Difference Approximation of a Derivative

Truncating (1.5) after the first derivative term gives

$$U(x_0 + h) = U(x_0) + U_x(x_0)h + O\left(h^2\right). \tag{1.6}$$

Rearranging (1.6) gives

$$\begin{aligned} U_x(x_0) &= \frac{U(x_0 + h) - U(x_0)}{h} - \frac{O(h^2)}{h}, \\ &= \frac{U(x_0 + h) - U(x_0)}{h} - O(h). \end{aligned}$$

Neglecting the $O(h)$ term gives

$$U_x(x_0) = \frac{U(x_0 + h) - U(x_0)}{h}. \tag{1.7}$$

Formula (1.7) is called a first order FD approximation since the approximation error$=$ $O(h)$ which depends on the first power of $h$. This approximation is called a **forward FD approximation** since we start at $x_0$ and step forward to the point $x_0 + h$, which $h$ is why called the step size ($h > 0$).

### Constructing a Finite Difference Toolkit

Now, we construct common FD approximations to common partial derivatives. For simplicity we suppose that $U$ is a function of only two variables $t$ and $x$. We will approximate the partial derivatives of $U$ with respect to $x$. As $t$ is held constant $U$ is effectively a function of the single variable $x$ so we can use Taylor's formula (1.5) where the ordinary derivative terms are now partial derivative and the arguments are $(t, x)$ instead of $x$. Finally, we will replace the step size $h$ by $\Delta x$ (to indicate a change in $x$) so that (1.5) becomes

$$U(x_0+\Delta x,t)=U(x_0,t)+\frac{\Delta x}{1!}U_x(x_0,t)+\frac{\Delta x^2 U_{xx}(x_0,t)}{2!}+\ldots+\frac{\Delta x^{n-1}}{(n-1)!}U_{(n-1)}(x_0,t)+O(\Delta x^n). \tag{1.8}$$

Truncating (1.8) to $O(\Delta x^2)$ gives

$$U(x_0+\Delta x,t)=U(x_0,t)+\Delta x U_x(x_0,t)+O\left(\Delta x^2\right). \tag{1.9}$$

Now we derive some FD approximations to partial derivatives. Rearranging (1.9) gives

$$\begin{aligned} U_x(x_0,t) &= \frac{U(x_0+\Delta x,t)-U(x_0,t)}{\Delta x}-\frac{O(\Delta x^2)}{\Delta x} \\ &= \frac{U(x_0+\Delta x,t)-U(x_0,t)}{\Delta x}-O(\Delta x). \end{aligned} \tag{1.10}$$

Equation (1.10) holds at any point $(t, x_0)$. In numerical schemes for solving PDEs we are restriced to a grid of discrete $x$ values $x_1, x_2, \ldots, x_N$ and discrete $t$ levels $0 = t_0, t_1, \ldots$. We will assume a constant grid spacing, $\Delta x$, in $x$, so that $x_{i+1} = x_i + \Delta x$. Evaluating equation (1.10) for a point $(t_j, x_i)$ on the grid gives

$$U_x(x_i,t_j)=\frac{U(x_{i+1},t_j)-U(x_i,t_j)}{\Delta x}-O(\Delta x). \tag{1.11}$$

We will use the common subscript/superscript notation

$$U_i^j=U(x_i,t_j), \tag{1.12}$$

so that droping the $O(\Delta x)$ error term, (1.11) becomes

$$U_x(x_i,t_j)\approx\frac{U_{i+1}^j-U_i^j}{\Delta x}. \tag{1.13}$$

Formula (1.13) is the first order **forward difference approximation** to $U_x(x_i, t_j)$ that we derived previously in approximation (1.7). Now, we derive another FD approximation to $U_x(x_i, t_j)$. Replacing $\Delta x$ by $-\Delta x$ in (1.9) gives

$$U(x_0 - \Delta x, t) = U(x_0, t) - \Delta x U_x(x_0, t) + O\left(\Delta x^2\right). \tag{1.14}$$

Evaluating (1.14) at $(t_j, x_i)$ and rearranging as previously gives

$$U_x(x_i, t_j) \approx \frac{U_i^j - U_{i-1}^j}{\Delta x}. \tag{1.15}$$

Formula (1.15) is the first order **backward difference approximmation** to $U_x(x_i, t_j)$.

Our first two FD approximmations are first order in $x$ but we can increase the order (and so make approximmation more accurate) by taking more terms in the Taylor series as follows. Truncating (1.8) to $O(\Delta x^3)$, then replacing $\Delta x$ by $-\Delta x$ and subtracting this new expression from (1.8) and evaluating at $(t_n, x_i)$ gives after some algebra

$$U_x(x_i, t_j) \approx \frac{U_{i+1}^j - U_{i-1}^j}{2\Delta x}. \tag{1.16}$$

Formula (1.16) is called the second order **central difference approximmation** to $U_x(x_i, t_j)$.

We could construct even higher order FD approximmations to $U_x$ by taking even more terms in Taylor series but we will stop at second order approximmation to first order derivatives.

Many PDEs of interest contain second order (and higher) partial derivatives so we need to derive approximation to them. We will restrict our attention to second order unmixed partial derivatives i.e. $U_{xx}$.

Truncating (1.8) to $O(\Delta x^4)$ gives

$$U(x_0 + \Delta x, t) = U(x_0, t) + \frac{\Delta x}{1!} U_x(x_0, t) + \frac{\Delta x^2}{2!} U_{xx}(x_0, t) + \frac{\Delta x^3}{3!} U_{xxx}(x_0, t) + O\left(\Delta x^4\right). \tag{1.17}$$

Replacing $\Delta x$ by $-\Delta x$ in (1.17) gives

$$U(x_0 - \Delta x, t) = U(x_0, t) - \frac{\Delta x}{1!} U_x(x_0, t) + \frac{\Delta x^2}{2!} U_{xx}(x_0, t) - \frac{\Delta x^3}{3!} U_{xxx}(x_0, t) + O\left(\Delta x^4\right). \tag{1.18}$$

Adding (1.17) and (1.18) gives

$$U(x_0 + \Delta x, t) + U(x_0 - \Delta x, t) = 2U(x_0, t) + \Delta x^2 U_{xx}(x_0, t) + O\left(\Delta x^4\right). \tag{1.19}$$

Evaluating (1.19) at $(x_i, t_j)$ and using our discrete notation gives

$$U_{i+1}^j + U_{i-1}^j = 2U_i^j + \Delta x^2 U_{xx}(x_i, t_j) + O\left(\Delta x^4\right). \tag{1.20}$$

Rearranging (1.20) and dropping the error term $O(\Delta x^2)$ gives

$$U_{xx}(x_i, t_j) \approx \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{\Delta x^2}. \tag{1.21}$$

Formula (1.21) is the second order symmetric difference approximation to $U_{xx}(x_i, t_j)$.

The above FD toolkit can be used to create a finite difference scheme (FDS) to obtain the approximate solution of a large number of PDEs simply by replacing each partial derivative by an appropriate FD approximation.

The following information is available at [47].

The parabolic partial differential equation we consider is the heat, or diffusion, equation

$$\frac{\partial U}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 U}{\partial x^2}(x,t), 0 < x < l, t > 0, \tag{1.22}$$

subject to the conditions

$$U(0,t) = U(l,t) = 0, t > 0 \;\text{ and }\; U(x,0) = f(x), 0 \leq x \leq l.$$

The approach we use to approximate the solution to this problem involves finite differences. First select an integer $m > 0$ and define the $x$-axis step size $h = l/m$. Then select a timestep size $k$. The grid points for this situation are $(x_i, t_j)$, where $x_i = ih = i\Delta x$, for $i = 0, 1, \ldots, m$, and $t_j = jk = j\Delta t$, for $j = 0, 1, \ldots$.

### 1.3.2 Forward Difference Method (Explicit Method)

We obtain the difference method using the Taylor series in $t$ to form the difference quotient

$$\frac{\partial U}{\partial t}(x_i, t_j) = \frac{U(x_i, t_j + k) - U(x_i, t_j)}{k} - \frac{k}{2}\frac{\partial^2 U}{\partial t^2}(x_i, \kappa_j), \quad \text{for some } \kappa_j \in (t_j, t_{j+1}), \tag{1.23}$$

and the Taylor series in $x$ to form the difference quotient

$$\frac{\partial^2 U}{\partial x^2}(x_i, t_j) = \frac{U(x_i + h, t_j) - 2U(x_i, t_j) + U(x_i - h, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 U}{\partial x^4}(\nu_i, t_j), \tag{1.24}$$

where $\nu_i \in (x_{i-1}, x_{i+1})$.

The parabolic partial differential equation (1.22) implies that at interior gridpoints $(x_i, t_j)$, for each $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots$, we have

$$\frac{\partial U}{\partial t}(x_i,t_j)-\alpha^2\frac{\partial^2 U}{\partial x^2}(x_i,t_j)=0,$$

so the difference method using the difference quotients (1.23) and (1.24) is

$$\frac{w_{i,j+1}-w_{ij}}{k}-\alpha^2\frac{w_{i,j+1}-2w_{ij}+w_{i,j-1}}{h^2}=0,\quad \text{where } w_{ij} \text{ approximates } U(x_i,t_j). \tag{1.25}$$

The local truncation error for this difference equation is

$$e_{ij}=\frac{k}{2}\frac{\partial^2 U}{\partial t^2}(x_i,\kappa_j)-\alpha^2\frac{h^2}{12}\frac{\partial^4 U}{\partial x^4}(\nu_i,t_j). \tag{1.26}$$

Solving (1.25) for $w_{i,j+1}$ gives

$$w_{i,j+1}=\left(1-\frac{2\alpha^2 k}{h^2}\right)w_{ij}+\alpha^2\frac{k}{h^2}(w_{i+1,j}+w_{i-1,j}),$$

for each $i=1,2,...,m-1$ and $j=1,2,....$

So, we have

$$w_{0,0}=f(x_0), w_{1,0}=f(x_1),...,w_{m,0}=f(x_m).$$

Then, we generate the next $t$-row by

$$\begin{aligned}
&w_{0,1}=U(0,t_1)=0,\\
&w_{1,1}=\left(1-\tfrac{2\alpha^2 k}{h^2}\right)w_{1,0}+\alpha^2\tfrac{k}{h^2}(w_{2,0}+w_{0,0}),\\
&w_{2,1}=\left(1-\tfrac{2\alpha^2 k}{h^2}\right)w_{2,0}+\alpha^2\tfrac{k}{h^2}(w_{3,0}+w_{1,0}),\\
&\cdots\\
&w_{m-1,1}=\left(1-\tfrac{2\alpha^2 k}{h^2}\right)w_{m-1,0}+\alpha^2\tfrac{k}{h^2}(w_{m,0}+w_{m-2,0}),\\
&w_{m,1}=U(m,t_1)=0.
\end{aligned}$$

Now we can use the $w_{i,1}$ values to generate all the $w_{i,2}$ values and so on.

The explicit nature of the difference method implies that the $(m-1)\times(m-1)$ matrix associated with this system can be written in the tridiagonal form

$$W=\begin{bmatrix}
(1-2\lambda) & \lambda & 0 & \cdots & 0\\
\lambda & (1-2\lambda) & \lambda & \ddots & \vdots\\
0 & \lambda & \ddots & \ddots & 0\\
\vdots & \ddots & \ddots & \ddots & \lambda\\
0 & \cdots & 0 & \lambda & (1-2\lambda)
\end{bmatrix},$$

where $\lambda = \alpha^2 \frac{k}{h^2}$. If we let

$$w^{(0)} = (f(x_1), f(x_2), ..., f(x_{m-1}))^t,$$

and

$$w^{(j)} = (w_{1j}, w_{2j}, ..., w_{m-1,j})^t, \text{for each } j = 1, 2, ...,$$

then the approximate solution is given by

$$w^{(j)} = W w^{(j-1)}, \text{for each } j = 1, 2, ....$$

So, $w^{(j)}$ is obtained from $w^{(j-1)}$ by a simple matrix multiplication. This is known as the Forward Difference method and the approximation at the cyan point shown in Figure 1.1 uses information from the other points marked on that figure. If the solution to the partial differential equation has four continuous partial derivatives in $x$ and two in $t$, then equation (1.26) implies that the method is of order $O(k + h^2)$.

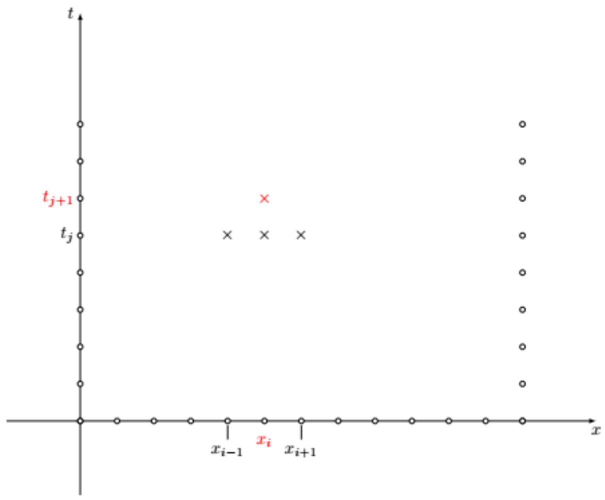


Figure 1.1: Forward Difference method

### 1.3.3 Backward Difference Method (Implicit Method)

To obtain a method that is **unconditionally** stable, we consider an implicit-difference method that results from using the backward-difference quotient for $(\partial U/\partial t)(x_i, t_j)$ in the form

$$\frac{\partial U}{\partial t}(x_i, t_j) = \frac{U(x_i, t_j) - U(x_i, t_{j-1})}{k} + \frac{k}{2}\frac{\partial^2 U}{\partial t^2}(x_i, \kappa_j), \quad \text{where } \kappa_j \in (t_j, t_{j+1}).$$

Substituting this equation, together with equation (1.24) for $\partial^2 U/\partial x^2$, into the partial differential equation gives

$$\frac{U(x_i,t_j)-U(x_i,t_{j-1})}{k}-\alpha^2\frac{U(x_{i+1},t_j)-2U(x_i,t_j)+U(x_{i-1},t_j)}{h^2}$$
$$=-\frac{k}{2}\frac{\partial^2 U}{\partial t^2}(x_i,\kappa_j)-\alpha^2\frac{h^2}{12}\frac{\partial^4 U}{\partial x^4}(\nu_i,t_j),$$

for some $\nu_i \in (x_{i-1},x_{i+1})$. The **Backward-Difference** method that results is

$$\frac{w_{ij}-w_{i,j-1}}{k}-\alpha^2\frac{w_{i,j+1}-2w_{ij}+w_{i,j-1}}{h^2}=0, \tag{1.27}$$

for each $i=1,2,...,m-1$ and $j=1,2,....$

The Backward Difference method involves the mesh points $(x_i,t_{j-1}),(x_{i-1},t_j)$, and $(x_{i+1},t_j)$ to approximate the value at $(x_i,t_j)$, as illustrated in Figure 1.2.

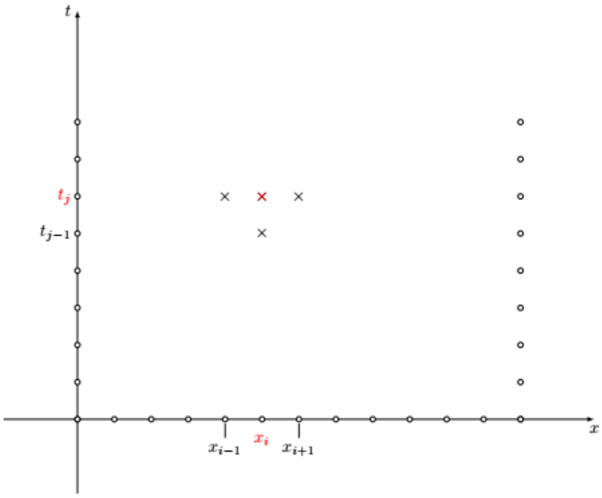


Figure 1.2: Backward Difference method

Since the boundary and initial conditions associated with the problem give information at the circled mesh points, the figure shows that no explicit procedures can be used to solve equation (1.27). Recall that in the Forward-Difference method (see Figure 1.2), approximations at $(x_{i-1},t_{j-1}),(x_i,t_{j-1})$, and $(x_{i+1},t_{j-1})$ were used to find the approximation at $(x_i,t_j)$. So an explicit method could be used to find the approximations, based on the information from the initial and boundary conditions.

If we again let $\lambda$ denote the quantity $\alpha^2\frac{k}{h^2}$, the Backward-Difference method becomes

$$(1+2\lambda)w_{ij}-\lambda w_{i+1,j}-\lambda w_{i-1,j}=w_{i,j-1} \text{ for each } i=1,2,...,m-1 \text{ and } j=1,2,....$$

Using the knowledge that $w_{i,0} = f(x_i)$, for each $i = 1, 2, ..., m-1$ and $w_{m,j} = w_{0,j} = 0$, for each $j = 1, 2, ...$ , this difference method has the matrix representation

$$\begin{bmatrix} (1+2\lambda) & -\lambda & 0 & \cdots & 0 \\ -\lambda & (1+2\lambda) & -\lambda & \ddots & \vdots \\ 0 & -\lambda & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\lambda \\ 0 & \cdots & 0 & -\lambda & (1+2\lambda) \end{bmatrix} \begin{bmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ w_{m-1,j} \end{bmatrix} = \begin{bmatrix} w_{1,j-1} \\ w_{2,j-1} \\ \vdots \\ w_{m-1,j-1} \end{bmatrix}$$

or

$$Ww^{(j)} = w^{(j-1)}, \quad \text{for each } i = 1, 2, ....$$

Hence, we must now solve a linear system to obtain $w^{(j)}$ from $w^{(j-1)}$. Note that $\lambda > 0$, so the matrix $W$ is positive definite and strictly diagonally dominant, as well as being tridiagonal.

# CHAPTER 2

## Reaction Diffusion Systems and Turing Instability

In this chapter, we are concerned with the reaction diffusion systems and the Turing instability. We make a general introduction to reaction diffusion systems, define it in the case of two dimensions and we simplify it by nondimensionalizing the variables. Then, we establish the equilibruim solution and the linearization of the system. Finally, we discuss the stability analysis, local stability in the ODE and the PDE senses. Also, we introduce the Turing instability and its conditions. We also give a description for activator–inhibitor type systems and complete the stability analysis from the previous section. We mention some methods to obtain the global asymptotic stability. At the end, we introduce the Degn Harrison model, present a brief history of it, and mention the most important works and research related to it. Also, we talk about the generalization of this model.

## 2.1 Reaction Diffusion Systems

Reaction–diffusion systems are mathematical models which correspond to several physical phenomena, the most common of which is the change in space and time of the concentration of one or more chemical substances, local chemical reactions in which the substances are transformed into each other, and diffusion which causes the substances to spread out over a surface in space. Reaction–diffusion systems are naturally applied in chemistry. However, the system can also describe dynamical processes of non-chemical nature. Many examples are found in biology, geology, physics (neutron diffusion theory) and ecology. Mathematically, reaction–diffusion systems take the form of semilinear parabolic partial differential equations. They can be represented by the general form

$$\partial_t U - D\Delta U = H(U), \tag{2.1}$$

where $U(x,t)$ denotes the unknown vector function, $D$ is a diagonal matrix of diffusion coefficients, and $H$ describes the reaction-diffusion mechanics of the system. The solutions of reaction–diffusion equations display a wide range of behaviours, including the formation of travelling waves and wave like phenomena as well as other self organized patterns like stripes, hexagons or more intricate structure like dissipative solitons. Each function, for which a reaction diffusion differential equation holds, represents in fact a concentration variable.

If $U = (u,v)$, the system (2.1) can be reduced to

$$\begin{cases} u_t - d_1\Delta u = F(u,v), \\ v_t - d_2\Delta v = G(u,v), \end{cases} \tag{2.2}$$

where $u, v$ are the chemical species and $d_1, d_2$ are the specific diffusion coeffecients.

### 2.1.1 Non-Dimensionalization

In (2.2), the nonlinear reaction functions, and the two species $u, v$ which are dependent on space $(x)$ and time $(t)$ have different diffusion coefficients. Depending on the reaction and diffusion of the system, the reaction kinetics can vary. To consider the kinetics of the system, we must first nondimensionalize the variables. Suppose $x = Ly$ where the domain is $x \in [0, L]$ which implies $y \in [0, 1]$. Then (2.2) can be rewritten as

$$\begin{cases} u_t - \frac{1}{L^2} d_1 \Delta u\left(y\left(x\right), t\right) = F\left(u, v\right), \\ v_t - \frac{1}{L^2} d_2 \Delta v\left(y\left(x\right), t\right) = G\left(u, v\right). \end{cases}$$

If we divide both sides by $d_1$ and multiply by $L^2$, we obtain

$$\begin{cases} \frac{L^2}{d_1} u_t - \Delta u\left(y\left(x\right), t\right) = \frac{L^2}{d_1} F\left(u, v\right), \\ \frac{L^2}{d_1} v_t - \frac{d_2}{d_1} \Delta v\left(y\left(x\right), t\right) = \frac{L^2}{d_1} G\left(u, v\right). \end{cases}$$

Finally, let $\tau = \frac{d_1}{L^2} t$, $d = \frac{d_2}{d_1}$ and $\gamma = \frac{L^2}{d_1}$. The dimensionless system of coupled nonlinear partial differential equations becomes

$$\begin{cases} u_\tau - \Delta u\left(y\left(x\right), \tau\right) = \gamma F\left(u, v\right), \\ v_\tau - d\Delta v\left(y\left(x\right), \tau\right) = \gamma G\left(u, v\right). \end{cases} \tag{2.3}$$

We will use (2.3) in the analysis, but since normally we use variables $(x, t)$ instead of $(y, \tau)$, we will use the old variables, i.e. the system becomes

$$\begin{cases} u_t - \Delta u\left(x, t\right) = \gamma F\left(u, v\right), \\ v_t - d\Delta v\left(x, t\right) = \gamma G\left(u, v\right). \end{cases} \tag{2.4}$$

### 2.1.2 Equilibrium Solution

One type of solution of particular interest is the equilibrium solution of a partial differential equation. Specifically of interest are attracting equilibrium solutions. These are time-independent solutions which are stable to small perturbations. Stability comes in many forms. We wish to classify equilibria which are linearly stable.

Equilibrium solutions to (2.4) are solutions $(u^*, v^*)^T$ such that $u_t = v_t = 0$. Thus, (2.4) turns into

$$\begin{cases} -\Delta u = \gamma F\left(u, v\right), \\ -d\Delta v = \gamma G\left(u, v\right). \end{cases} \tag{2.5}$$

A system without diffusion would have $\Delta u = \Delta v = 0$. Thus, (2.5) becomes

$$\begin{cases} 0 = F\left(u, v\right), \\ 0 = G\left(u, v\right). \end{cases}$$

So, for our model, equilibrium solutions in the absence of diffusion are those solutions $(u^*, v^*)^T$ which solve

$$F(u^*, v^*) = G(u^*, v^*) = 0.$$

Since (2.5) is a non-linear system, we must employ numerical methods.

Now, the question of stability of the equilibrium solutions is addressed. For this, we present the linearization of the system

$$\begin{cases} u_t = \gamma F\left(u, v\right), \\ v_t = \gamma G\left(u, v\right). \end{cases} \tag{2.6}$$

### 2.1.3 Linearization

**Definition 2.1** *An equilibrium solutions is linearly stable if its linearization attracts small perturbations.*

We define a perturbation of the equilibrium solution as

$$z = \begin{pmatrix} u - u^* \\ v - v^* \end{pmatrix}.$$

The functions $F$ and $G$ can be linearized using Taylor expansion about $(u^*, v^*)$

$$\begin{aligned} F(u, v) &\approx F(u^*, v^*) + F_u(u^*, v^*).(\mathrm{u} - u^*) + F_v(u^*, v^*).(v - v^*) \\ &= F_u(u^*, v^*).(\mathrm{u} - u^*) + F_v(u^*, v^*).(v - v^*), \end{aligned}$$

and

$$\begin{aligned} G(u, v) &\approx G(u^*, v^*) + G_u(u^*, v^*).(\mathrm{u} - u^*) + G_v(u^*, v^*).(v - v^*) \\ &= G_u(u^*, v^*).(\mathrm{u} - u^*) + G_v(u^*, v^*).(v - v^*). \end{aligned}$$

So, linearizing (2.6) about $(u^*, v^*)$, we obtain

$$\begin{cases} u_t = \gamma\left[F_u(u^*, v^*).(\mathrm{u} - u^*) + F_v(u^*, v^*).(v - v^*)\right], \\ v_t = \gamma\left[G_u(u^*, v^*).(\mathrm{u} - u^*) + G_v(u^*, v^*).(v - v^*)\right], \end{cases}$$

which can be written in matrix form as

$$\mathbf{z}_t = \gamma J \mathbf{z}, \tag{2.7}$$

where

$$J = \begin{pmatrix} F_u(u^*, v^*) & F_v(u^*, v^*) \\ G_u(u^*, v^*) & G_v(u^*, v^*) \end{pmatrix} = \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}_{(u^*,v^*)}. \tag{2.8}$$

Note that by linearizing (2.7), we have reduced the partial differential equation into a linear ordinary differential equation.

### 2.1.4 Stability Analysis

**Definition 2.2** *The solution* $\mathbf{z}$ *is said to be linearly stable if* $|\mathbf{z}| \to 0$ *as* $t \to \infty$.

We turn our attention to determine the conditions on the eigenvalues of $\gamma J$ which make the solution $z$ linearly stable.

**Theorem 2.3** *The solution* $\mathbf{z}$ *of equation (2.7) is linearly stable if and only if all eigenvalues of* $\gamma J$ *have negative real parts.*

#### Local Stability in the ODE Sense

Let us recall some of the fundamental ODE stability theory, see for more details [10] and [46]. The first important property is the asymptotic behavior of the solutions as $t \to +\infty$. It is well known that the asymptotic behavior is heavily dependent on the eigenvalues of $J$ denoted by $\xi_1$ and $\xi_2$. To calculate these eigenvalues, we simply solve the characteristic equation

$$\begin{aligned} |\gamma J - \xi I| &= \left| \begin{pmatrix} \gamma F_u - \xi & \gamma F_v \\ \gamma G_u & \gamma G_v - \xi \end{pmatrix} \right| = 0, \\ &\Rightarrow (\gamma F_u - \xi)(\gamma G_v - \xi) - \gamma^2 F_v G_u = 0, \\ &\Rightarrow \xi_{1,2} = \gamma \tfrac{(F_u+G_v) \pm \sqrt{(F_u+G_v) - 4(F_u G_v - F_v G_u)}}{2}. \end{aligned}$$

The linear stability is guaranteed if the trace of $J$ is negative and its determinant is positive, i.e.

$$\begin{cases} tr J = F_u + G_v < 0, \\ \det J = F_u G_v - F_v G_u > 0. \end{cases} \tag{2.9}$$

We conclude that the linearized system (2.7) is only stable subject to the real parts of the eigenvalues of $J$ being negative. If at least one eigenvalue is positive or has a positive real part, then $(u^*, v^*)$ is unstable.

**Local Stability in the PDE Sense**

**Properties of the Eigenvalues of the Laplace Operator** In order to study the local asymptotic stability in the PDE sense, one of the most commonly used methods is that of eigenfunction expansion [14]. It is important to recall some of the theory related to the eigenvalues of the Laplace operator. Let us denote these eigenvalues by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ and the corresponding normalized eigenfunctions in $\Omega$ by $\Phi_0, \cdots, \Phi_k, \cdots$, with Neumann boundary conditions. These eigenvalues and eigenfunctions satisfy the eigenvalue problem

$$-\Delta\Phi_k = \lambda_i \Phi_k, \quad \text{in} \quad \Omega, \tag{2.10}$$

with $\frac{\partial \Phi_k}{\partial \upsilon} = 0$, on $\partial\Omega$, and

$$\int_\Omega \Phi_k^2 (x)\, dx = 1. \tag{2.11}$$

In general, a two component reaction diffusion system is defined by the form

$$\partial_t U - D\Delta U = \gamma H (U), \tag{2.12}$$

where

$$U = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \text{ and } H(U) = \begin{pmatrix} F(U) \\ G(U) \end{pmatrix}.$$

We will now consider the full reaction-diffusion equation. Linearizing equation (2.12) about the steady-state $(u^*, v^*)$ in the same manner as done to derive equation (2.6) we get

$$\partial_t U - D\Delta U = \gamma J U, \tag{2.13}$$

where $D$ is the matrix of diffusion coefficients defined above and $J$ is the Jacobian matrix defined in (2.8). The eigenvalues of the Laplace operator $\Delta$ over the interval $[0, l]$ are the roots of the characteristic polynomial

$$\left|\gamma J - D\kappa^2 - \xi I\right| = 0, \text{ where } \kappa = \frac{2\pi}{l}. \tag{2.14}$$

According to [14], if the zero solution of the linearized form (2.13) is locally asymptotically stable, then so is the equilibrium of the original system (2.12). This leads us to the conditions for the stability of (2.12) as stated in the following theorem:

**Theorem 2.4** *[14]*

(**i**) *The equilibrium of (2.12) is globally asymptotically stable if for each nonnegative integer $n$ the eigenvalues of $J - \lambda_n D$ have negative real parts. Further, there exist positive* *constants $K$ and $\omega$ such that for any $t > 0$,*

$$\|u(t,x)\| \leq Ke^{-\omega t}\|\alpha(x)\|.$$

**(ii)** *The equilibrium of (2.12) is stable if for each nonnegative integer $n$ the eigenvalues of $J-\lambda_n D$ have nonpositive real parts and those with zero real parts have simple elementary divisors.*

**(iii)** *The equilibrium of (2.12) is unstable if for some $n$ there exists an eigenvalue of $J-\lambda_n D$ with either positive real part or zero real part with a nonsimple elementary divisor.*

For convenience, we have

$$\begin{aligned}
&|\xi I - \gamma J + D\kappa^2| = \left|\begin{pmatrix} \xi - \gamma F_u + \kappa^2 & -\gamma F_v \\ -\gamma G_u & \xi - \gamma G_v + \kappa^2 d \end{pmatrix}\right| = 0,\\
&\Rightarrow (\xi - \gamma F_u + \kappa^2)(\xi - \gamma G_v + d\kappa^2) - \gamma^2 F_v G_u = 0,\\
&\Rightarrow \xi^2 + \xi\left[\kappa^2(1+d) - \gamma(F_u + G_v)\right] + \gamma^2(F_u G_v - F_v G_u) - \gamma(dF_u + G_v)\kappa^2 + d\kappa^4 = 0.
\end{aligned}$$

So, the characteristic polynomial can be rewritten in the form

$$\xi^2 + P(\kappa^2)\xi + Q(\kappa^2) = 0, \tag{2.15}$$

where

$$P(\kappa^2) = \kappa^2(1+d) - \gamma tr J,$$

and

$$Q(\kappa^2) = d\kappa^4 - \gamma(dF_u + G_v)\kappa^2 + \gamma^2 \det J. \tag{2.16}$$

If $P > 0$ and $Q > 0$ then $\mathrm{Re}\,\xi < 0$ for all eigenvalues $\xi$. Consequently guarantee that the steady-state $(u^*, v^*)$ is locally asymptotically stable. If $P < 0$ or $Q < 0$, this implies the instability of $(u^*, v^*)$.

## 2.2 Turing Instability

One of the early uses of reaction-diffusion systems in science is pattern formation in natural creatures, for example, the spots on a leapard's skin. Understanding the development and arrangement of these patterns (called morphogenesis) is of generous significance for scientists and physicists the same. The British mathematician Alan Turing (1912-1954) is considered as one of the major pioneers of pattern formation hypothesis. In 1952, he studied the idea of morphogenesis and connected it to reaction-diffusion systems. The general idea that he put forward can be summarized in the following points:

• Active qualities in the natural cell are answerable for animating the generation and initiation of synthetic operators called morphogenesis.

• Chemical responses (reactions) are not sufficient for pattern formation as they are excessively symmetric.

• Instabilities authorized by the dissemination (diffusion) of chemical agents are the main thrust for primer pattern formation. The underlying patterns, at that point, experience certain advancements because of the response procedure.

Alan Turing posed two main inquiries. The simple inquiry is: can diffusion stabilize an otherwise unstable reactive (ODE) system? The appropriate response ends up being yes and that is somewhat simple to see. The second increasingly significant inquiry is: would diffusion be able to destabilize a stable system? Once more, the appropriate response ends up being truly, and this is the thing that Turing proposed just like the main thrust behind pattern formation. Turing's suggestion was thought to be comparatively radical and for quite a long time it stayed an untested hypothesis until the Chlorite-Iodide Malonic-Acid (CIMA) reaction was acknowledged by DeKepper in 1990, [19].

An interesting general definition of the diffusion–driven instability **"Turing Instability"** is given next.

**Definition 2.5** *A diffusion-driven instability, or Turing instability, occurs when a steady state, stable in the absence of diffusion, becomes unstable when diffusion is present.*

A good description of the conditions of Turing's instability can be found in Chapter. 4 of [21]. Going back to the characteristic polynomial of the general reaction–diffusion system (2.13) given in (2.14), we are interested in solutions that make the system unstable although it was stable in the ODE case, i.e. (2.9) is satisfied. However, by (2.9), $tr J < 0$, and since $d > 0$, $\kappa^2(1 + d) > 0$. So,

$$P\left(\kappa^2\right) = \kappa^2\left(1 + d\right) - \gamma tr J > 0.$$

Hence, the system becomes unstable only if $Q\left(\kappa^2\right)$ is negative. This leads to the only possibility

$$\det J < 0 \quad \text{or} \quad dF_u + G_v > 0,$$

but by (2.9), $\det J > 0$, which implies that the only condition is if $dF_u + G_v > 0$. We can see clearly that $d \neq 1$, since if it did, then $F_u + G_v > 0$, which contradicts (2.9). Thus, a third condition for Turing instability

$$dF_u + G_v > 0. \tag{2.17}$$

Condition (2.17) is sufficient but not necessary for $\operatorname{Re}\xi > 0$. For $Q(\kappa^2)$ to be negative for some $\kappa^2 > 0$, the minimum must be negative. With some simple calculus, we can calculate the minimum of $Q(\kappa^2)$ as follows

$$\begin{array}{l} Q(\kappa^2) = d\kappa^4 - \gamma(dF_u + G_v)\kappa^2 + \gamma^2 \det J, \\ \frac{dQ(\kappa^2)}{d\kappa^2} = 2d\kappa^2 - \gamma(dF_u + G_v). \end{array}$$

It is easy to see that the polynomial has an extremum at

$$\kappa_*^2 = \kappa^2 = \gamma\frac{dF_u + G_v}{2d}.$$

Thus $Q(\kappa^2)$ will attain it's minimum at $\kappa_*^2$, leading to

$$\begin{aligned} Q_{\min} &= Q(\kappa_*^2) \\ &= d\left(\gamma\frac{dF_u + G_v}{2d}\right)^2 - \gamma(dF_u + G_v)\left(\gamma\frac{dF_u + G_v}{2d}\right) + \gamma^2 \det J \\ &= \gamma^2\left[\det J - \frac{(dF_u + G_v)^2}{4d}\right]. \end{aligned}$$

The condition that $Q(\kappa^2) < 0$, for some $\kappa^2 \in \mathbb{N}$ is

$$\begin{array}{l} Q_{\min} < 0, \\ \gamma^2\left[\det J - \frac{(dF_u+G_v)^2}{4d}\right] < 0, \\ \det J < \frac{(dF_u+G_v)^2}{4d}. \end{array}$$

Consequently, the necessary and sufficient conditions for the existence of Turing instability in a linear 2–component reaction–diffusion system are

$$\begin{cases} trJ = F_u + G_v < 0, \\ \det J = F_uG_v - F_vG_u > 0 \\ dF_u + G_v > 0 \\ dF_u + G_v > 2\sqrt{d(F_uG_v - F_vG_u)} > 0. \end{cases}$$

### Activator–Inhibitor Nature

This part is avaible at [48].

As a rule, a reaction-diffusion system that is dependent upon diffusion-driven instability can have a place within one of two classes of systems exhibiting different behaviors, specifically activator-inhibitor and positive input. A helpful and concise description of these two classes and their attributes can be found in section 7.8 of [33]. Basically, when looking at the signs of the elements of the Jacobian evaluated at a certain steady state, we end up with two types of matrices of the form

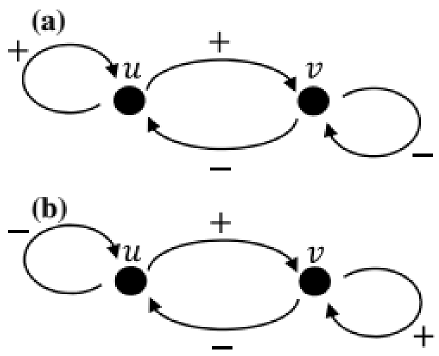


Figure 2.1: reaction–diffusion systems: **a** activator–inhibitor, and **b** positive feedback

$$\begin{pmatrix} + & - \\ + & - \end{pmatrix} \text{ or } \begin{pmatrix} + & + \\ - & - \end{pmatrix}.$$

These two types of matrices correspond directly to the activator–inhibitor and positive feedback classes as shown in Figure 2.1. In the first class, one substance is an activator in the sense that it enforces the formation of itself as well as the second substance, while the second is an inhibitor because it prevents the formation of both substances.

The significance of this activitaor-inhibitor property originates from the subject of the scientific drive behind pattern formation. Pattern formation is particularly examined in science and all the more decisively in morphogenesis, which we talked about before as the reason for Turing's progressive work. Patterns arise in science from spatially homogeneous states. For example, little zebras start from a homogeneous skin pigmentation and some way or another create various examples. This has pulled in the consideration of researcher and by expansion applied mathematicians. Things being what they are, the main thrust behind pattern formation is this activator-inhibitor property. Turing's instability alludes to the case that the inhibitor diffuses quicker than the activator by a given sum, ordinarily more than 10. This enormous distinction in diffusivities is the thing that frustrated the lab execution of Turing type chemical reactions and is definitely why the starch marker installed in the gel lattice was utilized in the CIMA response. It is important to note that Turing's instability is not sufficient for pattern formation. Certain nonlinearities in the reaction terms are required to ruin the solid positive input. More insights about pattern formation can be found in [48].

### Global Asymptotic Stability

**The Direct Lyapunov Method:**

One of the most important and powerful tools for studying the global asymptotic stability was coined by Russian mathematician Aleksandr Lyapunov in the early 1900s, referred to as the Lyapunov direct method, which is summarized in the following definition. For more on the method see, for instance, [10].

**Definition 2.6** *If $u^* \in \mathbb{R}^N$ is an equilibrium point of reaction diffusion system (2.12) and $\Omega \subseteq \mathbb{R}^N$ is an open set containing $u^*$, then the real valued function $V \in C^1(\Omega, \mathbb{R})$ is called a Lyapunov function if*

$$u \in \Omega, u \neq u^*, V(u) > V(u^*)$$

*and*

$$\frac{dV(u(t))}{dt} \leq 0, \text{ for all } u \in \Omega.$$

***Theorem 2.7 (Lyapunov stability theorem)*** *[10]*

(**i**) *If reaction diffusion system (2.12) has a Lyapunov function, then $u^*$ is stable.*

(**ii**) *If for all $u \neq 0$, $\frac{dV(u(t))}{dt} < 0$, then $u^*$ is asymptotically stable.*

For all $u \neq 0$ simply means that the Lyapunov function is nonincreasing when we travel along the trajectory $u(t)$.

The direct Lyapunov method is a powerful tool for establishing the global asymptotic stability. However, to the best of the author's knowledge, no systematic approach exists for finding Lyapunov functions and it is extremely difficult to select an appropriate function heuristically, i.e. through a trial and error process. In addition, this method is not sufficient to establish the global asymptotic stability, it is merely a tool that will be needed later on.

**The Negative Criteria:**

Among the methods used to establish the global asymptotic stability of solutions are Bendixson's and Dulac's criteria. Below is a summary of these criteria as described in [33].

It is important to note that Bendixson's criterion (Proposition 2.8) is merely a special case of Dulac's (Proposition 2.9).

**Proposition 2.8** ***Bendixson's criterion***

*Given the simply connected region $\Sigma$, if the expression*

$$C = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}$$

*is not zero for all $(x, y)$ in $\Sigma$ and does not change sign in $\Sigma$, then there are no limit cycles in $\Sigma$.*

**Proposition 2.9** ***Dulac's criterion***

*Given the simply connected region $\Sigma$, if there exists a function $B(x, y) \in \mathbb{C}^1$ such that*

$$C = \frac{\partial (BF)}{\partial x} + \frac{\partial (BG)}{\partial y}$$

*is not zero for all $(x, y)$ in $\Sigma$ and does not change sign in $\Sigma$, then there are no limit cycles in $\Sigma$.*

**The Poincare–Bendixson Theory:**

This theorem is based on the observation that two dimensional planes have some specific characteristics that may not exist in higher dimensions. Particularly, any trajectory may only have one of four limiting values: a critical point, a limit cycle, cycle graph, or infinite $xy$ values. Furthermore, if the trajectory is bounded, then it may only approach a critical point or a cycle graph. This is basis of the Poincare–Bendixson theory, which states that if a certain trajectory is bounded for $t \geq t_0$ and does not tend to a singular point, then it either is a limit cycle or tends to a limit cycle. For more on the theory, see [33].

The following theorem summarizes the Poincare–Benidixson theory:

**Theorem 2.10** *[13]*

*If an ODE system of the form*

$$\frac{du}{dt} = F(u),$$

*where $F$ is locally Lipschitz in $u$, has a solution $\phi$ that is bounded for $t \geq 0$, then either*

**(i)** *$\phi$ is periodic,*

**(ii)** *$\phi$ approaches a periodic solution, or*

**(iii)** *$\phi$ gets close to an equilibrium point infinitely often.*

**Theorem 2.11** ***(La Salle invariance theorem)*** *[34]*

*Let* $V : \Omega \rightarrow \mathbb{R}^+$ *be a function of* $\mathbb{C}^1$ *and suppose that* $\dot{V}(u) \leq 0$ *for all* $u \in \Omega$. *Define*

$$E = \{u \in \Omega : \dot{V}(u) = 0\}.$$

*Let* $L$ *be the largest invariant set contained in* $E$. *Then, any bounded solution tends to* $L$ *as the time goes to infinity. If, furthermore,* $L$ *reduce to* $u^*$, *then* $u^*$ *is asymptotically stable.*

**Definition 2.12** ***(Global Stability)*** *[9]*

Function $u$ is globally asymptotically stable on $\Omega$ if for all $u_0 \in \Omega$, the solution $u$ satisfies

$$\lim_{t \rightarrow \infty} \|u(t) - u^*\| = 0.$$

In addition to the above, let us present theorem and two lemmas from [17], which will come in handy at later chapters.

We consider the system of reaction-diffusion equations

$$u_t - D\nabla^2 u = \sum_{i=1}^{N} A_i(x,u)\frac{\partial u}{\partial x_i} + H(u), \text{ for } (x,t) \in \Omega \times \mathbb{R}^+, \tag{2.18}$$

where $\Omega \subset \mathbb{R}^N, N \geq 1$, is a bounded domain with reasonably smooth boundary, $\partial\Omega$, $u = (u_1, u_2, \ldots, u_N), N \geq 1$, $D$ is a constant positive definite matrix and the $A_i$'s are continuous matrix-valued functions. $U$ satisfies the initial condition (1.2) and the Neumann condition (1.3).

**Theorem 2.13** *[17]*

*Let* $\Sigma$ *be bounded invariant region of system (2.18) of the form*

$$\Sigma = \bigcap_{k=1}^{N} \left\{u \in \mathbb{R}^N : a_k \leq u \leq b_k\right\}, \text{ where } -\infty < a_k < b_k < \infty,$$

*and let* $\sigma$ *be positive and let* $u$ *be any solution such that all values of* $u_0$ *lie in* $\Sigma$. *Then there exist constants* $c_i > 0$, $i = 1, 2, 3, 4$, *such that*

$$\begin{aligned} \|\nabla_x u(.,t)\|_{\mathbb{L}^2(\Omega)} &\leq c_1 e^{-\sigma t}, \\ \|u(.,t) - \overline{u}(t)\|_{\mathbb{L}^2(\Omega)} &\leq c_2 e^{-\sigma t}, \end{aligned} \tag{2.19}$$

*where $\overline{u}$ the average of $u$ over $\Omega$, satisfies the system of ordinary differential equations*

$$\frac{d\overline{u}}{dt} = f\left(\overline{u}\right) + g\left(t\right), \overline{u}\left(0\right) = \frac{1}{|\Omega|}\int_{\Omega} u_0\left(x\right)dx,$$

*with*

$$|g\left(t\right)| \leq c_3 e^{-\sigma t}.$$

*If matrices $A_1, \cdots, A_n$ are zero or $A_1, \cdots, A_n$ and $D$ are diagonal, then (2.19) can be strengthened*

$$\left\|u\left(.,t\right) - \overline{u}\left(t\right)\right\|_{\mathbb{L}^{\infty}(\Omega)} \leq c_4 e^{-\sigma' t}, \sigma' < \sigma/m.$$

*The constants $c_1, c_2$ and $c_3$ are proportional to $\left\|\nabla u_0\right\|_{\mathbb{L}^2(\Omega)}$ while $c_4$ is proportional to $\left\|\nabla u_0\right\|_{\mathbb{L}^{\infty}(\Omega)}$.*

**Lemma 2.14** *[17]*
*Let $u$ be $\mathbb{H}^2\left(\Omega\right)$ function on $\Omega$ where $\partial u/\partial v$ on $\partial\Omega$. Then,*

$$\left\|\nabla^2 u\right\|^2_{\mathbb{L}^2(\Omega)} \geq \lambda \left\|\nabla u\right\|^2_{\mathbb{L}^2(\Omega)},$$

*where $\lambda$ is the smallest positive eigenvalue of $\left(-\nabla^2\right)$ with homogeneous Neumann boundary conditions on $\Omega$.*

**Lemma 2.15** *[17]*
*Let $u \in \mathbb{H}^2\left(\Omega\right), \partial u/\partial v = 0$ on $\partial\Omega$. Then*

$$\left\|\nabla u\right\|^2_{\mathbb{L}^2(\Omega)} \geq \lambda \left\|u - \overline{u}\right\|^2_{\mathbb{L}^2(\Omega)},$$

*where $\overline{u} = \frac{1}{|\Omega|}\int_{\Omega} u\left(x\right)dx$, and $\lambda$ is as in the previous Lemma.*

### 2.2.1 Degn Harrison Model

In our work, we are interested in the Degn-Harrison model, which is a Turing-type system. Our model was first proposed as early as 1969 by Degn and Harrison [18] to describe the respiratory behavior of the Klebsiella Aerogenes bacterial culture. The reaction being studied here is of the form

$$\begin{aligned} A &\rightharpoonup Y \\ B &\rightleftharpoons X \\ X + Y &\rightharpoonup P, \end{aligned}$$

where $X$ and $Y$ represent the concentrations of oxygen and nutrient being transmitted in the respiratory circle, $A$ and $B$ are "sources" or external parameters whose concentrations are to be kept at a constant level all over the reactor vessel, and $P$ is the final product in the reaction whose concentration is also assumed to be constant. In the reaction process, the last step is considered to be inhibited by excess of oxygen in the reactor. The first and last steps are assumed to be irreversible whereas the second step is reversible. For more background on this reaction scheme, one can refer to [18, 23, 24, 25, 11, 29, 31]. Degn and Harrison [18] first proposed that the last step followed a nonlinear rate equation of the type $XY/(1+qX^2)$, where $q$ measures the strength of the inhibitory law. With the homogeneous Neumann boundary condition, the above Degn–Harrison reaction scheme is governed by the following coupled nonlinear space-time differential equations in a dimensionless form

$$\begin{cases} X_t - D_1\Delta X = k_2 B - k_3 X - \frac{k_4 XY}{1+qX^2} & \text{in } \mathbb{R}^+ \times \Omega, \\ Y_t - D_2\Delta X = k_1 A - \frac{k_4 XY}{1+qX^2} & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial X}{\partial \nu} = \frac{\partial Y}{\partial \nu} = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \end{cases} \tag{2.20}$$

where $A, B, X$ and $Y$ denote dimensionless concentrations of the reactants; the constants $k_i$ $(i = 1, 2, 3, 4)$ are reaction rates, $D_1$ and $D_2$, respectively, denote the Fickian molecular diffusion coefficients of $X$ and $Y$, and they are assumed to be positive constants all over the reactor vessel. The rate and diffusion constants are parameters characteristic for a given system, and the concentrations $A$ and $B$ are variable parameters which can be controlled in the reaction process.For the detailed background of (2.20), one can refer to [18, 23, 25, 11, 29, 31].The Degn–Harrison reaction system (2.20) or (2.22) has been studied adequately by several authors, but most of the researches focus either on the corresponding ordinary differential equation system or on the reaction–diffusion system (2.20) in one-dimensional domain case. In [23], Farein and Velarde constructed the time-periodic limit cycle of the ODE system by using of the analytical, stochastic and computer-aided methods. Furthermore, in [31] Ibáñez, Farein and Velarde considered the linear stability of limit cycle and the dissipative Turing structure.

Later on, they discussed the steady state bifurcation and conducted the rigorous mathematical analysis for stability of spatially nonhomogeneous steady states of (2.20) arising from the steady state bifurcation in [24]. While in [29] Hemmer and Velarde explicitly constructed the existence of spatially nonhomogeneous steady states of (2.20) when the mass diffusions $D_2 \to \infty$ and $D_1 < \infty$. One can find more research studies on the Degn–Harrison model reported, for instance, in [11, 25, 51].

To simplify the reaction–diffusion system (2.20), Peng et al. [45] introduce the following dimensionless quantities

$$\begin{aligned} &\tau = k_3 t,\ u = \tfrac{k_4}{k_3}X,\ v = \tfrac{k_4}{k_3}Y,\ a = \tfrac{k_2 k_4}{k_3^2}B, \\ &b = \tfrac{k_1 k_4}{k_3^2}A,\ k = \tfrac{k_3^2}{k_4^2}q,\ d_1 = \tfrac{1}{k_3}D_1,\ d_2 = \tfrac{1}{k_3}D_2. \end{aligned} \tag{2.21}$$

With the rescaling (2.21), we can rewrite (2.20) as

$$\begin{cases} u_\tau - d_1 \Delta u = a - u - \dfrac{uv}{1+ku^2}, \\ v_\tau - d_2 \Delta v = b - \dfrac{uv}{1+ku^2}. \end{cases} \tag{2.22}$$

Peng et al. [45] recently considered the reaction–diffusion system (2.22) in $\mathbb{R}^N$. They studied the global stability of the constant steady state and gave the sufficient condition of the existence and nonexistence of the nonconstant steady states. Moreover, the Hopf and steady state bifurcations were also investigated. In [39], the authors investigated some fundamental analytic properties of nonconstant positive solutions, also they drived the stability of constant steady-state solution to both ordinary differential equation (ODE) and partial differential equation (PDE) systems and they established the global structure of steady-state bifurcations from simple eigenvalues by bifurcation theory and the local structure of the steady-state bifurcations from double eigenvalues by the techniques of space decomposition and implicit function theorem. On the other hand, in [20], Donga et all. derived The existence of Hopf bifurcation to ODE and PDE models by using the center manifold theory and the normal form method, we establish the bifurcation direction and stability of periodic solutions. Since, some numerical simulations are shown to support the analytical results. Lisena in [41] used the presence of contracting rectangles and the method of Lyapunov, to establish sufficient conditions for the global asymptotic stability of the unique constant steady state. In the work of [53], local asymptotic stability, Turing instability and existence of Hopf bifurcation for the only constant positive equilibrium solution are established by analyzing the relevant eigenvalue problem with numerical approximations.

In [56], Jun Zhou generalized Degn-Harrison model (3.1) by using $\varphi(u)v$ to replace $(uv/(1+ku^2))$. Therefore, he studied the Turing instability and showed the existence of periodic solutions of the PDE model and the ODE model by using Hopf bifurcation theory. Also numerical simulations are presented to verify and illustrate the theoretical results.

# CHAPTER 3

## On the Local and Global Asymptotic Stability of the Degn-Harrison Reaction-Diffusion Model

This chapter [1] contain the study of the well-known Degn-Harrison reaction diffusion model. It is concerned with the local and global asymptotic stability of the system, also, weaker conditions than those of previous studies are derived and validated by Matlab computer simulations.

## 3.1 Problem Formulation

We consider the following reaction–diffusion system based on the **Degn–Harrison** model

$$\begin{cases} u_t - d_1\Delta u = a - u - \dfrac{uv}{1+ku^2} := F(u,v) & \text{in } \mathbb{R}^+ \times \Omega, \\ v_t - d_2\Delta v = b - \dfrac{uv}{1+ku^2} := G(u,v) & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{3.1}$$

where $u(t)$ and $v(t)$ represent the dimensionless concentrations of oxygen and nutrient, respectivel, $a, b, d_1, d_2$ and $k$ are positive constants defined above, $\Omega \subset \mathbb{R}^N, N \geq 1$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta$ is the Laplacian operator on $\Omega$. With the initial condition

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x) \quad \text{in } \Omega,$$

where $u_0(x), v_0(x) \in \mathbb{C}^2(\Omega) \cap \mathbb{C}(\overline{\Omega})$, and the Neumann boundary condition

$$\frac{\partial u}{\partial \upsilon} = \frac{\partial u}{\partial \upsilon} = 0 \qquad \text{on } \partial\Omega,$$

where $\upsilon$ is the outward unit normal vector of the boundary $\partial\Omega$.

Now, before we present our analysis, let us recall the most relevant results reported in the literature in relation to the proposed system (3.1).

**Lemma 3.1** *[45]*

*System (3.1) has the unique steady state solution*

$$(u^*, v^*) = \left(\alpha, \frac{b}{\alpha}(1+k\alpha^2)\right), \text{ where } \alpha = a - b. \tag{3.2}$$

*If and only if $a > b$.*

**Proof 3.2** *An equilibrium point $(u^*, v^*)$ of the ODE of (3.1) satisfes the system*

$$\begin{cases} 0 = a - u - \dfrac{uv}{1+ku^2} \\ 0 = b - \dfrac{uv}{1+ku^2}, \end{cases} \tag{3.3}$$

[1] Article published in Mathematical Methods in the Applied Sciences 42, pp. 567–577, (2019). View [1]

*by the second equation of (3.3), we obtain*

$$v = \frac{b\left(1+ku^2\right)}{u}, \tag{3.4}$$

*we compensate (3.4) in the first equation of (3.3), we find*

$$u = a - b. \tag{3.5}$$

*We substitue (3.5) in (3.4), we find (3.2).*

**Lemma 3.3** *The Jacobian of the Degn-Harrison model (3.1) is given by*

$$J = \begin{pmatrix} F_0 & -G_0 \\ 1+F_0 & -G_0 \end{pmatrix}, \quad \text{where} \tag{3.6}$$

$$F_0 = -\frac{a+\alpha^2\left(a-2b\right)k}{\alpha\left(1+k\alpha^2\right)} \quad \text{and} \quad G_0 = \frac{\alpha}{1+k\alpha^2}, \tag{3.7}$$

*with*

$$\alpha = a - b. \tag{3.8}$$

**Proof 3.4** *The Jacobian matrix is*

$$J = \begin{pmatrix} F_u\left(u,v\right) & G_u\left(u,v\right) \\ F_v\left(u,v\right) & G_v\left(u,v\right) \end{pmatrix} = \begin{pmatrix} -1-\frac{v\left(1+ku^2\right)-2ku^2v}{\left(1+ku^2\right)^2} & -\dfrac{u}{1+ku^2} \\ \frac{v\left(1+ku^2\right)-2ku^2v}{\left(1+ku^2\right)^2} & -\dfrac{u}{1+ku^2} \end{pmatrix},$$

*so, the Jacobian matrix associated to the ODE of (3.1) evaluated at* $(u^*, v^*)$ *is*

$$J = \begin{pmatrix} -1-\frac{\frac{b}{\alpha}\left(1+k\alpha^2\right)\left(1-k\alpha^2\right)}{\left(1+k\alpha^2\right)^2} & -\dfrac{\alpha}{1+k\alpha^2} \\ \frac{b\left(1-k\alpha^2\right)}{\alpha\left(1+k\alpha^2\right)} & -\dfrac{\alpha}{1+k\alpha^2} \end{pmatrix} = \begin{pmatrix} -1-\frac{b}{\alpha}\frac{\left(1-k\alpha^2\right)}{\left(1+k\alpha^2\right)} & -\frac{\alpha}{1+k\alpha^2} \\ \frac{b}{\alpha}\frac{\left(1-k\alpha^2\right)}{\left(1+k\alpha^2\right)} & -\frac{\alpha}{1+k\alpha^2} \end{pmatrix}.$$

*This completes the proof.*

**Lemma 3.5** *[41]*

*System (3.1) has an invariant rectangle which is the form*

$$\Re = [\overline{u}, a] \times [2b\sqrt{k}, \overline{v}], \tag{3.9}$$

*where*

$$\overline{u} = \frac{bu^*}{a(1+ka^2)} \quad \text{and} \quad \overline{v} = \frac{\left(a-\overline{u}\right)\left(1+k\overline{u}^2\right)}{\overline{u}}. \tag{3.10}$$

**Proof 3.6** *First, since $b < a$ we are allowed to write $b = ta, 0 < t < 1$, so that*

$$bu^* = t(1-t)a^2 \leq \frac{a^2}{4}, t \in \left]0,1\right[.$$

*Consequently*

$$\overline{u} \leq \frac{a}{4\left(1+ka^2\right)} \leq \frac{1}{8\sqrt{k}} < \frac{1}{\sqrt{k}},$$

*because $\frac{a}{(1+ka^2)} \leq \frac{1}{2\sqrt{k}}$. Observe that $\frac{b}{\varphi_k(u)} = \frac{b\left(1+ku^2\right)}{u}$ and $f_{a,k}(u) = \frac{a-u}{\varphi_k(u)}$ are strictly decreasing in $\left]0,\overline{u}\right]$ (for each $k$).*

*As second step let us verify that*

$$\frac{b}{\varphi_k\left(a\right)} < f_{a,k}(\overline{u}). \tag{3.11}$$

*Indeed*

$$f_{a,k}(\overline{u}) = \left(\frac{a}{\overline{u}} - 1\right)\left(1+k\overline{u}^2\right) > \left(4\left(1+ka^2\right)-1\right)\left(1+k\overline{u}^2\right) > 4ka^2+3,$$

*and*

$$\frac{b}{\varphi_k\left(a\right)} = \frac{b}{a}\left(1+ka^2\right) < 1+ka^2.$$

*Hence (3.11) easily follows. Previous estimates prove, in particular, that $(u^*, v^*)$ lies in the interior of $\Re$. At this point, we can state that, on the boundary of $\Re$, the vector field $(F(u,v), G(u,v))$, defined in (3.1), does not point outwards. Indeed*

$$\begin{aligned} F(\overline{u}, v) &> 0 \text{ and } F(a,v) < 0 \quad \text{for} \quad 2b\sqrt{k} < v < \overline{v}, \\ G(u, 2b\sqrt{k}) &> 0 \text{ and } G(u,\overline{v}) < 0 \quad \text{for} \quad \overline{u} < u < a. \end{aligned}$$

*Therefore rectangle $\Re$ is an invariant region.*

**Lemma 3.7** *[39]*

*The steady-state solution $(u^*, v^*)$ is locally asymptotically stable in the PDE sense subject to*

$$b < a < 2b \tag{3.12}$$

*and*

$$\begin{cases} \lambda_1 d_1 \geq F_0, \\ \lambda_1 d_1 < F_0 \quad \text{and} \quad 0 < d_2 < \tilde{d}_2. \end{cases} \tag{3.13}$$

*Alternatively, if $b < a < 2b$ and*

$$\lambda_1 d_1 < F_0 \quad \text{and} \quad d_2 > \tilde{d}_2, \tag{3.14}$$

*then $(u^*, v^*)$ is locally asymptotically unstable.*

**Proof 3.8** *Because the proof is long and for simplicity, we have omitted it. Interested readers may look it up in [39].*

## 3.2 Asymptotic Stability

In this section, we examine the asymptotic stability of the steady state solution $(u^*, v^*)$, which is the main concern of this section.

Before that, we start by defining some of the necessary notation and definitions. Considering the Laplacian operator $(-\Delta)$ with Neumann boundaries on $\Omega$, its infinite sequence of eigenvalues is denoted by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ Each eigenvalue $\lambda_i$ is assumed to have an algebraic multiplicity $m_i \geq 1$. The normalized eigenfunctions corresponding to $\lambda_i$ are denoted by $\Phi_{ij}, 1 \leq j \leq m_i$. It is important to note that as $i \to \infty$, $\lambda_i$ tends to $\infty$ and that $\Phi_0 = \text{const}$. From standard eigenfunction theory, we know that

$$-\Delta \Phi_{ij} = \lambda_i \Phi_{ij}$$

in $\Omega$, with

$$\frac{\partial \Phi_{ij}}{\partial v} = 0$$

in $\partial\Omega$, and

$$\int_\Omega \Phi_{ij}^2 dx = 1.$$

Consequently, the set $\{\Phi_{ij} : i \geq 0, 1 \leq \lambda_i \leq m_i\}$ forms a complete orthonormal basis in $L^2(\Omega)$.

If the inequality

$$d_1 \lambda_1 < F_0 \tag{3.15}$$

is satisfied, then let us define $i_\alpha = i_\alpha(\alpha, \Omega)$ as the largest positive integer guaranteeing

$$d_1 \lambda_i < F_0 \tag{3.16}$$

for all $i \leq i_\alpha$. Also, observe that when (3.15) is satisfied, it is inherent that $1 \leq i_\alpha < \infty$.

We, then, define

$$\tilde{d}_2 = \min_{1 \le i \le i_\alpha} \tilde{d}_i, \quad \text{where}$$

$$\tilde{d}_i = \frac{G_0(d_1\lambda i + 1)}{\lambda_i(F_0 - d_1\lambda_i)}.$$

With this notation in mind, let us now examine the local and global asymptotic stability of the Degn-Harrison system (3.1) separately.

### 3.2.1 Local Asymptotic Stability

**Proposition 3.9** *If $F_0 \le 0$, then $(u^*, v^*)$ is asymptotically stable as a steady state of (3.1). Alternatively, if*

$$0 < F_0 < G_0, \tag{3.17}$$

*then $(u^*, v^*)$ is asymptotically stable if*

$$\begin{cases} \lambda_1 d_1 \ge F_0 \quad \textit{or} \\ \lambda_1 d_1 < F_0 \ \textit{and} \ \begin{cases} \frac{d_2}{d_1} \le \frac{G_0}{F_0}, \\ \textit{or} \\ \frac{G_0}{F_0} < \frac{d_2}{d_1} < \wp, \end{cases} \end{cases} \tag{3.18}$$

*where $\wp$ is the solution of*

$$(F_0 x + G_0)^2 = 4(1 + F_0)G_0 x. \tag{3.19}$$

**Proof 3.10** *We start by reformulating (3.1) in its vectorial form given by*

$$\frac{\partial \mathbf{z}}{\partial t} - D\Delta\mathbf{z} = \mathbf{F}(\mathbf{z}), \ \textit{where} \tag{3.20}$$

$$\mathbf{z} = \begin{pmatrix} u \\ v \end{pmatrix}, D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \ \textit{and} \ \mathbf{F}(\mathbf{z}) = \begin{pmatrix} a - u - \frac{uv}{1+ku^2} \\ b - \frac{uv}{1+ku^2} \end{pmatrix}. \tag{3.21}$$

*It is well known that the solution $(u^*, v^*)$ is asymptotically stable for (3.20) if $\mathbf{z} = 0$ is asymptotically stable as a steady-state solution of the linearized system*

$$\frac{\partial \mathbf{z}}{\partial t} - D\Delta\mathbf{z} = J\mathbf{z}, \tag{3.22}$$

*where $J$ represents the Jacobian matrix evaluated at the steady state, ie,*

$$J(u^*, v^*) = \begin{pmatrix} -\frac{a+(a-2b)(a-b)^2k}{(a-b)(1+k(a-b)^2)} & -\frac{a-b}{1+k(a-b)^2} \\ 1-\frac{a+(a-2b)(a-b)^2k}{(a-b)(1+k(a-b)^2)} & -\frac{a-b}{1+k(a-b)^2} \end{pmatrix}.$$

*In the revolutionary work of Casten et al [14], Theorem 2.4 states that if all the eigenvalues of $J - \lambda_n D$ for all non-negative integers, $n$ have negative real parts, then the zero steady state is asymptotically stable for (3.22). In fact, it suffices to ensure that the trace is negative and the determinant is positive. First, consider the case $F_0 \leq 0$. We have*

$$\begin{aligned} J - \lambda_n D &= \begin{pmatrix} -\frac{a+(a-2b)(a-b)^2k}{(a-b)(1+k(a-b)^2)} & \frac{a-b}{1+k(a-b)^2} \\ 1-\frac{a+(a-2b)(a-b)^2k}{(a-b)(1+k(a-b)^2)} & -\frac{a-b}{1+k(a-b)^2} \end{pmatrix} - \lambda_n \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \\ &= \begin{pmatrix} F_0 - \lambda_n d_1 & -G_0 \\ 1 + F_0 & -G_0 - \lambda_n d_2 \end{pmatrix}. \end{aligned}$$

*The trace and determinant are given by*

$$tr(J - \lambda_n D) = F_0 - G_0 - \lambda_n d_1 - \lambda_n d_2 \leq 0,$$

*and*

$$\begin{aligned} \det(J - \lambda_n D) &= (F_0 - \lambda_n d_1)(-G_0 - \lambda_n d_2) - (1 + F_0)(-G_0) \\ &= G_0 + \lambda_n^2 d_1 d_2 + \lambda_n(-F_0)d_2 + \lambda_n G_0 d_1 \geq 0, \end{aligned}$$

*respectively. Clearly, regardless of $n$, the eigenvalues of $J - \lambda_n D$ have negative real parts. This proves the first part of our proposition.*

*In the second part, (3.17) is satisfied. For $\lambda_0 = 0$, the matrix $J - \lambda_n D$ reduces to $A$ and we know that*

$$tr J = F_0 - G_0 < 0,$$

*and*

$$\det J = G_0 > 0.$$

*Now, let $\lambda_1 d_1 \geq F_0$ yielding $\lambda_n d_1 \geq F_0$, and thus*

$$tr(J - \lambda_n D) = (F_0 - \lambda_n d_1) - G_0 - \lambda d_2 \leq 0,$$

*and*

$$\det(J - \lambda_n D) = G_0 + (\lambda_n d_1 - F_0)\lambda_n d_2 + \lambda_n G_0 d_1 \geq 0.$$

*Hence, the steady state is locally asymptotically stable.*

*Finally, if* $\lambda_1 d_1 < F_0$*, then for those eigenvalues* $\lambda_n, n > 1$*, that satisfy* $\lambda_n d_1 \geq F_0$*, we end up with the same result as before and* $J - \lambda_n D$ *has eigenvalues with negative real parts. For the remaining eigenvalues, we denote one of these eigenvalues by* $\theta$*, ie,* $\frac{F_0}{d_1}$*. The trace is straight forward as*

$$\begin{aligned} tr(J-\theta D) &= F_0 - G_0 - \theta d_1 - \theta d_2, \\ &= (F_0 - G_0) - \theta\,(d_1 + d_2) < 0. \end{aligned}$$

*As for the determinant, we have*

$$\begin{aligned} \det(J-\theta D) &= G_0 + \theta^2 d_1 d_2 - \theta F_0 d_2 + \theta G_0 d_1, \\ &= \theta^2 d_1 d_2 - \theta(F_0 d_2 - G_0 d_1) + G_0. \end{aligned}$$

*If we set*

$$\frac{d_2}{d_1} \leq \frac{G_0}{F_0},$$

*we end up with* $\det(J - \theta D) > 0$*. The last case is where*

$$\frac{d_2}{d_1} > \frac{G_0}{F_0}.$$

*Notice that the trinomial*

$$\theta^2 d_1 d_2 - \theta(F_0 d_2 - G_0 d_1) + G_0$$

*is positive if its discriminant is negative, that is,*

$$(F_0 d_2 - G_0 d_1)^2 - 4 d_1 d_2 G_0 < 0.$$

*which is,*

$$\begin{aligned} F_0^2 d_2^2 - 2F_0 G_0 d_1 d_2 + G_0^2 d_1^2 - -4 d_1 d_2 G_0 &< 0 \\ \left(F_0 \frac{d_2}{d_1}\right)^2 + 2F_0 G_0 \frac{d_2}{d_1} + G_0^2 - 4\,(1 + F_0)\, G_0 \frac{d_2}{d_1} &< 0. \end{aligned}$$

*This can be rearranged to the form*

$$(F_0 \frac{d_2}{d_1} + G_0)^2 < 4\,(1 + F_0)\, G_0 \frac{d_2}{d_1}.$$

*In the interval* $[0, +\infty)$*, between the parabola* $y = (F_0 x + G_0)^2$ *and the line* $y = 4(1 + F_0)G_0 x$*, it is easy to see that, at point* $\bar{x} = \frac{G_0}{F_0}$*, we have*

$$(F_0 \bar{x} + G_0)^2 < 4 (1 + F_0) G_0 \bar{x}$$

*and the line intersects the parabolic curve at two points* $x_1; x_2$ *such that* $0 < x_1 < x < x_2$*. Setting* $D = x_2$*, we obtain that* $D$ *is the solution of Equation (3.19) satisfying* $D > \frac{G_0}{F_0}$*. In addition, the inequality*

$$(F_0 x + G_0)^2 < 4 (1 + F_0) G_0 x$$

*holds for* $\frac{G_0}{F_0} < x < D$*. We conclude that* $\det(A - \theta D)$ *is positive if*

$$\frac{d_2}{d_1} \leq \frac{G_0}{F_0} \text{ or } \frac{G_0}{F_0} < \frac{d_2}{d_1} < D.$$

*The poof is complete.*

### 3.2.2 Global Asymptotic Stability

In this subsection, we present the findings of this study. First, we show that, subject to a specific condition, the proposed system has a rectangular invariant region. For convenience, let us define the function

$$\varphi_k(u) = \frac{u}{1 + ku^2}. \tag{3.23}$$

Using (3.23), system (3.1) may now be rewritten in the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = \left( \frac{a-u}{\varphi_k(u)} - v \right) \varphi_k(u) := F(u, v) \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v = \left( \frac{b}{\varphi_k(u)} - v \right) \varphi_k(u) := G(u, v). \end{cases} \tag{3.24}$$

We start with a proposition of sufficient conditions for the existence of an invariant rectangle for (3.1).

**Proposition 3.11** *If*

$$a > \frac{1}{k} \tag{3.25}$$

*and*

$$b < \frac{a}{1 + ka^2} \left( a \sqrt[3]{ak^2} - 1 \right), \tag{3.26}$$

*system (3.1) has the invariant region*

$$\Re = \left[\sqrt[3]{\frac{a}{k}}, a\right] \times \left[2b\sqrt{k}, a\sqrt[3]{ak^2} - 1\right]. \tag{3.27}$$

**Proof 3.12** *First of all, on the left boundary of $u$, we obtain*

$$\begin{aligned} F\left(\sqrt[3]{\frac{a}{k}}, v\right) &= a - \sqrt[3]{\frac{a}{k}} - v\varphi_k\left(\sqrt[3]{\frac{a}{k}}\right) = \left(\frac{a - \sqrt[3]{\frac{a}{k}}}{\varphi_k\left(\sqrt[3]{\frac{a}{k}}\right)} - v\right)\varphi_k\left(\sqrt[3]{\frac{a}{k}}\right) \\ &= \left(\left(a\sqrt[3]{ak^2} - 1\right) - v\right)\varphi_k\left(\sqrt[3]{\frac{a}{k}}\right) > 0, \end{aligned}$$

*for all $2b\sqrt{k} < v < a\sqrt[3]{ak^2} - 1$. Similarly, on the right boundary, we have for all*

$2b\sqrt{k} < v < a\sqrt[3]{ak^2} - 1$

$$\begin{aligned} F(a, v) &= \left(\frac{a - a}{\varphi_k(a)} - v\right)\varphi_k(a) \\ &= -v\varphi_k(a) < 0. \end{aligned}$$

*As for the boundaries of $v$, the left boundary yields*

$$\begin{aligned} G\left(u, 2b\sqrt{k}\right) &= b - 2b\sqrt{k}\varphi_k(u) \\ &= \varphi_k(u)\, b\left(\frac{1}{\varphi_k(u)} - 2\sqrt{k}\right) > 0, \end{aligned}$$

*for $\sqrt[3]{\frac{a}{k}} < u < a$, and for the right boundary, we have*

$$\begin{aligned} G\left(u, a\sqrt[3]{ak^2} - 1\right) &= b - \left(a\sqrt[3]{ak^2} - 1\right)\varphi_k(u) \\ &= \varphi_k(u)\left(\frac{b}{\varphi_k(u)} - \left(a\sqrt[3]{ak^2} - 1\right)\right) < 0, \end{aligned}$$

*for $\sqrt[3]{\frac{a}{k}} < u < a$. Finally, since*

$$\left(a\sqrt[3]{ak^2} - 1\right) > \frac{b}{\frac{a}{1+ka^2}},$$

*then*

$$\left(a\sqrt[3]{ak^2} - 1\right) > \frac{b}{\varphi_k(u)},$$

*and the proof is complete.*

That we have determined the invariant region for system (3.1), we move to derive sufficient conditions for its global asymptotic stability. Now, with the aim of simplifying the proofs, we rewrite the proposed system in the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = \left[\left(\frac{a-u}{\varphi_k(u)} - \frac{a-\alpha}{\varphi_k(\alpha)}\right) - \left(v - \frac{b}{\varphi_k(\alpha)}\right)\right] \varphi_k(u) & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v = \left[\left(\frac{b}{\varphi_k(u)} - \frac{b}{\varphi_k(\alpha)}\right) - \left(v - \frac{b}{\varphi_k(\alpha)}\right)\right] \varphi_k(u) & \text{in } \mathbb{R}^+ \times \Omega. \end{cases} \tag{3.28}$$

The following theorem constitutes the main finding of study of this subsection:

**Theorem 3.13** *If*

$$\frac{a}{(a-b)} > ku(b-u), \tag{3.29}$$

*and*

$$3\sqrt[3]{ak^2} - ak + 1 \leq 0 \tag{3.30}$$

*are satisfied. Then for any solution* $(u, v) \in \Re$ *to* (3.1)*, we get*

$$\lim_{t\to\infty} \|u(x,t) - u^*\|_{\mathbb{L}^2(\Omega)} = \lim_{t\to\infty} \|v(x,t) - v^*\|_{\mathbb{L}^2(\Omega)} = 0. \tag{3.31}$$

Before stating the proof of our main theorem, the following lemmas and proposition are necessary to complete the proof.

**Lemma 3.14** *Consider the function* $H$ *defined as*

$$H(u) = \int_\alpha^u \left(\frac{b}{\varphi_k(r)} - \frac{b}{\varphi_k(\alpha)}\right) dr \geq 0. \tag{3.32}$$

*It follows that*

$$\frac{d}{du} H(u) = \frac{b}{\varphi_k(u)} - \frac{b}{\varphi_k(\alpha)}.$$

**Proposition 3.15** *Let* $(u(t,.), v(t,.))$ *be a solution of* (3.1) *and let*

$$V(t) = \int_\Omega E(u(x,t), v(x,t))\, dx, \tag{3.33}$$

*where*

$$E(u,v) = H(u) + \frac{1}{2}(v - v^*)^2. \tag{3.34}$$

*Then, subject to* (3.29) $V(t)$ *is a Lyapunov functional.*

**Proof 3.16** *First of all, substituting* (3.34) *in* (3.33) *yields*

$$V(t) = \int_\Omega \left[H(u) + \frac{1}{2}(v - v^*)^2\right] dx.$$

*Differentiating the functional $V(t)$ wrt $t$ leads to*

$$\begin{aligned}\frac{d}{dt}V(t) &= \int_\Omega \left[\left(\frac{b}{\varphi_k(u)}-\frac{b}{\varphi_k(u^*)}\right)\left(d_1\Delta u+\varphi_k(u)\left(\left(\frac{a-u}{\varphi_k(u)}-\frac{a-u^*}{\varphi_k(u^*)}\right)-(v-v^*)\right)\right)\right]dx \\ &\quad +\int_\Omega (v-v^*)\left(d_2\Delta v+\varphi_k(u)\left[\left(\frac{b}{\varphi_k(u)}-\frac{b}{\varphi_k(u^*)}\right)-(v-v^*)\right]\right)dx. \\ &= \left[\int_\Omega d_1\left(\frac{b}{\varphi_k(u)}-\frac{b}{\varphi_k(u^*)}\right)\Delta u dx++\int_\Omega d_2(v-v^*)\Delta v dx\right] \\ &\quad +\left[\int_\Omega \varphi_k(u)\left(\left(\frac{b}{\varphi_k(u)}-\frac{b}{\varphi_k(u^*)}\right)\left(\frac{a-u}{\varphi_k(u)}-\frac{a-u^*}{\varphi_k(u^*)}\right)-(v-v^*)^2\right)dx\right].\end{aligned}$$

*To simplify things, we split the derivative into two parts*

$$\frac{d}{dt}V(t)=I+J, \tag{3.35}$$

*where*

$$I=-bd_1\int_\Omega\left(k-\frac{1}{u^2}\right)|\nabla u|^2dx-d_2\int_\Omega|\nabla v|^2dx,$$

*and*

$$J=\int_\Omega \varphi_k(u)\left[b\left(k-\frac{1}{\gamma^2}\right)(u-u^*)\left(\frac{a-u}{\varphi_k(u)}-\frac{a-u^*}{\varphi_k(u^*)}\right)-(v-v^*)^2\right]dx.$$

*It follows directly from (3.25) that $I\le 0$. If condition (3.29) is satisfied, then*

$$\begin{aligned} u &\le u^*\Rightarrow (u-u^*)\left(\frac{a-u}{\varphi_k(u)}-\frac{a-u^*}{\varphi_k(u^*)}\right)\le 0, \\ u &\ge u^*\Rightarrow (u-u^*)\left(\frac{a-u}{\varphi_k(u)}-\frac{a-u^*}{\varphi_k(u^*)}\right)\le 0.\end{aligned}$$

*It is easy to see that $J\le 0$, and therefore*

$$\frac{d}{dt}V(t)\le 0.$$

*This concludes the proof of the proposition.*

Now that we have established that $V(t)$ is a valid Lyapunov functional and conditions (3.25), (3.26), and (3.29), we move to state the sufficient conditions for the global asymptotic stability of (3.1).

**Proof 3.17** *Going back to (3.35), the positive-definite functional $V(t)$ has a nonpositive derivative and if $(u,v)\in\Re$ is a solution of (3.1), for which $\frac{d}{dt}V(t)=0$, then $u$ and $v$ must necessarily be spatially homogeneous as $|\nabla u|^2=|\nabla v|^2=0$. Hence, $(u,v)$ satisfies* *the ODE system corresponding to (3.1). Since, for the differential system (3.24), $(u^*, v^*)$ is the largest invariant subset*

$$\left\{(u,v) \in \dot{\Re} \mid \frac{d}{dt}V(t) = 0\right\},$$

*one gets (see Lisena [40] and Yi et al[54]) via La Salle invariance theorem*

$$\lim_{t\to\infty} |u(x,t) - u^*| = \lim_{t\to\infty} |v(x,t) - v^*| = 0, \tag{3.36}$$

*uniformly in $x$. Hence,*

$$\lim_{t\to\infty} \int_\Omega (u-u^*)^2 (x,t)\, dx = \lim_{t\to\infty} \int_\Omega (v-v^*)^2\, dx = 0. \tag{3.37}$$

*The equalities in (3.37) yield (3.36).*

## 3.3 NUMERICAL EXAMPLES

Practically speaking, the previous results gathered from Peng et al [45] and Lisena [41] mean that if we want to choose parameters guaranteeing the global asymptotic stability of solutions, the following four conditions must be satisfied

$$\begin{cases} (C_1) & \frac{a}{2} < (\alpha = a-b) < a \Longleftrightarrow 0 < b < \frac{a}{2} \\ (C_2) & a^2 > \frac{27}{4}, \\ (C_3) & b \leq \frac{a}{1+ka^2}\left(k\frac{a^2}{4}+1\right) = \frac{a}{4}\left(\frac{4+a^2k}{1+a^2k}\right), \\ (C_4) & 3\sqrt[3]{ak^2} - ak + 1 \leq 0. \end{cases} \tag{3.38}$$

Obviously, condition (C4) implies (C2) as

$$\begin{aligned} 3\sqrt[3]{ak^2} - ak + 1 &\leq 0 \Longrightarrow 3\sqrt[3]{ak^2} \leq ak - 1 \\ &\Rightarrow 3\sqrt[3]{ak^2} < ak \\ &\Rightarrow 27ak^2 < a^3k^3 \\ &\Rightarrow 27 < a^2k. \end{aligned} \tag{3.39}$$

Hence, (C2) may be ignored. The main finding of this study is that we can replace the conditions in (3.39) by our new weaker conditions

$$\begin{cases} (C_4) & 3\sqrt[3]{ak^2} - ak + 1 \leq 0, \\ (C_5) & b \leq \frac{a}{1+ka^2}\left(a\sqrt[3]{ak^2} - 1\right), \\ (C_6) & \frac{a}{a-b} > ku\left(b-u\right). \end{cases} \tag{3.40}$$

Note that the region of $(u, v)$, we operate within is $\Re$. For instance, let us consider the parameters

$$a = 1.2371, k = 19.974, \text{ and } b = 0.34. \tag{3.41}$$

Clearly (C1) is satisfied. As for (C4), we see that

$$\begin{aligned} 3\sqrt[3]{ak^2} - ak + 1 &= 3\sqrt[3]{1.2371 \times 19.974^2} - 1.2371 \times 19.974 + 1 \\ &= -2.2378 \times 10^{-4} \leq 0. \end{aligned}$$

However, condition (C3) is not satisfied as

$$\begin{aligned} \frac{a}{1+ka^2}\left(k\frac{a^2}{4} + 1\right) &= \frac{1.237}{1 + 19.974 \times 1.2374^2}(19.974 \times \frac{1.2374^2}{4} + 1) \\ &= 0.33864 \leq b. \end{aligned}$$

Therefore, the sufficient conditions stated Peng et al[45] and Lisena[41] cannot guarantee the global asymptotic stability. As for the new derived conditions, it is easy to see that (C5) is fulfilled as

$$\begin{aligned} \frac{a}{1+ka^2}\left(a\sqrt[3]{ak^2} - 1\right) &= \frac{1.237}{1 + 19.974 \times 1.2374^2}(1.237 \times \sqrt[3]{1.237 \times 19.974^2} - 1) \\ &= 0.34392 \geq b. \end{aligned}$$

The last condition (C6) requires closer attention. Keeping in mind that $u \in [\sqrt[3]{\frac{a}{k}}, a]$, Figure 3.1 (top) shows the quantity $\frac{a}{(a-b)} - ku(b-u)$, which has to be strictly positive for (C6) to be satisfied. It is easy to see that this is in fact the case. Figure 3.1 (bottom) shows the functions

$$f\left(u\right) = \frac{\left(a-u\right)}{u}\left(ku^2 + 1\right),$$

and

$$g\left(u\right) = \frac{b}{u}\left(ku^2 + 1\right).$$

What condition (C6) guarantees for us is that the two functions intersect at the unique point $u = \alpha$ and that before the intersection $f(u) > g(u)$, while after it $f(u) < g(u)$.

The Degn-Harrison system has been solved numerically by means of the implicit finite difference method using Matlab. Figures 3.2 and 3.3 show the solutions of the system using the parameters in (3.41) in the ODE and one-dimensional cases, respectively. In the 1D case, the diffusion constants were set to $d_1 = 3$ and $d_2 = 2$. Clearly, the solutions are globally asymptotically stable as suggested by our new conditions.

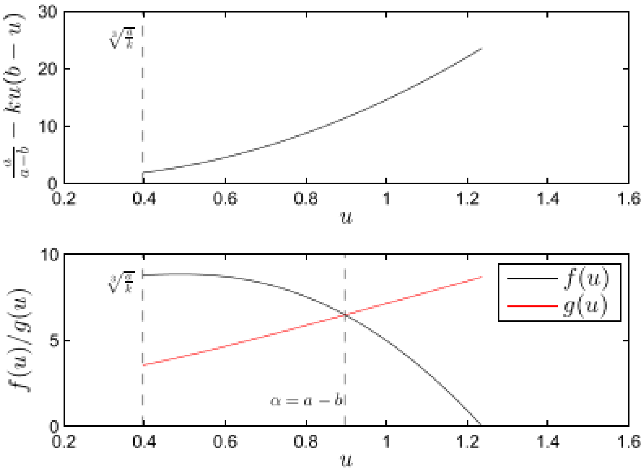


Figure 3.1: (Top) Condition (C9) states that $\frac{a}{a-b} - ku(b-u) > 0$.(Bottom) The functions $f(u)$ and $g(u)$ in the range $u \in \left[\sqrt[3]{\frac{a}{k}}, a\right]$ .The parameters chosen here are stated in (3.41).

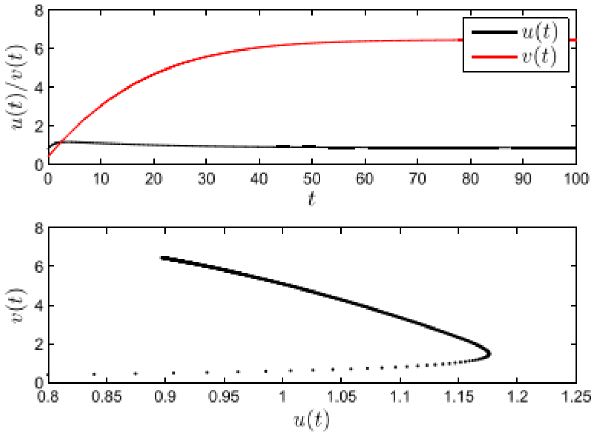


Figure 3.2: Solutions of the Degn-Harrison model (4.1) in the diffusion-free case. The parameters chosen here are stated in (4.87). The initial states are set at $u_0 = 0.8$ and $v_0 = 0.4$.

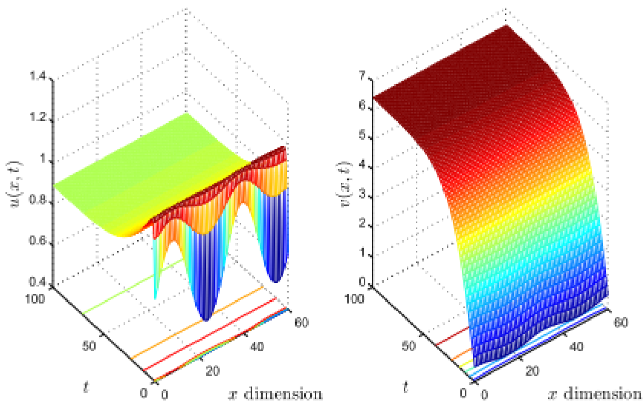


Figure 3.3: Solutions of the Degn-Harrison model (4.1) in the diffusion-free case. The parameters chosen here are stated in ( 4.87). The initial states are set at $u(x, 0) = 0.8[1 + 0.3\sin(0.2x)]$ and $v(x, 0) = 0.4[1 + 0.3\cos(0.2x)]$.

# CHAPTER 4

## A Generalized Degn-Harrison Reaction-Diffusion System: Asymptotic Stability and Non-Existence Results

In this chapter [1], we study the Degn–Harrison system with a generalized reaction term. Once proved the global existence and boundedness of a unique solution, we address the asymptotic behavior of the system. The conditions for the global asymptotic stability of the steady state solution are derived using the appropriate techniques based on the eigen-analysis, the Poincaré–Bendixson theorem and the direct Lyapunov method. Numerical simulations are also shown to corroborate the asymptotic stability predictions. Moreover, we determine the constraints on the size of the reactor and the diffusion coefficient such that the system does not admit non-constant positive steady state solutions.

## 4.1 Problem Formulation

Before we start to introduce our works, we consider the general Degn-Harrison reaction diffusion system

$$\begin{cases} u_t - d_1\Delta u = a - u - \lambda\varphi(u)\,v, & x \in \Omega, t > 0, \\ v_t - d_2\Delta v = b - \lambda\varphi(u)\,v, & x \in \Omega, t > 0. \end{cases} \tag{4.1}$$

To simplify the reaction–diffusion system (4.1), we create the new dimensionless parameters

$$u = RU, v = TV, t = \frac{\lambda^2}{d_1}s, x = \lambda y, \tag{4.2}$$

next, we use the **Chain rule** to derive a new differential system

$$\begin{aligned} \frac{du}{dt} &= \frac{du}{dU}.\frac{dU}{ds}.\frac{ds}{dt} = R\frac{d_1}{\lambda^2}\frac{dU}{ds}, \\ \frac{dv}{dt} &= \frac{dv}{dV}.\frac{dV}{ds}.\frac{ds}{dt} = T\frac{d_1}{\lambda^2}\frac{dV}{ds}. \end{aligned} \tag{4.3}$$

System (4.1) in one-dimensional becomes

$$\begin{cases} R\dfrac{d_1}{\lambda^2}\dfrac{dU}{ds} - d_1\dfrac{R}{\lambda^2}\dfrac{d^2U}{dy^2} = a - RU - \lambda\varphi(U)\,TV \\ T\dfrac{d_1}{\lambda^2}\dfrac{dV}{ds} - d_2\dfrac{T}{\lambda^2}\dfrac{d^2V}{dy^2} = b - \lambda\varphi(U)\,TV, \end{cases} \tag{4.4}$$

we devide both sides of equation one in (4.4) by $(Rd_1)$ and we multiply by $\lambda^2$ (similair to equation one we devide equation two by $(Td_1)$ and multiply by $\lambda^2$)

[1] Article published in Nonlinear Analysis: Real World Applications 57 (103191), pp. 1-28, (2021). View [2]

$$\begin{cases} \dfrac{dU}{ds} - \dfrac{d^2U}{dy^2} = \dfrac{\lambda^2}{d_1}\left[\dfrac{a}{R} - U - \lambda\varphi\left(U\right)\dfrac{T}{R}V\right] \\ \dfrac{dV}{ds} - \dfrac{d_2}{d_1}\dfrac{d^2V}{dy^2} = \dfrac{\lambda^2}{d_1}\left[\dfrac{b}{T} - \lambda\varphi\left(U\right)V\right], \end{cases} \tag{4.5}$$

we put $R = T, A = \dfrac{a}{R}, B = \dfrac{b}{R}, d = \dfrac{d_2}{d_1}$ and $\gamma = \dfrac{\lambda^2}{d_1}$, we obtain

$$\begin{cases} \dfrac{dU}{ds} - \dfrac{d^2U}{dy^2} = \gamma\left[A - U - \lambda\varphi\left(U\right)V\right] := F\left(U,V\right) \\ \dfrac{dV}{ds} - d\dfrac{d^2V}{dy^2} = \gamma\left[B - \lambda\varphi\left(U\right)V\right] := G\left(U,V\right). \end{cases} \tag{4.6}$$

In our study, we will use (4.6) in the analysis, but since normally we use variables $(u, v, x, t)$ instead of $(U, V, y, t)$ and $(a, b)$ instead of $(A, B)$. Thus we rewrite (4.6) in the old variables

$$\begin{cases} u_t - \Delta u = \gamma\left[a - u - \lambda\varphi\left(u\right)v\right] := F\left(u,v\right), & x \in \Omega, t > 0, \\ v_t - d\Delta v = \gamma\left[b - \lambda\varphi\left(u\right)v\right] := G\left(u,v\right), & x \in \Omega, t > 0, \end{cases} \tag{4.7}$$

where $u$ and $v$ represents the dimensionless concentrations of the reactants. The parameters $a, b, \lambda, \gamma$ and $d$ are positive constants and the inhibitory function $\varphi \in \mathbb{C}^1(0,\infty) \cap \mathbb{C}[0,\infty)$ satisfies the following conditions

$$\varphi(0) = 0, \tag{4.8}$$

and for $u \in [\delta, a]$

$$\varphi(u) > 0, \tag{4.9}$$

with

$$0 < \delta < a - b. \tag{4.10}$$

The system (4.7), defined in the bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$ with smooth boundary $\partial\Omega$, is supplemented with the initial data

$$u\left(x,0\right) = u_0\left(x\right) \geq 0, \qquad v\left(x,0\right) = v_0\left(x\right) \geq 0, \qquad x \in \Omega \tag{4.11}$$

and the following Neumann boundary conditions

$$\frac{\partial u}{\partial \upsilon} = \frac{\partial v}{\partial \upsilon} = 0, \quad x \in \partial\Omega, \qquad t > 0, \tag{4.12}$$

where $\upsilon$ is the outward unit normal vector of the boundary $\partial\Omega$.

**Lemma 4.1** *The system (4.7) has a unique constant steady state*

$$(u^*, v^*) = \left(a - b, \frac{b}{\lambda\varphi\,(a-b)}\right), \tag{4.13}$$

*if and only if* $a > b$.

**Proof 4.2** *An equilibrium point* $(u^*, v^*)$ *of (4.7) satisfies*

$$\begin{cases} F\,(u,v) = \gamma\,[a - u - \lambda\varphi\,(u)\,v] = 0, \\ G\,(u,v) = \gamma\,[b - \lambda\varphi\,(u)\,v] = 0, \end{cases} \tag{4.14}$$

*by the second equation of (4.14), we obtain*

$$v^* = \frac{b}{\lambda\varphi\,(u)}, \tag{4.15}$$

*we compensate (4.15) in the first equation of (4.14), we find*

$$u^* = a - b,$$

*then, we conclude*

$$(u^*, v^*) = \left(a - b, \frac{b}{\lambda\varphi\,(a-b)}\right).$$

## 4.2 Global Existence of a Unique Bounded Solution

In this section, we shall show that the system (4.7) has a unique solution $(u(x,t), v(x,t))$, defined for all $t > 0$, which is bounded by some positive constants depending on the system parameters, the arbitrary function $\varphi(u)$ and the initial conditions $u_0$ and $v_0$. The existence of a unique bounded global solution will be proved applying the theory of invariant regions as was developed in [52].

**Lemma 4.3** *For any* $d > 0$, *the system (4.7) admits a unique solution* $(u,v) = (u(x,t), v(x,t))$ *for all* $x \in \Omega$ *and* $t > 0$. *Moreover, there exist two positive constants* $C_1$ *and* $C_2$, *depending on the initial conditions* $(u_0, v_0)$, *the system parameters* $a, b, \lambda$ *and the arbitrary function* $'\varphi$ *such that*

$$C_1 < u(x,t), v(x,t) < C_2. \tag{4.16}$$

**Proof 4.4** *The local existence and uniqueness of the solution for the system (4.7) are classical. In order to prove the global existence and the boundedness, we consider the* *following rectangular region*

$$\Re = (u_1, u_2) \times (v_1, v_2)$$

*and evaluate the functions $F(u,v)$ and $G(u,v)$ at their boundaries in such a way it is a contracting rectangle [49]. By condition (4.8), there exists a function $\varphi_1(u)$ such that $\varphi(u) = u\varphi_1(u)$. Let $u = u_1$ and $v_1 \le v \le v_2$ then*

$$F(u,v) = \gamma\left(a - u_1 - \lambda\varphi\left(u_1\right)v\right) \ge \gamma a - \gamma u_1\left[1 + \lambda \sup_{u\in[u_1,u_2]} \varphi_1(u)v_2\right].$$

*Hence, $F(u,v) \ge 0$ if*

$$u_1 \le \frac{a}{1 + \lambda \sup\limits_{u\in[u_1,u_2]} \varphi_1(u)v_2}. \tag{4.17}$$

*Let us evaluate $F(u,v)$ at the second boundary, i.e. $u = u_2$ and $v_1 \le v \le v_2$*

$$F(u,v) = \gamma\left(a - u_2 - \lambda\varphi\left(u_2\right)v\right) \le \gamma\left(a - u_2\right).$$

*Therefore $F(u,v) < 0$ if*

$$u_2 \ge a. \tag{4.18}$$

*From (4.17) and (4.18), it follows that $F(u,v)$ points inside the rectangle $\Re$ with*

$$u_1 = \min\left\{\frac{a}{1 + \lambda \sup\limits_{u\in[u_1,u_2]} \varphi_1(u)v_2}, \min u_0\left(x\right)\right\},$$

*and*

$$u_2 = \max\left\{a, \max u_0\left(x\right)\right\}.$$

*Evaluating the function $G(u,v)$ at the boundary $v = v_1$ and $u_1 \le u \le u_2$, we obtain*

$$\begin{aligned} G(u,v) &= \gamma\left[b - \lambda\varphi\left(u\right)v_1\right] \\ &= \gamma\left[b - \lambda u\varphi_1\left(u\right)v_1\right] \\ &\ge \gamma\left[b - \lambda u v_1 \sup_{u\in[u_1,u_2]} \varphi_1\left(u\right)\right], \end{aligned}$$

*and, since $u \le u_2$, it follows*

$$G(u,v) \ge \gamma\left[b - \lambda u_2 v_1 \sup_{u\in[u_1,u_2]} \varphi_1\left(u\right)\right].$$

*A sufficient condition for $G(u,v) \geq 0$ can then be formulated as*

$$v_1 \leq \frac{b}{\lambda u_2 \sup\limits_{u\in[u_1,u_2]} \varphi_1(u)}. \tag{4.19}$$

*At the last boundary $v = v_2$ and $u_1 \leq u \leq u_2$ we have*

$$G(u,v) = \gamma\left[b - \lambda\varphi(u)\, v_2\right] \leq \gamma\left[b - \lambda \min_{u\in[u_1,u_2]} \varphi(u)\, v_2\right],$$

*thus $G(u,v) \leq 0$ is satisfied when*

$$v_2 \geq \frac{b}{\lambda \min\limits_{u\in[u_1,u_2]} \varphi(u)}. \tag{4.20}$$

*From (4.19) and (4.20), it follows that $G(u,v)$ points inside the rectangle $\Re$ with*

$$v_1 = \min\left\{\frac{b}{\lambda u_2 \sup\limits_{u\in[u_1,u_2]} \varphi_1(u)}, \min v_0(x)\right\},$$

*and*

$$v_2 = \max\left\{\frac{b}{\lambda \min\limits_{u\in[u_1,u_2]} \varphi(u)}, \max v_0(x)\right\}.$$

*Therefore, the rectangle $\Re$ is an invariant rectangle for the system (4.7). Finally, the constants $C_1$ and $C_2$ in (4.16) can be defined as follows*

$$C_1 = \min\{u_1, v_1\} > 0 \quad \textit{and} \quad C_2 = \max\{u_2, v_2\} > 0. \tag{4.21}$$

Let us now prove the boundedness of the solutions.

**Lemma 4.5** *Let $(u,v) = (u(x,t), v(x,t))$ be the unique solution of (4.7). Then, for all $x \in \overline{\Omega}$*

$$\lim_{t\to\infty} \sup u < a, \quad \lim_{t\to\infty} \sup v < \frac{a-\delta}{\lambda\varphi(\delta)}. \tag{4.22}$$

**Proof 4.6** *Let $\varepsilon$ be a constant such that*

$$\varepsilon < \lambda\varphi(u)\, v, \tag{4.23}$$

*and* $\tilde{u} = \tilde{u}(t)$ *be the unique solution of the following Cauchy problem*

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = \gamma(\tilde{a} - \tilde{u}), \\ \\ \tilde{u}(0) = 2\max\limits_{x\in\overline{\Omega}} u_0(x), \end{cases} \tag{4.24}$$

*with*

$$\tilde{a} = a - \frac{\varepsilon}{2}.$$

*Let us also define the variable* $\hat{u} = u - \tilde{u}$. *From (4.7), we obtain*

$$\hat{u}_t - \Delta\hat{u} + [\tilde{u}_t - \gamma(\tilde{a} - \tilde{u})] = \gamma[a - \tilde{a} - \hat{u} - \lambda\varphi(u)v],$$

*and from (4.24),*

$$\hat{u}_t - \Delta\hat{u} = \gamma[a - \tilde{a} - \hat{u} - \lambda\varphi(u)v].$$

*So,*

$$-\hat{u}_t + \Delta\hat{u} - \gamma\hat{u} = \gamma[\lambda\varphi(u)v - a + \tilde{a}],$$

*by (4.23), we obtain*

$$\begin{cases} -\hat{u}_t + \Delta\hat{u} - \gamma\hat{u} = \gamma\left[\lambda\varphi(u)v - \frac{\varepsilon}{2}\right] > 0, \\ \hat{u}(x,0) < 0. \end{cases}$$

*Using the maximum principle for parabolic equations and the Neumann boundary conditions (4.12), we get*

$$\hat{u}_t < 0 \text{ and } \hat{u} < 0,$$

*so,*

$$\hat{u}(x,t) < 0 \quad \Rightarrow \quad u(x,t) < \tilde{u}(t) \qquad \text{for all } t > 0 \text{ and } x \in \overline{\Omega}. \tag{4.25}$$

*The maximum principle for parabolic equations cannot be directly used for the solution* $v = v(x,t)$, *therefore we define* $\tilde{v}(t)$ *as the solution of the following Cauchy problem*

$$\begin{cases} \dfrac{d\tilde{v}}{dt} = \gamma\tilde{g}(\tilde{u}, \tilde{v}), \\ \\ \tilde{v}(x,0) = 2\max\limits_{x\in\overline{\Omega}} v_0(x), \end{cases} \tag{4.26}$$

*where*

$$\tilde{g}(\tilde{u}, \tilde{v}) = \sup_{C_1 < \xi < \tilde{u}} \left[\tilde{b} - \lambda(\tilde{v} - \varepsilon_0)\right]\varphi(\xi), \tag{4.27}$$

*with* $\varepsilon_0 > 0$, $\tilde{b} > b$ *and*

$$\frac{\tilde{b}}{\lambda\varphi(\tilde{a})} + \varepsilon_0 < \frac{a-\delta}{\lambda\varphi(\delta)}.$$

*Let* $\hat{v} = v - \tilde{v}$. *It follows straightforwardly that* $\hat{v}(x,0) < 0$. *Hence, we may prove by contradiction that for all* $x \in \overline{\Omega}$ *and* $t > 0$

$$\hat{v}(x,t) < 0. \tag{4.28}$$

*If we let* $\hat{v}(x,t) < 0$, *then there exists* $T > 0$ *such that* $\hat{v}(x,t) < 0$ *for* $(x,t) \in \overline{\Omega} \times (0,T)$ *and* $\hat{v}(x,t) = 0$ *for some* $x \in \overline{\Omega}$, *which leads to*

$$\max_{x\in\overline{\Omega}} \hat{v}(x,t) = 0.$$

*If there exists* $x_1 \in \Omega$ *such that* $\hat{v}(x_1,T) = 0$, *then* $\hat{v}_t(x_1,T) \geq 0$ *and* $\Delta\hat{v}(x_1,T) \leq 0$ *and thus we have*

$$-\hat{v}_t(x_1,T) + d\Delta\hat{v}(x_1,T) \leq 0. \tag{4.29}$$

*However, we combine (4.7) and (4.26) for point* $(x_1,T)$, *we end up with*

$$\begin{aligned}
\hat{v}_t - d\Delta\hat{v} + \tilde{v}_t &= \gamma\left[b - \lambda\varphi(u)\,v\right] \\
\hat{v}_t - d\Delta\hat{v} + \left[\tilde{v}_t - \gamma\tilde{g}(\tilde{u},\tilde{v})\right] &= \gamma\left[b - \lambda\varphi(u)\,v\right] - \gamma\tilde{g}(\tilde{u},\tilde{v}), \\
\hat{v}_t - d\Delta\hat{v} &= \gamma\left[b - \lambda\varphi(u)\,v\right] - \gamma\tilde{g}(\tilde{u},\tilde{v}).
\end{aligned}$$

*So,*

$$-\hat{v}_t + d\Delta\hat{v} = \gamma\left[\tilde{g}(\tilde{u},\tilde{v}) - \left[b - \lambda\varphi(u)\,v\right]\right]. \tag{4.30}$$

*Setting* $\tilde{v} = v$ *and* $\tilde{u} > u$ *yields*

$$\begin{aligned}
\tilde{g}(\tilde{u},\tilde{v}) &= \sup_{C_1<\xi<\tilde{u}} \left[\tilde{b} - \lambda(\tilde{v} - \varepsilon_0)\right]\varphi(\xi), \\
&= \sup_{C_1<\xi<\tilde{u}} \left[\tilde{b} - \lambda(v - \varepsilon_0)\right]\varphi(\xi), \\
&> \sup_{C_1<\xi<\tilde{u}} \left[b - \lambda v\right]\varphi(\xi), \\
&\geq \sup_{C1<\xi<u} \left[b - \lambda v\right]\varphi(\xi), \\
&\geq \left[b - \lambda v\right]\varphi(u).
\end{aligned}$$

*Therefore*

$$\tilde{g}(\tilde{u},\tilde{v}) - \left[b - \lambda v\right]\varphi(u) \geq 0,$$

*and consequently*

$$-\hat{v}_t(x_1, T) + d\Delta\hat{v}(x_1, T) > 0,$$

*which contradicts the result in (4.29). Hence, (4.28) holds and we conclude that there exists some* $x_1 \in \partial\Omega$ *such that* $\hat{v}(x_1, T) = 0$ *leading to a positive right–hand side of (4.30) at* $(x_1, T)$*. By continuity, we know that it remains positive in* $\Omega' \times \{T\}$ *for any* $\Omega'$ *being a sub–domain of* $\Omega$ *and* $x_1 \in \Omega'$*. Hence, we get*

$$-\hat{v}_t(x_1, T) + d\Delta\hat{v}(x_1, T) > 0,$$

*on* $\Omega' \times \{T\}$*. Up to this point, we cannot state whether or not this inequality holds for* $\Omega\times(0, T]$*. Using Hopf's boundary lemma on (4.30) in* $\overline{\Omega}' \times \{T\}$*, we get* $\partial\hat{v} = \partial\hat{v}(x_1, T) > 0$*, which contradicts the Neumann boundary conditions and thus*

$$\hat{v}\,(x, t) < 0 \quad \Rightarrow \quad v(x, t) < \tilde{v}\,(t) \qquad \text{for all } x \in \overline{\Omega} \text{ and } \ t > 0. \tag{4.31}$$

*Finally, we consider the ODEs system*

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = \gamma\,(\tilde{a} - \tilde{u})\,, \\ \\ \dfrac{d\tilde{v}}{dt} = \gamma\tilde{g}\,(\tilde{u}, \tilde{v})\,, \end{cases}$$

*in* $\Re$*. From (4.27), we find that*

$$\begin{cases} \tilde{g}\,(\tilde{u}, \tilde{v}) < 0 \quad \text{for } \tilde{v} > \frac{\tilde{b}}{\lambda\varphi(\tilde{u})} + \varepsilon_0, \\ \tilde{g}\,(\tilde{u}, \tilde{v}) > 0 \quad \text{for } \tilde{v} < \frac{\tilde{b}}{\lambda\varphi(\tilde{u})} + \varepsilon_0. \end{cases}$$

*Hence,* $\tilde{v} = \dfrac{\tilde{b}}{\lambda\varphi\,(\tilde{u})} + \varepsilon_0$ *constitutes the nullcline of* $\tilde{g}$ *and the system admits the unique equilibrium*

$$(\tilde{u}, \tilde{v}) = (\tilde{a}, \frac{\tilde{b}}{\lambda\varphi\,(\tilde{a})} + \varepsilon_0).$$

*Since* $\lim_{t\to\infty} \tilde{u}\,(t) = \tilde{a}$*, it follows that* $(\tilde{u}, \tilde{v})$ *is globally asymptotically stable in* $\Re$*, which implies that*

$$\lim_{t\to\infty} \tilde{v}\,(t) = \frac{\tilde{b}}{\lambda\varphi\,(\tilde{a})} + \varepsilon_0.$$

*By (4.25) and (4.31), being* $\tilde{a} < a$ *and* $\frac{\tilde{b}}{\lambda\varphi(\tilde{a})} + \varepsilon_0 < \frac{a-\delta}{\lambda\varphi(\delta)}$*, the Lemma is proved.*

## 4.3 Asymptotic Stability

In this Section we shall study the asymptotic behaviour of the generalized Degn–Harrison system (4.7). In particular, we will find the conditions on the system parameters and the arbitrary function $\varphi(u)$ which guarantee the attractivity of the unique homogeneous steady state solution (4.13) and therefore prevent pattern formation. The asymptotic analysis shall be performed at first for the local dynamics using the eigenfunction expansion method, then we will drive a suitable conditions for the global asymptotic stability. Also, we will discus the global asymptotic stability by other method "the direct Lyapunov method".

### 4.3.1 Local Asymptotic Stability

At first let us perform the linear stability analysis of the equilibrium $(u^*, v^*)$ in (4.13).

**Proposition 4.7** *Given the following ODEs system associated to the generalized Degn–Harrison system (4.7)*

$$\begin{cases} \dfrac{du}{dt} = \gamma \left[a - u - \lambda\varphi\left(u\right)v\right], \quad t > 0 \\ \\ \dfrac{dv}{dt} = \gamma \left[b - \lambda\varphi\left(u\right)v\right], \quad t > 0, \end{cases} \tag{4.32}$$

*the solution $(u^*, v^*)$ is locally asymptotically stable as an equilibrium of (4.32) if*

$$-\left[\varphi\left(a-b\right) + b\varphi'\left(a-b\right)\right] < \lambda\varphi^2\left(a-b\right). \tag{4.33}$$

**Proof 4.8** *The Jacobian matrix associated to the system (4.32) and evaluated in the equilibrium $(u^*, v^*)$ is computed as*

$$J\left(u^*, v^*\right) = \gamma \begin{pmatrix} F_0 & -G_0 \\ 1 + F_0 & -G_0 \end{pmatrix}, \tag{4.34}$$

*with*

$$F_0 = -1 - b\frac{\varphi'\left(a-b\right)}{\varphi\left(a-b\right)} \quad \text{and} \quad G_0 = \lambda\varphi\left(a-b\right). \tag{4.35}$$

*The equilibrium $(u^*, v^*)$ is locally asymptotically stable if the eigenvalues of the jacobian matrix $J\left(u^*, v^*\right)$ are both with negative real parts. The following characteristic polynomial associated to $J\left(u^*, v^*\right)$*

$$\sigma^2 - tr J\left(u^*, v^*\right)\sigma + \det J\left(u^*, v^*\right)$$

*admits roots with negative real parts if* $\det J(u^*, v^*) > 0$ *and* $tr J(u^*, v^*) < 0$ . *The determinant*

$$\det J(u^*, v^*) = \gamma^2 G_0 = \gamma^2 \lambda\varphi(a-b) \tag{4.36}$$

*is positive by (4.9). The trace*

$$tr J(u^*, v^*) = \gamma(F_0 - G_0) = -\gamma\left[1 + b\frac{\varphi'(a-b)}{\varphi(a-b)} + \lambda\varphi(a-b)\right]. \tag{4.37}$$

*Therefore,* $(u^*, v^*)$ *is locally asymptotically stable when the condition (4.33) is satisfied.*

Using the eigenvalue/eigenfunction notation defined at the end of the Introduction, if

$$\lambda_1 < \gamma F_0 = -\gamma - \gamma b\frac{\varphi'(a-b)}{\varphi(a-b)}, \tag{4.38}$$

then $i_\alpha = (\alpha, \Omega)$ is defined as the largest positive integer such that

$$\lambda_i < \gamma F_0 \text{ for } i \leq i_\alpha. \tag{4.39}$$

Clearly, if (4.38) holds, then $1 \leq i_\alpha < \infty$. In this case, we define the constant

$$\tilde{d} = \min_{1\leq i\leq i_\alpha} d_i,\ d_i = \frac{\gamma^2 G_0(\lambda_i + 1)}{\lambda_i(\gamma F_0 - \lambda_i)}. \tag{4.40}$$

The following Theorem can now be formulated for the local stability of $(u^*, v^*)$ as a steady state of (4.7).

**Theorem 4.9** *Let us assume that condition (4.33) holds. The constant steady state* $(u^*, v^*)$ *is locally asymptotically stable for the system (4.7) if*

$$\begin{cases} \lambda_i \geq \gamma F_0 & \textit{or} \\ \lambda_i < \gamma F_0 & \textit{and} \quad 0 < d = \frac{d_2}{d_1} < \tilde{d}. \end{cases} \tag{4.41}$$

*If*

$$\lambda_i < \gamma F_0 \quad \textit{and} \quad d > \tilde{d},$$

*then* $(u^*, v^*)$ *is locally asymptotically unstable.*

**Proof 4.10** *Let* $L$ *be the linearized operator associated to the system (4.7) in* $(u^*, v^*)$

$$L = \begin{pmatrix} \Delta + \gamma F_0 & -\gamma G_0 \\ 1 + \gamma F_0 & d\Delta - \gamma G_0 \end{pmatrix}.$$

*The constant steady state* $(u^*, v^*)$ *is said to be locally asymptotically stable for the system* (4.7) *if and only if all the eigenvalues of* $L$ *have negative real parts. Denoting* $(\phi_1(x), \phi_2(x))$ *the eigenfunction associated with the eigenvalue* $\xi$ $(\phi_1(x), \phi_2(x))$*, we get*

$$[L - \xi I](\phi_1(x), \phi_2(x))^t = (0, 0)^t,$$

*which explicitly reads*

$$\begin{pmatrix} \Delta + \gamma F_0 - \xi & -\gamma G_0 \\ \gamma(1 + F_0) & d\Delta - \gamma G_0 - \xi \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

*Defining* $(\phi_1(x), \phi_2(x))$ *in sequence form as follows*

$$\phi_1 = \sum_{0 \le i \le \infty, 1 \le j \le m_i} a_{ij} \Phi_{ij} \quad \text{and} \quad \phi_2 = \sum_{0 \le i \le \infty, 1 \le j \le m_i} b_{ij} \Phi_{ij},$$

*we obtain*

$$\sum_{0 \le i \le \infty, 1 \le j \le m_i} \begin{pmatrix} \gamma F_0 - \lambda_i - \xi & -\gamma G_0 \\ \gamma(1 + F_0) & -\gamma G_0 - d\lambda_i - \xi \end{pmatrix} \begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \Phi_{ij} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

*Then,* $\xi$ *is an eigenvalue of* $L$ *if for some* $i \ge 0$ *the following equation is satisfied*

$$\xi^2 + P_i \xi + Q_i = 0,$$

*where*

$$P_i = \lambda_i (d + 1) + \gamma (G_0 - F_0),$$

*and*

$$Q_i = \lambda_i d (\lambda_i - \gamma F_0) + \gamma^2 G_0 (\lambda_i + 1).$$

*Since the condition* (4.33) *holds, then* $P_i > 0$*. Moreover, being* $G_0 = \frac{1}{\gamma^2} \det J(u^*, v^*)$*, it is clear that* $Q_0 > 0$ *for* $\lambda_0 = 0$*. Let us now check the sign of* $Q_i$ *if the conditions* (4.41) *of the Theorem* 4.9 *are satisfied*

*If* $\lambda_i \ge \gamma F_0$*, then* $Q_i > 0$ *for* $i \ge 1$*.*

*If* $\lambda_i < \gamma F_0$ *and* $0 < d < \tilde{d}$*, then*

$$\lambda_i < \gamma F_0 \text{ and } 0 < d < d_i, \text{ for } i \in [1, i_\alpha].$$

*Hence,* $Q_i > 0$ *for* $i \in [1, i_\alpha]$*. Furthermore, if* $i \ge i_\alpha$*, then* $\lambda_i \ge \gamma F_0$ *and* $Q_i > 0$*.*

*Therefore, when* (4.33) *and* (4.41) *hold, we get* $P_i > 0$ *and* $Q_i > 0$ *for all* $i \ge 0$*, which* *implies that all the eigenvalues $\xi$ have negative real part, and the steady–state $(u^*, v^*)$ is locally asymptotically stable.*

*Finally, if $\lambda_i < \gamma F_0$ and $d > \tilde{d}$, we assume that the minimum in (4.40) is obtained for some $k \in [1, i_\alpha]$*

$$d > d_k, \tag{4.42}$$

*therefore $Q_k < 0$ and $(u^*, v^*)$ is locally asymptotic unstable.*

**Theorem 4.11** *The homogeneous steady state $(u^*, v^*)$ is locally asymptotically stable for the system (4.7) if $F_0 \leq 0$ or*

$$0 < F_0 < G_0, \tag{4.43}$$

*and*

$$\begin{cases} \lambda_1 \geq \gamma F_0 \quad or \\ \lambda_1 < \gamma F_0 \ and \ \begin{cases} d \leq \frac{G_0}{F_0} \quad or \\ \frac{G_0}{F_0} < d < \wp, \end{cases} \end{cases} \tag{4.44}$$

*where $\wp$ is the solution of the following equation*

$$(F_0 x + G_0)^2 = 4(1 + F_0) G_0 x. \tag{4.45}$$

**Proof 4.12** *First of all, let us rewrite (4.7) in vector form as follows*

$$\frac{\partial \mathbf{z}}{\partial t} - D\Delta \mathbf{z} = \mathbf{F}(\mathbf{z}), \quad where \tag{4.46}$$

$$\mathbf{z} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \ and \ \ \mathbf{F}(\mathbf{z}) = \gamma \begin{pmatrix} a - u - \lambda\varphi(u) v \\ b - \lambda\varphi(u) v \end{pmatrix}.$$

*In order to establish the local asymptotic stability of $(u^*, v^*)$ as the steady–state solution of (4.46) it suffices to show that $(0, 0)$ is asymptotically stable as a steady state solution of the linearized system*

$$\frac{\partial \mathbf{z}}{\partial t} - D\Delta \mathbf{z} = J, \tag{4.47}$$

*where $J$ the Jacobian matrix associated at $(u^*, v^*)$.*

*The local asymptotic stability of $(0, 0)$ for (4.47) requires the eigenvalues of $J - \lambda_n d$ to have negative real parts for all $n \geq 0$. Since the system is $2 \times 2$, it suffices that the trace of $J - \lambda_n d$ is negative and the determinant is positive. In what follows we check the signs of trace and determinant in the various cases when the hypotheses of the Theorem 4.11 are satisfied* *let* $F_0 \leq 0$, *then*

$$J - \lambda_n D = \begin{pmatrix} \gamma F_0 - \lambda_n & -\gamma G_0 \\ \gamma (1 + F_0) & -\gamma G_0 - d\lambda_n \end{pmatrix}.$$

*Via a straightforward computation, being* $G_0 > 0$, *we get*

$$\begin{aligned} \det (J - \lambda_n d) &= (\gamma F_0 - \lambda_n) (-\gamma G_0 - d\lambda_n) - (-\gamma G_0) (\gamma (1 + F_0)) \\ &= \lambda_n^2 d + \gamma \lambda_n d (-F_0) + \gamma \lambda_n G_0 + \gamma^2 G_0 \geq 0, \end{aligned}$$

*and*

$$\begin{aligned} tr (J - \lambda_n d) &= \gamma F_0 - \lambda_n - \gamma G_0 - d\lambda_n \\ &= -\lambda_n (d + 1) + \gamma (F_0 - G_0) \leq 0. \end{aligned}$$

*Hence, all the eigenvalues of* $J - \lambda_n d$ *have negative real parts and the steady–state is locally asymptotic stable. Let (4.43) and the first condition in (4.44) be satisfied. For the first eigenvalue* $\lambda_0 = 0$, *we have* $J - \lambda_0 d = J$ *and therefore*

$$\det J = \gamma^2 G_0 > 0,$$

*and*

$$tr J = \gamma (F_0 - G_0) < 0.$$

*Being* $\lambda_1 \geq \gamma F_0$, *we have* $\lambda_n \geq \gamma F_0$, *which leads to*

$$\det (J - \lambda_n d) = \lambda_n d (\lambda_n - \gamma F_0) + \gamma \lambda_n G_0 + \gamma^2 G_0 \geq 0,$$

*and*

$$tr (J - \lambda_n d) = (\gamma F_0 - \lambda_n) - \gamma G_0 - \lambda_n d \leq 0.$$

*Therefore, the steady state is locally asymptotically stable. Let* $\lambda_1 < \gamma F_0$ *and* $d \leq \frac{G_0}{F_0}$. *For the eigenvalues* $\lambda_n$, $n > 1$ *such that* $\lambda_n \geq \gamma F_0$, *with the same arguments as above we can conclude that* $J - \lambda_n D$ *has eigenvalues with negative real parts. Let* $\theta$ *one of the remaining eigenvalues such that* $\theta < \gamma F_0$. *Since (4.43) holds, the trace is still negative*

$$tr (J - \theta D) = \gamma (F_0 - G_0) - \theta (d + 1) < 0, \tag{4.48}$$

*and, being* $d \leq \frac{G_0}{F_0}$*, the determinant is positive*

$$\begin{aligned} \det(J - \theta D) &= \gamma^2 G_0 + \theta^2 d - \gamma\theta d F_0 + \gamma\theta G_0 \\ &= \theta^2 d - \gamma\theta (dF_0 - G_0) + \gamma^2 G_0 > 0. \end{aligned} \tag{4.49}$$

*Therefore, the steady state is locally asymptotically stable. Let* $\lambda_1 < \gamma F_0$ *and* $\frac{G_0}{F_0} d < \wp$*, with* $\wp$ *given in the statement of the Theorem. The trace of an eigenvalue* $\theta$ *is computed as in (4.48) and it is always negative under the condition (4.43). Therefore, we must check the sign of the determinant*

$$\theta^2 d - \gamma\theta (dF_0 - G_0) + \gamma^2 G_0. \tag{4.50}$$

*If*

$$(\gamma (dF_0 - G_0))^2 - 4d\gamma^2 G_0 < 0, \tag{4.51}$$

*then the determinant in (4.50) is positive for all* $\theta$*. Let us rewrite condition (4.51) in the following form*

$$\begin{aligned} \gamma^2 \left((dF_0)^2 + G_0^2 - 2dF_0G_0\right) &< 4d\gamma^2 G_0 \\ (F_0 d + G_0)^2 &< 4(1 + F_0) G_0 d. \end{aligned}$$

*In the interval* $[0, +\infty)$*, between the parabola* $y = (F_0 x + G_0)^2$ *and the line* $y = 4(1 + F_0) G_0 x$*, it is easy to see that, at the point* $\bar{x} = \frac{G_0}{F_0}$*, we have*

$$(F_0 \bar{x} + G_0)^2 < 4(1 + F_0) G_0 \bar{x}.$$

*The line intersects the parabola at two points* $x_1$ *and* $x_2$ *such that* $0 < x_1 < \bar{x} < x_2$*. Setting* $\wp = x_2$*, we obtain that* $\wp$ *is the solution of (4.45) satisfying*

$$\wp > \frac{G_0}{F_0}.$$

*In addition, the inequality:*

$$(F_0 x + G_0)^2 < 4(1 + F_0) G_0 x,$$

*holds for*

$$\frac{G_0}{F_0} < x < \wp.$$

*We can again conclude that the steady state is locally asymptotically stable.*

### 4.3.2 Global Asymptotic Stability

In this Section, we shall obtain sufficient conditions to achieve global asymptotic stability of the steady state solution (4.13). At first, we will apply the Poincaré–Bendixson theorem [13] at the ODEs system associated with (4.7) in order to obtain global stability for the local dynamics. Then, in Theorem 3, we shall find suitable conditions to guarantee the global stability of the steady state for the PDEs system (4.7).

The global stability of the equilibrium solution (4.13) will be also discussed performing the well-known direct Lyapunov method. Further conditions ensuring that the steady state solution is globally asymptotically stable for the system (4.7) are obtained in Theorem 6.

Let us first find the invariant rectangle $\Re_\delta$ defined as in (4.52).

**Proposition 4.13** *The following rectangle*

$$\Re_\delta = [\delta, a] \times \left[ \frac{b}{\lambda \sup\limits_{u\in[\delta,a]} \varphi(u)}, \frac{a-\delta}{\lambda\varphi(\delta)} \right], \tag{4.52}$$

*with*

$$\frac{a-\delta}{\varphi(\delta)} > \frac{b}{\inf\limits_{u\in[\delta,a]} \varphi(u)}, \tag{4.53}$$

*is an invariant rectangle for the system (4.7).*

**Proof 4.14** *According to the Definition 1.3, we just evaluate the vector field* $(F, G)$ *given in (4.7) at the boundaries of* $\Re_\delta$*. Let*

$$\frac{b}{\lambda \sup\limits_{u\in[\delta,a]} \varphi(u)} < v < \frac{a-\delta}{\lambda\varphi(\delta)},$$

*then it straight for wardly results*

$$F(\delta, v) = \gamma(a - \delta - \lambda\varphi(\delta)v) > 0,$$

*and*

$$F(a, v) = \gamma(a - a - \lambda\varphi(a)v) = -\gamma\lambda\varphi(a)v < 0.$$

*Similarly, assuming $\delta < u < a$ leads to*

$$\begin{aligned} G\left(u, \frac{b}{\lambda \sup\limits_{u\in[\delta,a]} \varphi(u)}\right) &= \gamma\left[b - \lambda\varphi(u)\frac{b}{\lambda \sup\limits_{u\in[\delta,a]} \varphi(u)}\right] \\ &= \gamma\varphi(u)\left(\frac{b}{\varphi(u)} - \frac{b}{\sup\limits_{u\in[\delta,a]} \varphi(u)}\right) > 0, \end{aligned}$$

*and*

$$\begin{aligned} G\left(u, \frac{a-\delta}{\lambda\varphi(\delta)}\right) &= \gamma\left(b - \varphi(u)\frac{a-\delta}{\varphi(\delta)}\right) \\ &< \gamma\left(b - \inf_{u\in[\delta,a]} \varphi(u)\frac{a-\delta}{\varphi(\delta)}\right) < 0, \end{aligned}$$

*where the last inequality follows by condition (4.53).*

Now, we state the following Theorem which gives the conditions for the global asymptotic stability of $(u^*, v^*)$ as a solution of the reduced ODEs system associated to (4.7).

**Theorem 4.15** *Given the ODEs system (4.32), let us define $f(u) = \frac{a-u}{\varphi(u)}$ and $u_i$, $i = 1, \ldots, N$ be the inflection points of $f(u)$. If the following condition holds*

$$\max\left\{\max_{i=1,\ldots,N} f'(u_i), f'(\delta), f'(a)\right\} < \lambda, \tag{4.54}$$

*then the equilibrium $(u^*, v^*)$ given in (4.13) is globally asymptotically stable for the system (4.32).*

**Proof 4.16** *Let us rewrite the system (4.32) in terms of the function $f(u)$*

$$\begin{cases} u_t = F(u,v) = \gamma\varphi(u)\left(\frac{a-u}{\varphi(u)} - \lambda v\right) = \gamma\varphi(u)(f(u) - \lambda v), \\ v_t = G(u,v) = \gamma\varphi(u)\left(\frac{b}{\varphi(u)} - \lambda v\right). \end{cases} \tag{4.55}$$

*We would like to apply the Dulac criterion to the plane system (4.55) in the invariant region $\Re_\delta$ defined in (4.52).*

*Let $\psi = \frac{1}{\gamma\varphi(u)}$ be the Dulac function candidate. We shall check the sign of the following divergence*

$$\frac{\partial\psi F}{\partial u} + \frac{\partial\psi G}{\partial v} = \frac{-\varphi(u) - (a-u)\varphi'(u)}{(\varphi(u))^2} - \lambda = f'(u) - \lambda. \tag{4.56}$$

*If $f(u)$ is decreasing, then $f'(u) < 0$ and the sign of the divergence in (4.56) is negative. If $f(u)$ is not decreasing, then*

$$f'(u) < \max\left\{\max_{i=1,N} f'(u_i), f'(\delta), f'(a)\right\} \quad \text{in } [\delta, a],$$

*which implies*

$$f'(u) - \lambda < \max\left\{\max_{i=1,\dots,N} f'(u_i), f'(\delta), f'(a)\right\} - \lambda < 0,$$

*where the last inequality holds under the hypothesis (4.54) of the Theorem. Therefore, the divergence in (4.56) has the same negative sign in $\Re_\delta$ and, according to the Dulac criterion, there are no closed orbits lying entirely in $\Re_\delta$.*

*To complete the proof it suffices to show that $(u^*, v^*)$ is locally asymptotic stable. Since $f'(u^*) < \max\left\{\max_{i=1,\dots,N} f'(u_i), f'(\delta), f'(a)\right\}$, using the assumption (4.33), it follows that*

$$f'(u^*) < \lambda. \tag{4.57}$$

*The condition in (4.57) is equivalent to the assumption (4.33) which guarantees the local asymptotic stability of the equilibrium $(u^*, v^*)$. Therefore, using the absence of periodic solutions and the Poincaré- Bendixson theorem, we complete the proof.*

Let $\alpha$ denote the following quantity

$$\alpha = \max_{(u,v)\in\Re_\delta} \varsigma(u,v), \tag{4.58}$$

where $\varsigma(u,v)$ is the greatest real eigenvalue of the symmetric matrix $J^H$

$$J^H = \frac{1}{2}\left(J + J^T\right),$$

with $J(u,v)$ the Jacobian matrix associated to the system (4.32) and $J^T$ its transpose matrix.

**Theorem 4.17** *Assume that*

$$f'(u^*) > 0 \quad \text{and} \quad \lambda_1 > \frac{\alpha}{\beta}, \tag{4.59}$$

*where $f(u) = \frac{a-u}{\varphi(u)}$ as in the previous theorem, $\alpha$ is defined by (4.58) and $\beta = \min\{1, d\}$. Let $\mathbf{z}(x,t)$ be a solution of the Neumann boundary value problem associated with the* *linearized system (4.47). Then*

$$\lim_{t\to\infty} \|\nabla \mathbf{z}(.,t)\|_{\mathbb{L}^2(\Omega)} = 0. \tag{4.60}$$

**Proof 4.18** *In order to prove (4.60), we show that there exist two constants $T$ and $C$ such that*

$$\|\nabla \mathbf{z}(.,t)\|_{\mathbb{L}^2(\Omega)} \leq Ce^{-(\beta\lambda_1-\alpha)t}, \quad \text{for } t > T. \tag{4.61}$$

*In fact, the inequality (4.61) together with the assumption $\lambda_1 > \frac{\alpha}{\beta}$ in (4.59) will directly imply (4.60).*

*At first, we observe that the assumption $f'(u^*) > 0$ is equivalent to $F_0 > 0$.*

*Let us evaluate the matrix $J^H$ at the steady state $(u^*, v^*)$*

$$J^H(u^*,v^*) = \gamma \begin{pmatrix} F_0 & \frac{1}{2}(1+F_0-G_0) \\ \frac{1}{2}(1+F_0-G_0) & -G_0 \end{pmatrix}.$$

*Being $F_0 > 0$, it follows that*

$$\det J^H(u^*,v^*) = -F_0G_0 - \frac{1}{4}(1+F_0-G_0)^2 < 0,$$

*therefore the constant $\alpha$ in (4.58) is positive*

$$\alpha \geq \varsigma(u^*,v^*) > 0.$$

*For the linearized system (4.47), there exist $T > 0$ such that*

$$\mathbf{z}(x,t) = (u(x,t), v(x,t)) \in \Re_\delta, \quad t > T.$$

*Let us define the following function*

$$\begin{aligned} \Phi(t) &= \frac{1}{2}\|\nabla \mathbf{z}(.,t)\|^2_{\mathbb{L}^2(\Omega)} \\ &= \frac{1}{2}\int_\Omega \langle \nabla \mathbf{z}(x,t), \nabla \mathbf{z}(x,t)\rangle\, dx, \quad \text{for } t > T, \end{aligned} \tag{4.62}$$

*where $\langle .,. \rangle$ denotes the inner product in $\mathbb{R}^2$. The derivative of $\Phi(t)$ is, thus, given by*

$$\begin{aligned} \frac{d\Phi(t)}{dt} &= \int_\Omega \langle \nabla \mathbf{z}, \nabla \mathbf{z}_t\rangle\, dx \\ &= -\int_\Omega \langle \Delta \mathbf{z}, D\Delta \mathbf{z}\rangle\, dx + \int_\Omega \left\langle \nabla \mathbf{z}, J^H(\mathbf{z})\nabla \mathbf{z}\right\rangle dx. \end{aligned} \tag{4.63}$$

*Using Lemma A.1 of [17], we deduce the following inequality*

$$\int_{\Omega} \langle \Delta \mathbf{z}, D\Delta \mathbf{z} \rangle \, dx \geq \beta\lambda_1 \int_{\Omega} |\nabla \mathbf{z}|^2 \, dx. \tag{4.64}$$

*Using the definition (4.58) and the properties of the symmetric matrix $J^H$, the inequality in (4.64) can be rearranged as follows*

$$\left\langle \nabla \mathbf{z}, J^H(\mathbf{z}) \nabla \mathbf{z} \right\rangle \leq \varsigma(\mathbf{z}) |\nabla \mathbf{z}|^2 \leq \alpha |\nabla \mathbf{z}|^2 . \tag{4.65}$$

*Using (4.65) into (4.63), we obtain*

$$\frac{d\Phi(t)}{dt} \leq -(\beta\lambda_1 - \alpha) \int_{\Omega} |\nabla \mathbf{z}|^2 \, dx, \qquad t > T.$$

*Hence, the function $\Phi$ satisfies the following differential inequality*

$$\Phi'(t) \leq -2(\beta\lambda_1 - \alpha)\Phi(t), \qquad \textit{for} \;\; t > T. \tag{4.66}$$

*From (4.66) we can state that there exists a constant $c_1 > 0$ such that*

$$\Phi(t) \leq c_1 e^{-2(\beta\lambda_1 - \alpha)t},$$

*and by the definition in (4.62), the (4.61) trivially follows with $C = 2c_1$.*

The final result of the paper concerns the global asymptotic stability of the steady–state solution $(u^*, v^*)$ for the system (4.7).

**Theorem 4.19** *Under the same assumptions of the Theorems 2.4 and 4.17, we have*

$$\lim_{t\to\infty} \|u(x,t) - u^*\|_{\mathbb{L}^2(\Omega)} = \lim_{t\to\infty} \|v(x,t) - v^*\|_{\mathbb{L}^2(\Omega)} = 0. \tag{4.67}$$

**Proof 4.20** *Let $\mathbf{z} = (u(x,t), v(x,t))$ be a solution of the system (4.7). As demonstrated in Lemma A.2 of [17], we may use the Poincaré inequality to obtain*

$$\|\mathbf{z}(.,t) - \overline{\mathbf{z}}(.,t)\|^2_{\mathbb{L}^2(\Omega)} \leq \frac{1}{\lambda_1} \|\nabla \mathbf{z}(.,t)\|^2_{\mathbb{L}^2(\Omega)}, \tag{4.68}$$

*where*

$$\overline{\mathbf{z}}(t) = \frac{1}{|\Omega|} \int_{\Omega} \mathbf{z}(x,t) \, dx.$$

*Using the inequality (4.68) and (4.60) we obtain*

$$\lim_{t\to\infty}\|u(x,t)-\overline{u}(t)\|_{\mathbb{L}^2(\Omega)}=\lim_{t\to\infty}\|v(x,t)-\overline{v}(t)\|_{\mathbb{L}^2(\Omega)}=0, \tag{4.69}$$

*where $\overline{u}(t)$ and $\overline{v}(t)$ denote, respectively, the averages on $\Omega$ of $u(x,t)$ and $v(x,t)$. Now, using Theorem 3.1 in [17] again, we deduce that the pair $(\overline{u}(t),\overline{v}(t))$ satisfies the following ODEs system*

$$\begin{cases} u'=F(u,v)+q_1(t)\\ v'=G(u,v)+q_2(t)\\ u(0)=\frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx, \quad v(0)=\frac{1}{|\Omega|}\int_\Omega v_0(x)\,dx, \end{cases} \tag{4.70}$$

*where for some $k>0, t>T$, and $i=1,2$, we have*

$$|q_i(t)|\le ke^{-(\beta\lambda_1-\alpha)t}. \tag{4.71}$$

*From (4.71) it follows that as $t\to\infty$,*

$$\int_t^{t+1} q_i(s)\,ds\to 0, \quad \textit{for } i=1,2.$$

*Moreover, Theorem 4.15 guarantees that the constant steady state solution is globally asymptotically stable for the ODE system. At this stage, we apply Theorem 5.5.7 of [9] to show that every solution of (4.70) converges to $(u^*,v^*)$, thus*

$$\lim_{t\to\infty}|\overline{u}(t)-u^*|=\lim_{t\to\infty}|\overline{v}(t)-v^*|=0. \tag{4.72}$$

*Since the following inequalities hold*

$$\|u(.,t)-u^*\|_{\mathbb{L}^2(\Omega)}\le\|u(.,t)-\overline{u}(t)\|_{\mathbb{L}^2(\Omega)}+|\Omega|^{\frac{1}{2}}|\overline{u}(t)-u^*|,$$

*and*

$$\|v(.,t)-v^*\|_{\mathbb{L}^2(\Omega)}\le\|v(.,t)-\overline{v}(t)\|_{\mathbb{L}^2(\Omega)}+|\Omega|^{\frac{1}{2}}|\overline{v}(t)-v^*|,$$

*using (4.69) and (4.72) we end up the proof of the Theorem.*

In order to state the final Theorem 4.25 for the global asymptotic stability of the equilibrium (4.13), we need to formulate the following lemmas and propositions.

**Lemma 4.21** *If $u\in[\delta,a]$, then there exists a constant $\mu$ between $u$ and $u^*$ such that*

$$\frac{b}{\varphi(u)}-\frac{b}{\varphi(u^*)}=(u-u^*)\left(\frac{b}{\varphi(\mu)}\right)'. \tag{4.73}$$

**Lemma 4.22** *The derivative of the function*

$$H(u(x,t)) = \int_{\alpha}^{u} \left( \frac{b}{\varphi(r)} - \frac{b}{\varphi(u^*)} \right) dr \geq 0, \tag{4.74}$$

*is given by*

$$\frac{d}{du} H(u) = \frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)}. \tag{4.75}$$

**Proposition 4.23** *Let*

$$V(t) = \int_{\Omega} E(u(x,t), v(x,t))\, dx, \tag{4.76}$$

*where*

$$E(u,v) = H(u) + \frac{\lambda}{2}(v - v^*) \tag{4.77}$$

*and $(u(x,t), v(x,t))$ is a solution of the system (4.7). If $\varphi(u)$ is a decreasing function and*

$$(u^* - u)\left( \frac{a-u}{\varphi(u)} - \frac{a-u^*}{\varphi(u^*)} \right) > 0 \quad \text{for} \quad u \in [\delta, u^*) \cup (u^*, a], \tag{4.78}$$

*then $V(t)$ is a Lyapunov functional.*

**Proof 4.24** *Let us rewrite the system (4.7) in the following convenient form*

$$\begin{cases} u_t - \Delta u = \gamma\varphi(u) \left[ \left( \frac{a-u}{\varphi(u)} - \frac{a-u^*}{\varphi(u^*)} \right) - \lambda \left( v - \frac{b}{\lambda\varphi(u^*)} \right) \right], \\ v_t - d\Delta v = \gamma\varphi(u) \left[ \left( \frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)} \right) - \lambda \left( v - \frac{b}{\lambda\varphi(u^*)} \right) \right], \end{cases} \tag{4.79}$$

*with $u^* = a - b$ and $x \in \Omega, t > 0$.*

*Differentiating the functional $V(t)$ with respect to $t$ yields*

$$\begin{aligned} \dot{V}(t) &= \lambda \int_{\Omega} \left[ (v - v^*) \left( d\Delta v + \gamma\varphi(u) \left( \left( \frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)} \right) - \lambda (v - v^*) \right) \right) \right] dx \\ &+ \int_{\Omega} \left[ \left( \frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)} \right) \left( \Delta u + \gamma\varphi(u) \left( \left( \frac{a-u}{\varphi(u)} - \frac{a-u^*}{\varphi(u^*)} \right) - \lambda (v - v^*) \right) \right) \right] dx, \end{aligned}$$

*which we rewrite as follow*

$$\dot{V}(t) = I + J, \tag{4.80}$$

*where*

$$I = \int_{\Omega} \left( \frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)} \right) \Delta u dx + d\lambda \int_{\Omega} (v - v^*) \Delta v dx,$$

*and*

$$J = \int_{\Omega} \gamma\varphi(u) \left[ \left( \frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)} \right) \left( \frac{a-u}{\varphi(u)} - \frac{a-u^*}{\varphi(u^*)} \right) - \lambda^2 (v - v^*)^2 \right] dx.$$

*We now check the sign of $I$ and $J$*

$$\begin{aligned} I &= \int_{\Omega}\left(\frac{b}{\varphi(u)}-\frac{b}{\varphi(u^*)}\right)\Delta u dx + d\lambda\int_{\Omega}(v-v^*)\Delta v dx \\ &= -\int_{\Omega}\nabla\left(\frac{b}{\varphi(u)}-\frac{b}{\varphi(u^*)}\right)\nabla u dx - d\lambda\int_{\Omega}\nabla(v-v^*)\nabla v dx \\ &= -\int_{\Omega}\left(\frac{b}{\varphi(u)}\right)'|\nabla u|^2 dx - d\lambda\int_{\Omega}|\nabla v|^2 dx \le 0. \end{aligned}$$

*For $J$ we get*

$$\begin{aligned} J &= \int_{\Omega}\gamma\varphi(u)\left[\left(\frac{b}{\varphi(u)}-\frac{b}{\varphi(u^*)}\right)\left(\frac{a-u}{\varphi(u)}-\frac{a-u^*}{\varphi(u^*)}\right)-\lambda^2(v-v^*)^2\right]dx \\ &= \int_{\Omega}\gamma\varphi(u)\left[\left(\frac{b}{\varphi(u)}\right)'_{u=\mu}(u-u^*)\left(\frac{a-u}{\varphi(u)}-\frac{a-u^*}{\varphi(u^*)}\right)-\lambda^2(v-v^*)^2\right]dx\,. \end{aligned}$$

*The condition (4.78) leads to*

$$u \le u^* \Longrightarrow (u-u^*)\left(\frac{a-u}{\varphi(u)}-\frac{a-u^*}{\varphi(u^*)}\right)\le 0, \tag{4.81}$$

$$u \ge u^* \Longrightarrow (u-u^*)\left(\frac{a-u}{\varphi(u)}-\frac{a-u^*}{\varphi(u^*)}\right)\le 0. \tag{4.82}$$

*Using (4.81), it is straightforward to show that $J \le 0$. Therefore*

$$\dot{V}(t) \le 0$$

*and $V$ is a Lyapunov functional.*

**Theorem 4.25** *Let $\varphi(u)$ be a decreasing function and assume that (4.78) holds. Then, for any solution $(u,v)$ of (4.4) in $\Re_\delta$ we have*

$$\lim_{t\to\infty}\|u(x,t)-u^*\|_{\mathbb{L}^2(\Omega)} = \lim_{t\to\infty}\|v(x,t)-v^*\|_{\mathbb{L}^2(\Omega)} = 0. \tag{4.83}$$

**Proof 4.26** *If $(u,v)\in\Re_\delta$ is a solution of (4.7) for which $\frac{d}{dt}V(t)=0$, where $V(t)$ is the Lyapunov functional defined in (4.76), then $u$ and $v$ must be spatially homogeneous. Therefore, $(u,v)$ satisfies the ODE system (4.54). Noting that $(u^*,v^*)$ is the largest invariant subset for the system (4.32)*

$$\left\{(u,v)\in\Re_\delta \mid \frac{d}{dt}V(t)=0\right\},$$

*we can employ the La Salle's invariance theorem [40, 54] to obtain*

$$\lim_{t\to\infty} |u(x,t) - u^*| = \lim_{t\to\infty} |v(x,t) - v^*| = 0,$$

*uniformly in $x$. Hence*

$$\lim_{t\to\infty} \int_\Omega (u(x,t) - u^*)^2 \, dx = \lim_{t\to\infty} \int_\Omega (v(x,t) - v^*)^2 \, dx = 0, \tag{4.84}$$

*which implies (4.83).*

## 4.4 Nonconstant Positive Solutions

Let us now analyze the following elliptic boundary value problem

$$\begin{cases} \Delta u + \gamma [a - u - \lambda\varphi(u) v] = 0, & x \in \Omega, \\ d\Delta v + \gamma [b - \lambda\varphi(u) v] = 0, & , \end{cases} \tag{4.85}$$

supplemented with the following Neumann boundary conditions

$$\frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = 0 \text{ for all } x \in \partial\Omega, \tag{4.86}$$

in such a way to determine a priori estimates for the nonconstant steady state solution and discuss its properties. Also to find conditions for the nonexistence of nonconstant positive solutions.

### 4.4.1 A Priori Estimates of the Nonconstant Steady State Solution

**Proposition 4.27** *(A priori estimates) Let $(u,v) = (u(x), v(x))$ be a positive solution to the elliptic boundary value problem (4.85). Assumimg*

$$\min_{u\in[\delta,a]} \varphi(u) > b, \tag{4.87}$$

*the following estimates hold for all $x \in \Omega$*

$$\begin{cases} \frac{a}{\sup\limits_{u\in[\delta,a]} \varphi_1(u)}\left(1 - \frac{b}{\min\limits_{u\in[\delta,a]} \varphi(u)}\right) < u(x) < a, \\ \frac{b}{\lambda \max\limits_{u\in[\delta,a]} \varphi(u)} < v(x) < \frac{b}{\lambda\left(\min\limits_{u\in[\delta,a]} \varphi(u) - b\right)}. \end{cases} \tag{4.88}$$

**Proof 4.28** *If the function $u$ has a maximum over $\overline{\Omega}$ at some point in space, then by applying Proposition 1.6 to the boundary value problem (4.85), we obtain*

$$a - u - \lambda\varphi(u)\, v \geq 0,$$

*then*

$$a - u > a - u - \lambda\varphi(u)\, v \geq 0,$$

*which implies the following upper bound for the solution $u$*

$$u < a. \tag{4.89}$$

*Similarly, if $v$ has a maximum over $\overline{\Omega}$ at some point, then by Proposition 1.6 it follows*

$$b - \lambda\varphi(u)\, v \geq 0.$$

*Being*

$$b - \lambda \min \varphi(u)\, v + \lambda b v > b - \lambda\varphi(u)\, v \geq 0,$$

*by condition (4.87) we get*

$$b - \lambda v \left(\min_{u\in[\delta,a]} \varphi(u) - b\right) > 0,$$

*leading to the following upper bound for the function $v$*

$$v < \frac{b}{\lambda\left(\min\limits_{u\in[\delta,a]} \varphi(u) - b\right)}. \tag{4.90}$$

*In order to find the lower bounds in (4.88), we consider the case in which $u$ has a minimum* *over $\overline{\Omega}$ at some point, then by Proposition 1.6 it follows*

$$\begin{aligned} a &\leq u+\lambda\varphi(u)v \\ &= u+\lambda u\varphi_1(u)v \\ &< u\sup_{u\in[\delta,a]}\varphi_1(u)(1+\lambda v). \end{aligned}$$

*Then, taking into account the bound (4.90), we get*

$$a < u\sup_{u\in[\delta,a]}\varphi_1(u)\left(1+\lambda\frac{b}{\lambda\left(\min_{u\in[\delta,a]}\varphi(u)-b\right)}\right),$$

*which implies*

$$a\left(\min_{u\in[\delta,a]}\varphi(u)-b\right) < u\sup_{u\in[\delta,a]}\varphi_1(u)\left(\min_{u\in[\delta,a]}\varphi(u)\right),$$

*and thus the following lower bound for $u$ is obtained*

$$u > \frac{a}{\sup_{u\in[\delta,a]}\varphi_1(u)}\left(1-\frac{b}{\min_{u\in[\delta,a]}\varphi(u)}\right). \tag{4.91}$$

*Assuming that $v$ admits a minimun at some point over $\overline{\Omega}$ leads to*

$$b-\lambda\varphi(u)v \leq 0,$$

*which implies*

$$b-\lambda\max_{u\in[\delta,a]}\varphi(u)v \leq b-\lambda\varphi(u)v \leq 0,$$

*then the lower bound for $v$ is given by*

$$\frac{b}{\lambda\max_{u\in[\delta,a]}\varphi(u)} \leq v. \tag{4.92}$$

Notice that the estimates in (4.88) guarantee that there exist two positive constants $c_1$ depending on $b$ and $\gamma$, and $c_2$ depending on $a$ and $\gamma$ such that

$$|G(u,v)| = |\gamma[b-\lambda\varphi(u)v]| \leq c_1, \tag{4.93}$$

and

$$|F(u,v)| = |\gamma [a - u - \lambda\varphi(u) v]| \leq c_2. \tag{4.94}$$

Let us now define the averages of a given pair of solutions $(u,v) = (u(x), v(x))$ to the elliptic problem (4.85) over $\Omega$ as follows

$$\overline{u} = \frac{1}{|\Omega|}\int_{\Omega} u(x)\,dx \quad \text{and} \quad \overline{v} = \frac{1}{|\Omega|}\int_{\Omega} v(x)\,dx,$$

where $|\Omega|$ is the volume of $\Omega$.

**Lemma 4.29** *The average of $u(x)$ over $\Omega$ is given by*

$$\overline{u} = a - b. \tag{4.95}$$

**Proof 4.30** *Let us define the following change of variable*

$$w(x) = dv(x) - u(x). \tag{4.96}$$

*From (4.85), we get*

$$\begin{aligned} \Delta w(x) &= d\Delta v - \Delta u \\ &= -\gamma [b - \lambda\varphi(u) v] + \gamma [a - u - \lambda\varphi(u) v] \\ \Delta w(x) &= \gamma [a - b - u]. \end{aligned} \tag{4.97}$$

*Integrating (4.97) over $\Omega$ yields*

$$\gamma \int_{\Omega} [a - b - u]\,dx = \int_{\Omega} \Delta w(x)\,dx = \int_{\Omega} \frac{\partial w}{\partial \nu} ds = 0,$$

*then,*

$$\frac{1}{|\Omega|}\int_{\Omega} u(x)\,dx = a - b,$$

*therefore*

$$\overline{u} = a - b.$$

Let us denote

$$\phi = u - \overline{u} \quad \text{and} \quad \psi = v - \overline{v}, \tag{4.98}$$

then

$$\int_{\Omega} \phi = \int_{\Omega} \psi = 0. \tag{4.99}$$

If $(u, v)$ is not a constant solution, then $\phi$ and $\psi$ must not be trivial and their signs should alternate in $\Omega$. The following Lemma shows that the product $\phi\psi$ has a positive average over $\Omega$.

**Lemma 4.31** *Let $(u, v)$ be a noncostant solution of (4.85) and $(\phi, \psi)$ defined as in (4.98). Then*

$$\int_\Omega \phi\psi > 0 \quad \text{and} \quad \int_\Omega \nabla\phi\nabla\psi > 0. \tag{4.100}$$

**Proof 4.32** *Equation (4.97) can be rewritten as*

$$\begin{aligned} \Delta w(x) &= \gamma [a - b - u] \\ \Delta w(x) &= \gamma [\overline{u} - u] \\ -\Delta w &= \gamma\phi. \end{aligned} \tag{4.101}$$

*Multiplying (4.101) by $w = dv - u$ and integrating by parts lead to*

$$\begin{aligned} \int_\Omega |\nabla w|^2 &= \gamma \int_\Omega \phi w \\ &= \gamma \int_\Omega \phi (dv - u) \\ &= \gamma d \int_\Omega \phi v - \gamma \int_\Omega \phi u, \end{aligned}$$

*so,*

$$\begin{aligned} \int_\Omega |\nabla w|^2 &= \gamma d \int_\Omega \phi v - \gamma d \int_\Omega \phi\overline{v} + \gamma d \int_\Omega \phi\overline{v} - \gamma \int_\Omega \phi u + \gamma \int_\Omega \phi\overline{u} - \gamma \int_\Omega \phi\overline{u} \\ &= \gamma d \int_\Omega \phi (v - \overline{v}) + \gamma d \int_\Omega \phi\overline{v} - \gamma \int_\Omega \phi (u - \overline{u}) - \gamma \int_\Omega \phi\overline{u} \\ &= \gamma d \int_\Omega \phi\psi + \gamma d \int_\Omega \phi\overline{v} - \gamma \int_\Omega \phi^2 - \gamma \int_\Omega \phi\overline{u}, \end{aligned}$$

*by (4.99), we obtain*

$$\int_\Omega \phi\overline{u} = 0 \quad \text{and} \quad \int_\Omega \phi\overline{v} = 0,$$

*which implies*

$$\int_\Omega |\nabla w|^2 = \gamma d \int_\Omega \phi\psi - \gamma \int_\Omega \phi^2.$$

*Therefore*

$$\int_\Omega \phi\psi = \frac{1}{\gamma d} \int_\Omega |\nabla w|^2 + \frac{1}{d} \int_\Omega \phi^2 > 0 \tag{4.102}$$

*and the first inequality in (4.100) is proved. Multiplying (4.101) by $\phi$ and integrating by* *parts yields*

$$\begin{aligned}\gamma\int_{\Omega}\phi^{2} &= \int_{\Omega}\nabla\phi\nabla w\\ &= \int_{\Omega}\nabla\phi\nabla\left(dv-u\right)\\ &= d\int_{\Omega}\nabla\phi\nabla\psi-\int_{\Omega}\nabla\phi^{2},\end{aligned}$$

*which implies the second inequality in (4.100)*

$$\int_{\Omega}\nabla\phi\nabla\psi=\frac{\gamma}{d}\int_{\Omega}\phi^{2}+\frac{1}{d}\int_{\Omega}\nabla\phi^{2}>0. \tag{4.103}$$

**Lemma 4.33** *There exists a constant $C_G$ depending on $b$, $\gamma$ and $\Omega$ such that*

$$\int_{\Omega}\psi^{2}+\int_{\Omega}\left|\nabla\psi\right|^{2}\leq C_{G}d^{-2}. \tag{4.104}$$

**Proof 4.34** *From (4.85),*

$$\begin{aligned}-d\Delta v &= G\left(u,v\right)\\ -d\Delta v+d\Delta\overline{v}-d\Delta\overline{v} &= G\left(u,v\right)\\ -d\Delta\psi-d\Delta\overline{v} &= G\left(u,v\right).\end{aligned} \tag{4.105}$$

*Miltiplying (4.105) by $\psi$ and integrating by parts*

$$d\int_{\Omega}\left|\nabla\psi\right|^{2}=\int_{\Omega}G\left(u,v\right)\psi,$$

*using the Cauchy–Schwarz inequality and condition (4.93), we obtain*

$$d\int_{\Omega}\left|\nabla\psi\right|^{2}=\int_{\Omega}G\left(u,v\right)\psi\leq c_{1}\sqrt{\left|\Omega\right|}\left(\int_{\Omega}\left|\psi\right|^{2}\right)^{1/2}. \tag{4.106}$$

*The Poincaré inequality yields*

$$\int_{\Omega}\psi^{2}\leq\frac{1}{\lambda_{1}}\int_{\Omega}\left|\nabla\psi\right|^{2}, \tag{4.107}$$

*where $\lambda_1 > 0$ is the first positive eigenvalue of $(-\Delta)$. Therefore, under the Neumann* *boundary conditions, from (4.106) it follows*

$$d\int_{\Omega}|\nabla\psi|^2 \le c_1\sqrt{\frac{|\Omega|}{\lambda_1}}\left(\int_{\Omega}|\nabla\psi|^2\right)^{1/2},$$

$$\left(d\int_{\Omega}|\nabla\psi|^2\right)^2 \le \left(c_1\sqrt{\frac{|\Omega|}{\lambda_1}}\left(\int_{\Omega}|\nabla\psi|^2\right)^{1/2}\right)^2,$$

*and consequently*

$$\int_{\Omega}|\nabla\psi|^2 \le \frac{|\Omega|\,c_1^2}{\lambda_1 d^2}. \tag{4.108}$$

*Adding up (4.107) and (4.108) and using once again the inequality in (4.108) leads to*

$$\int_{\Omega}\psi^2 + \int_{\Omega}|\nabla\psi|^2 \le C_G d^{-2},$$

*where*

$$C_G = c_1^2\,|\Omega|\left(\frac{1+\lambda_1}{\lambda_1^2}\right).$$

**Lemma 4.35** *There exists a constant $C_F$ depending on $a, \gamma$ and $\Omega$ such that*

$$\int_{\Omega}\phi^2 + \int_{\Omega}|\nabla\phi|^2 \le C_F. \tag{4.109}$$

**Proof 4.36** *The proof follows the same lines of the previous Lemma. By (4.85)*

$$\begin{aligned} -\Delta u &= F(u,v) \\ -\Delta u + \Delta\overline{u} - \Delta\overline{u} &= F(u,v) \\ -\Delta\phi - \Delta\overline{u} &= F(u,v). \end{aligned} \tag{4.110}$$

*Miltiplying (4.110) by $\phi$ and integrating by parts*

$$\int_{\Omega}|\nabla\phi|^2 = \int_{\Omega}F(u,v)\,\phi.$$

*Applying the Cauchy–Schwarz inequality to (4.85) and using (4.94) yields*

$$\int_{\Omega}|\nabla\phi|^2 = \int_{\Omega}F(u,v)\,\phi \le c_2\sqrt{|\Omega|}\left(\int_{\Omega}|\phi|^2\right)^{1/2}. \tag{4.111}$$

*The Poincaré inequality assures that*

$$\int_\Omega \phi^2 \le \frac{1}{\lambda_1} \int_\Omega |\nabla\phi|^2, \tag{4.112}$$

*where* $\lambda_1 > 0$ *is the first positive eigenvalue of* $(-\Delta)$. *Hence from (4.111) it follows*

$$\int_\Omega |\nabla\phi|^2 \le c_2 \sqrt{\frac{|\Omega|}{\lambda_1}} \left( \int_\Omega |\nabla\phi|^2 \right)^{1/2}$$

$$\left( \int_\Omega |\nabla\phi|^2 \right)^2 \le \left( c_2 \sqrt{\frac{|\Omega|}{\lambda_1}} \left( \int_\Omega |\nabla\phi|^2 \right)^{1/2} \right)^2,$$

*implying that*

$$\int_\Omega |\nabla\phi|^2 \le \frac{|\Omega|\, c_2^2}{\lambda_1}. \tag{4.113}$$

*Adding up (4.112) and (4.113) and using once again (4.113) leads to*

$$\int_\Omega \phi^2 + \int_\Omega |\nabla\phi|^2 \le C_F,$$

*where*

$$C_F = c_2^2\, |\Omega| \left( \frac{1+\lambda_1}{\lambda_1^2} \right).$$

**Lemma 4.37** *Let* $(u, v)$ *be a nonconstant solution of the problem (4.85). Then, the following inequalities hold*

$$\frac{\lambda_1^2}{\gamma^2 + 2\lambda_1 (\lambda_1 + \gamma)} \le \frac{\int_\Omega |\nabla\phi|^2}{d^2 \int_\Omega |\nabla\psi|^2} \le 1, \tag{4.114}$$

$$\frac{\lambda_1^3}{(\lambda_1 + 1)(2\lambda_1 (\lambda_1 + \gamma) + \gamma^2)} < \frac{\int_\Omega \left( |\nabla\phi|^2 + 2\gamma\phi^2 \right)}{d^2 \int_\Omega \left( |\nabla\psi|^2 + \psi^2 \right)} < 1, \tag{4.115}$$

*where* $\phi$ *and* $\psi$ *are defined in (4.98) and* $\lambda_1$ *is the first positive eigenvalue of* $-\Delta$.

**Proof 4.38** *Let* $w = dv - u$. *Using the definitions in (4.98), we get*

$$\begin{aligned} \int_\Omega |\nabla w|^2 &= \int_\Omega |\nabla (dv - u)|^2 \\ &= d^2 \int_\Omega |\nabla\psi|^2 + \int_\Omega |\nabla\phi|^2 - 2d \int_\Omega \nabla\phi\nabla\psi. \end{aligned}$$

*Using (4.103) leads to*

$$\int_\Omega |\nabla w|^2 = d^2 \int_\Omega |\nabla \psi|^2 - \int_\Omega |\nabla \phi|^2 - 2\gamma \int_\Omega \phi^2, \tag{4.116}$$

*which implies*

$$d^2 \int_\Omega |\nabla \psi|^2 \geq \int_\Omega |\nabla \phi|^2 + 2\gamma \int_\Omega \phi^2 \geq \int_\Omega |\nabla \phi|^2 . \tag{4.117}$$

*Therefore, the second inequality in (4.114) is obtained, i.e.*

$$\frac{\int_\Omega |\nabla \phi|^2}{d^2 \int_\Omega |\nabla \psi|^2} \leq 1. \tag{4.118}$$

*Next, we use (4.102) and (4.115), we obtain*

$$\begin{aligned} \int_\Omega \phi\psi &= \frac{1}{d}\int_\Omega \phi^2 + \frac{1}{\gamma d}\left[d^2 \int_\Omega |\nabla \psi|^2 - \int_\Omega |\nabla \phi|^2 - 2\gamma \int_\Omega \phi^2\right] \\ &= \frac{1}{d}\left[\int_\Omega \phi^2 + \frac{1}{\gamma}\left[d^2 \int_\Omega |\nabla \psi|^2 - \int_\Omega |\nabla \phi|^2 - 2\gamma \int_\Omega \phi^2\right]\right], \end{aligned}$$

*so*

$$d\int_\Omega \phi\psi = \frac{d^2}{\gamma}\int_\Omega |\nabla \psi|^2 - \frac{1}{\gamma}\int_\Omega |\nabla \phi|^2 - \int_\Omega \phi^2,$$

*therefore, we compute*

$$\frac{d^2}{\gamma}\int_\Omega |\nabla \psi|^2 = \frac{1}{\gamma}\int_\Omega |\nabla \phi|^2 + \int_\Omega \phi^2 + d\int_\Omega \phi\psi.$$

*Using the* $\epsilon-$*Young inequality* $ab \leq \frac{1}{4\epsilon}a^2 + \epsilon b^2$ *leads to*

$$\frac{d^2}{\gamma}\int_\Omega |\nabla \psi|^2 \leq \frac{1}{\gamma}\int_\Omega |\nabla \phi|^2 + \int_\Omega \phi^2 + \frac{1}{4\epsilon}\int_\Omega \phi^2 + \epsilon d^2 \int_\Omega \psi^2.$$

*Then, the Poincaré inequality gives*

$$\frac{d^2}{\gamma}\int_\Omega |\nabla \psi|^2 \leq \frac{1}{\gamma}\int_\Omega |\nabla \phi|^2 + \frac{1}{\lambda_1}\int_\Omega |\nabla \phi|^2 + \left(\frac{1}{4\lambda_1\epsilon}\right)\int_\Omega |\nabla \phi|^2 + \frac{\epsilon d^2}{\lambda_1}\int_\Omega |\nabla \psi|^2,$$

*and the last conditions can be simplified to the following form*

$$\left(\frac{1}{\gamma} - \frac{\epsilon}{\lambda_1}\right) d^2 \int_\Omega |\nabla \psi|^2 \leq \left(\frac{1}{\gamma} + \frac{1}{\lambda_1} + \frac{1}{4\lambda_1\epsilon}\right)\int_\Omega |\nabla \phi|^2 .$$

*Setting* $\epsilon = \frac{1}{2\gamma}\lambda_1$ *leads to*

$$\begin{aligned}
\frac{1}{2\gamma} d^2 \int_\Omega |\nabla\psi|^2 &\leq \left(\frac{1}{\gamma} + \frac{1}{\lambda_1} + \frac{\gamma}{2\lambda_1^2}\right) \int_\Omega |\nabla\phi|^2, \\
d^2 \int_\Omega |\nabla\psi|^2 &\leq 2\gamma \left(\frac{2\lambda_1^2 + 2\lambda_1\gamma + \gamma^2}{2\lambda_1^2\gamma}\right) \int_\Omega |\nabla\phi|^2,
\end{aligned}$$

*which gives the first inequality in* (4.114)

$$\frac{\lambda_1^2}{2\lambda_1(\lambda_1+\gamma)+\gamma^2} \leq \frac{\int_\Omega |\nabla\phi|^2}{d^2 \int_\Omega |\nabla\psi|^2}.$$

*Let us now prove the inequalities in* (4.115)*. The Poincaré inequality leads to*

$$\int_\Omega \left(|\nabla\psi|^2 + \psi^2\right) \leq \left(\frac{(\lambda_1+1)}{\lambda_1}\right) \int_\Omega |\nabla\psi|^2.$$

*Therefore, we compute*

$$\frac{\int_\Omega \left(|\nabla\phi|^2 + 2\gamma\phi^2\right)}{d^2 \int_\Omega \left(|\nabla\psi|^2 + \psi^2\right)} \geq \left(\frac{\lambda_1}{\lambda_1+1}\right) \frac{\int_\Omega \left(|\nabla\phi|^2 + 2\gamma\phi^2\right)}{d^2 \int_\Omega \left(|\nabla\psi|^2\right)} > \left(\frac{\lambda_1}{\lambda_1+1}\right) \frac{\int_\Omega |\nabla\phi|^2}{d^2 \int_\Omega |\nabla\psi|^2},$$

*and the left hand side of inequality* (4.115) *follows from* (4.114)*. Moreover, we have*

$$\frac{\int_\Omega |\nabla\phi|^2 + 2\gamma \int_\Omega \phi^2}{d^2 \int_\Omega \left(|\nabla\psi|^2 + \psi^2\right)} < \frac{\int_\Omega |\nabla\phi|^2 + 2\gamma \int_\Omega \phi^2}{d^2 \int_\Omega |\nabla\psi|^2},$$

*and using* (4.117) *we obtain the right hand side of the inequality in* (4.115)*.*

### 4.4.2 Nonexistence of Nonconstant Positive Solutions

Now, we shall concern the nonexistence of nonconstant positive solutions of (4.85).

Our results show that the size of the reactor (reflected by its first eigenvalue $\lambda_1$), and the diffusion coefficient $d$ play a critical role in obtaining the nonexistence of nonconstant positive solutions. In particular, in Theorem 4.39 the nonexistence of non-constant positive solutions will be proved when the diffusion coefficient is below a threshold proportional to the size of the reactor; in Theorem 4.41 the nonexistence of nonconstant positive solutions will be achieved when the size of the reactor is large enough.

**Theorem 4.39** *If the diffusion coefficient $d$ satisfies the following condition subject to*

$$0 < d < d_0, \qquad \text{where } d_0 = \frac{4\lambda_1 C_2(a,b,\gamma,\lambda)}{C_1^2(a,b,\gamma,\lambda)}.$$

*Then the problem (4.85)-(4.86) does not admit a nonconstant solution*

**Proof 4.40** *Multiplying $\psi$ in the second equation of (4.85) and integrating by parts yields*

$$d\int_\Omega |\nabla\psi|^2 = \gamma b\int_\Omega \psi - \gamma\lambda\int_\Omega \varphi(u)\, v\psi$$

*by using (4.99) and as $\varphi(u) = u\varphi_1(u)$,*

$$\begin{aligned} d\int_\Omega |\nabla\psi|^2 &= -\gamma\lambda\int_\Omega \varphi(u)\, v\psi \\ &= -\gamma\lambda\int_\Omega \left[(u\varphi_1(u)\, v - \overline{u}\varphi_1(u)\, v) + (\overline{u}\varphi_1(u)\, v - \overline{u}\varphi_1(u)\,\overline{v})\right]\psi \\ &\quad -\gamma\lambda\int_\Omega \left[(\overline{u}\varphi_1(u)\,\overline{v} - u\varphi_1(u)\,\overline{v}) + u\varphi_1(u)\,\overline{v}\right]\psi \\ &\leq -\gamma\lambda\int_\Omega \overline{u}\varphi_1(u)(v-\overline{v})\,\psi + \gamma\lambda\int_\Omega (\overline{u}-u)\,\varphi_1(u)\,\overline{v}\psi \\ &= \gamma\lambda\int_\Omega \varphi_1(u)\,\overline{v}\phi\psi - \gamma\lambda\int_\Omega \overline{u}\varphi_1(u)\,\psi^2. \end{aligned}$$

*From the a priori estimates in Proposition 4.27 it follows that*

$$\begin{aligned} d\int_\Omega |\nabla\psi|^2 &\leq \gamma\lambda C_1(a,b)\int_\Omega \phi\psi - \gamma\lambda C_2(a,b)\int_\Omega \psi^2 \\ &\leq C_1(a,b,\gamma,\lambda)\int_\Omega \phi\psi - C_2(a,b,\gamma,\lambda)\int_\Omega \psi^2. \end{aligned} \tag{4.119}$$

*By the Cauchy-Schwarz inequality and Using the $\epsilon$−Young inequality*

$$\begin{aligned} C_1\int_\Omega \phi\psi &\leq C_1\left(\int_\Omega |\phi|^2\right)^{1/2}\left(\int_\Omega |\psi|^2\right)^{1/2} \\ &\leq \frac{C_1^2}{4\epsilon}\int_\Omega |\phi|^2 + \epsilon\int_\Omega |\psi|^2, \end{aligned}$$

*substiting in (4.119) and puting $C_2 = \epsilon$, we get*

$$d\int_\Omega |\nabla\psi|^2 \leq \frac{C_1^2}{4C_2}\int_\Omega |\phi|^2.$$

*By the Poincaré inequality, we have*

$$d\int_{\Omega}|\nabla\psi|^2 \leq \frac{C_1^2}{4\lambda_1 C_2}\int_{\Omega}|\nabla\phi|^2 . \tag{4.120}$$

*It follows from (4.114) and (4.120) that*

$$\int_{\Omega}|\nabla\psi|^2 \leq \frac{d}{d_0}\int_{\Omega}|\nabla\psi|^2 , \tag{4.121}$$

*where* $d_0 = d_0(a,b,\gamma,\lambda,\lambda_1) = \frac{4\lambda_1 C_2(a,b,\gamma,\lambda)}{C_1^2(a,b,\gamma,\lambda)}$. *Therefore, if* $d < d_0$, *by (4.121), then*

$$\int_{\Omega}|\nabla\psi|^2 = 0,$$

*and so*

$$\int_{\Omega}|\nabla\phi|^2 = 0,$$

*by (4.114). Hence,* $|\nabla\phi| = |\nabla\psi| \equiv 0$ *over* $\Omega$, *which verifies the assertion.*

**Theorem 4.41** *There is a positive constant* $\Lambda \equiv \Lambda(a,b,\gamma,\lambda)$ *such that the problem (4.85) does not admit nonconstant positive solutions when* $\lambda_1(\Omega) > \Lambda$.

**Proof 4.42** *Multiplying equation (4.85) by* $\phi$ *and integrating by parts we have*

$$\int_{\Omega}|\nabla\phi|^2 = \gamma a\int_{\Omega}\phi - \gamma\int_{\Omega}\phi^2 - \gamma\lambda\int_{\Omega}\varphi(u)\,v\phi.$$

*Using (4.99) and as* $\varphi(u) = u\varphi_1(u)$,

$$\begin{aligned}\int_{\Omega}|\nabla\phi|^2 &= -\gamma\int_{\Omega}\phi^2 - \gamma\lambda\int_{\Omega}\varphi(u)\,v\phi \\ &= -\gamma\int_{\Omega}\phi^2 - \gamma\lambda\int_{\Omega}\left[(u\varphi_1(u)\,v - \overline{u}\varphi_1(u)\,v) + (\overline{u}\varphi_1(u)\,v - \overline{u}\varphi_1(u)\,\overline{v})\right]\phi \\ &\quad -\gamma\lambda\int_{\Omega}\left[(\overline{u}\varphi_1(u)\,\overline{v} - u\varphi_1(u)\,\overline{v}) + u\varphi_1(u)\,\overline{v}\right]\phi \\ &\leq -\gamma\int_{\Omega}\phi^2(1+\lambda\varphi_1(u)\,v) - \gamma\lambda\int_{\Omega}\overline{u}\varphi_1(u)\,\psi\phi + \gamma\lambda\int_{\Omega}\varphi_1(u)\,\overline{v}\phi^2,\end{aligned}$$

*it follows that*

$$\int_{\Omega}|\nabla\phi|^2 \leq \gamma\lambda\int_{\Omega}\varphi_1(u)\,\overline{v}\phi^2 - \gamma\lambda\int_{\Omega}\overline{u}\varphi_1(u)\,\phi\psi.$$

*Applying the a priori estimates in Proposition 4.27, we obtain the estimate*

$$\int_{\Omega} |\nabla\phi|^2 \leq C_3 \int_{\Omega} \phi^2 + C_3 \int_{\Omega} |\phi\psi|\,, \tag{4.122}$$

*where $C_3$ stands for a generic constant depending on $(a, b, \gamma, \lambda)$ in this proof. By the Caushy-Schwarz inequality and the Poincaré inequality, we have*

$$\int_{\Omega} \phi\psi \leq \left(\int_{\Omega} |\phi|^2\right)^{1/2} \left(\int_{\Omega} |\psi|^2\right)^{1/2} \leq \frac{1}{\lambda_1} \left(\int_{\Omega} |\nabla\phi|^2\right)^{1/2} \left(\int_{\Omega} |\nabla\psi|^2\right)^{1/2}.$$

*Thus, by (4.120), we have*

$$\int_{\Omega} \phi\psi \leq C_4 \lambda^{-3/2} d^{-1/2} \int_{\Omega} |\nabla\phi|^2\,, \quad \textit{where} \quad C_4 = \frac{C_1}{2\sqrt{C_2}}.$$

*Combining this with (4.122), we see that*

$$\int_{\Omega} |\nabla\phi|^2 \leq \frac{C}{\lambda_1} \left(1 + \frac{1}{(\lambda_1 d)^{1/2}}\right) \int_{\Omega} |\nabla\phi|^2\,, \tag{4.123}$$

*where $C(a, b, \gamma, \lambda) = \max\{C_3(a, b, \gamma, \lambda), C_3(a, b, \gamma, \lambda) \times C_4(a, b, \gamma, \lambda)\}$. Now, if $d$ is not small $(d \geq 1)$, then we can choose so large that for $\lambda_1 > \Lambda$,*

$$\frac{C}{\lambda_1} \left(1 + \frac{1}{(\lambda_1 d)^{1/2}}\right) \leq \frac{C}{\lambda_1} \left(1 + \frac{1}{(\lambda_1)^{1/2}}\right) < 1.$$

*By (4.123), we obtain $\int_{\Omega} |\nabla\phi|^2 = 0$, which means that $u$ and $v$ have to be constants. On the other hand, if $d < 1$, by the expression of $d_0$, then we can also choose $\lambda_1$ so large that $d_0 > 1$, which gives the nonexistence again by previous theorem.*

## 4.5 NUMERICAL EXAMPLES

In this Section, we aim to validate the analytical findings regarding the asymptotic stability of the equilibrium (4.13). We choose the following form of the arbitrary function $\varphi(u)$

$$\varphi(u) = \frac{u^p}{k + u^q} =: \varphi_k(u), \tag{4.124}$$

with $p, q \geq 0$ and $k \geq 0$. We also assume that $\lambda = \gamma = 1$, $p = \frac{1}{2}$, and $q = 1$. Substituting these parameters into (4.7), we get

$$\begin{cases} u_t - \Delta u = a - u - \frac{\sqrt{u}}{k+u} v, \\ v_t - d\Delta v = b - \frac{\sqrt{u}}{k+u} v, \end{cases} \tag{4.125}$$

which admits the following unique equilibrium

$$(u^*, v^*) = \left( a - b, \frac{b\,(k + a - b)}{\sqrt{a - b}} \right). \tag{4.126}$$

The invariant region for the system (4.125) is

$$\Re_\delta = [\delta, a] \times \left[ \frac{b\,(k + \delta)}{\sqrt{\delta}}, \frac{(a - \delta)\,(k + \delta)}{\sqrt{\delta}} \right].$$

Letting $b = \frac{a}{2} = \frac{1}{8}$, since condition (4.10) must hold, then it should be

$$\delta < \frac{a}{2}. \tag{4.127}$$

We choose the value $\delta = \frac{1}{10}$, which clearly satisfies the condition (4.127). With the above choices for the system parameters, the steady state is given by $(u^*, v^*) = \left(\frac{1}{8}, 2\sqrt{2}\left(\frac{1}{8}k + \frac{1}{64}\right)\right)$. Since the chosen function $\varphi(u)$ is decreasing over $[\delta, a]$, then (4.52) is satisfied .

The equilibrium solution (4.126) is asymptotically stable for the ODEs system (4.32) if the condition (4.33) holds. Substituting the chosen parameters into (4.33), we have

$$-\,(24k + 1) < 4\sqrt{2},$$

which is always satisfied regardless of $k$. Therefore, for the chosen parameter set, we should achieve asymptotic stability of the ODEs system for any $k > 0$. We perform two different numerical tests. At the top of Figure 4.1, for $k = 0.05$ and initial conditions $(u_0, v_0) = (0.2, 0.06)$, it is shown that the solutions converge towards the equilibrium $(u^*, v^*) = \left(\frac{1}{8}, \frac{7}{160}\sqrt{2}\right)$ . Analogously, at the bottom of Figure 4.1, for $k = 0.1$ and initial conditions $(u_0, v_0) = (0.2, 0.09)$, we can see that the solution of the ODes system asymptotically converges towards the steady state $(u^*, v^*) = \left(\frac{1}{8}, \frac{9}{160}\sqrt{2}\right)$. The same solutions are plotted in Figure 4.2 in the $u - v$ phase plane to better show the asymptotic evolution towards the steady state.Let us, now, consider the one–dimensional reaction-diffusion system (4.7) using the same parameters as above. The initial conditions are chosen as the following

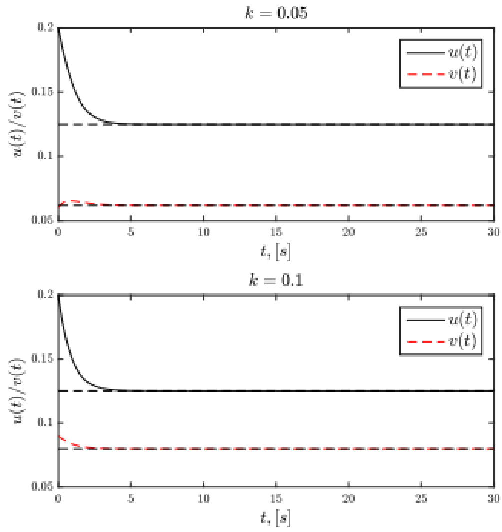


Figure 4.1: Time evolution of the solutions of (4.125) in the ODE case with $(a, b, d, k) = (\frac{1}{4}, \frac{1}{8}, 5, 0.05)$and $(u_0, v_0) = (0.2, 0.06)$ (top) and $(a, b, d, k) = (\frac{1}{4}, \frac{1}{8}, 5, 0.1)$ and $(u_0, v_0) = (0.2, 0.09)$ (bottom).

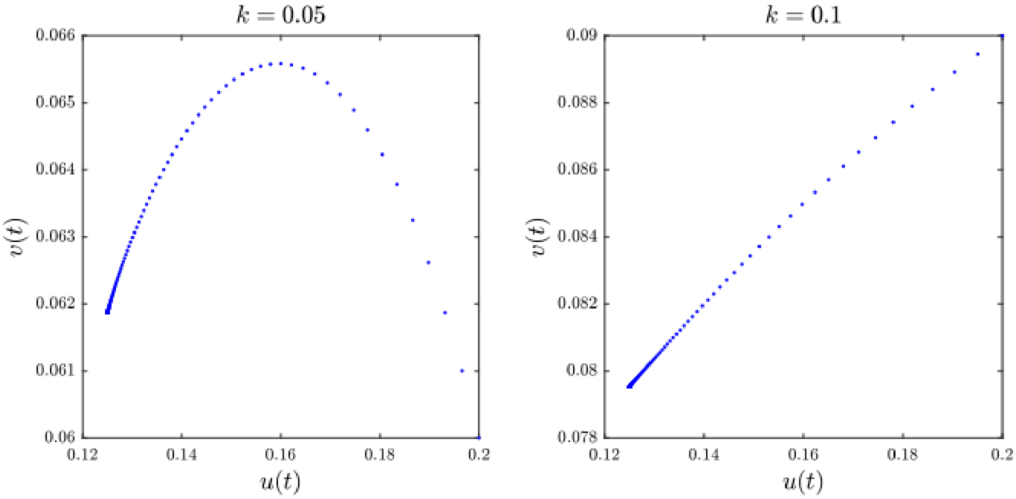


Figure 4.2: The solutions of (4.125) in the ODE case plotted in the $u - v$ phase plane with $(a, b, d, k) = \left(\frac{1}{4}, \frac{1}{8}, 5, 0.05\right)$ and $(u_0, v_0) = (0.2, 0.06)$ (left) and $(a, b, d, k) = \left(\frac{1}{4}, \frac{1}{8}, 5, 0.1\right)$ and $(u_0, v_0) = (0.2, 0.09)$ (right).

sinusoidal disturbance

$$\begin{cases} u(x,0) = u_0 \times (1 + \sin(50x)), \\ v(x,0) = v_0 + (1 + \cos(50x)). \end{cases} \tag{4.128}$$

Being $\varphi(u)$ a decreasing function, in order to achieve the global asymptotic stability of the solutions the condition (4.78) must hold. If the following function

$$f(u) = \frac{a-u}{\varphi_k(u)} = \frac{a-u}{\frac{\sqrt{u}}{k+u}}$$

is also decreasing, then (4.78) holds. It is easy to check that if

$$k \le \min\left\{\delta = \frac{1}{10}, 7 - \sqrt{48}a = 7 - \sqrt{3}\right\}, \tag{4.129}$$

then the function $f(u)$ is descreasing. We again perform two numerical tests choosing respectively $k = 0.05$ and $k = 0.1$, as both these values satisfies (4.129). The numerical simulations of the one dimensional reaction-diffusion system are respectively given in Figures 4.3 and 4.4 showing that the solutions converge towards the spatially homogeneous steady state.

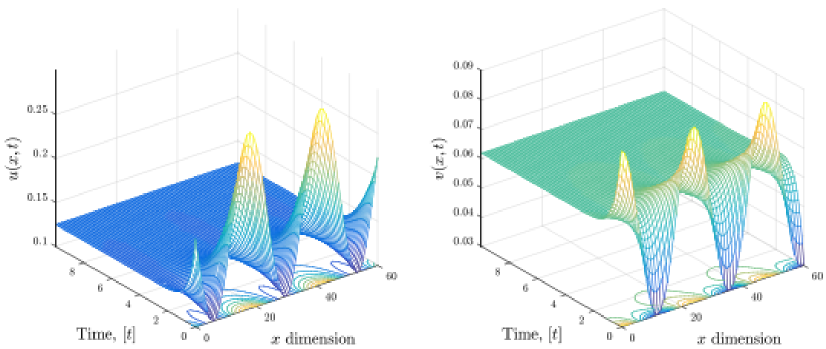


Figure 4.3: The solutions of (4.125) in a one-dimensional spatial domain. Here $k = 0.05$.

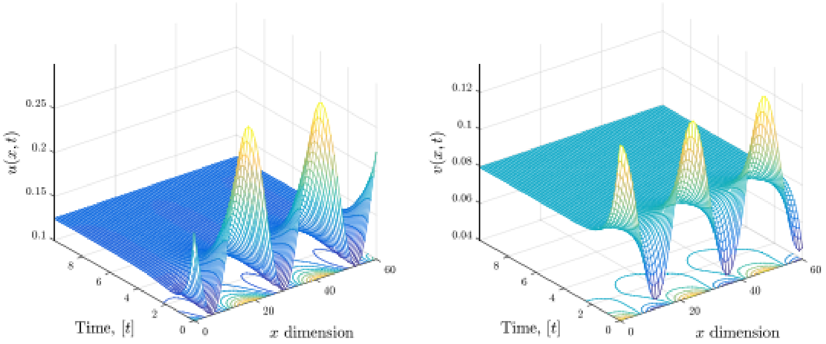


Figure 4.4: The solutions of (4.125) in a one-dimensional spatial domain Here $k = 0.1$.

# General Conclusion

In this thesis, we have studied the dynamics of the Turing-type Degn-Harrison reaction-diffusion model. The study is concerned with the derivation of local and global asymptotic stability conditions for the proposed dimensionless system. The derived conditions have been shown to be weaker than those reported in previous publications. The theoretical results derived herein have been validated by means of Matlab simulations carried out using the finite difference numerical scheme.

A reaction-diffusion system with a generalized reaction term based on that of the Degn-Harrison model has also been considered. Once the global existence and boundedness of the unique solution was established for the generalized model, the study addressed the system's asymptotic behavior. We derived conditions for the global asymptotic stability of the steady state solution by means of theoretical tools related to eign-analysis, the Poincare-Bendixon theorem and the direct Lyapunov method. Numerical simulation results were presented to corroborate the theoretical asymptotic stability predictions. In terms of the chemical reaction behind the model, the study has established theoretical constraints on the size of the reactor and the diffusion coefficient required to ensure that the system does not admit non-constant positive steady state solutions.

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