People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research
Larbi Tebessi University - Tebessa

Faculty of Exact Sciences, Natural, and Life Sciences

Department: Mathematics and Computer Science

Doctoral Thesis

Option: Applied Mathematics

Theme

EXISTENCE AND NON-EXISTENCE RESULTS OF SOLUTIONS OF CERTAIN CLASSES OF PARTIAL DIFFERENTIAL EQUATIONS AND SYSTEMS

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Academic year 2021-2022

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Acknowledgement

First of all, I want to thank Almighty God for giving me the courage and patience to be able to complete this work.

I would like to express my gratitude to my supervisors. Professor **AKROUT Kamel** and Professor **Guefaifia Rafik** from the university of Larbi Tébessi- Tebessa, for their framing, their valuable advice, their encouragement and their human qualities as well as their constant support throughout the development of this project. I express my deep gratitude. It was a pleasure to work with them.

I thank Mr. **ABDELMALEK Salem,** Professor from the university of Larbi Tébessi-Tebessa for having accepted to chair the jury.

My warmest thanks also go to Mr. **KHELLAF Hacene** Professor at the University of Constantine 2 , Mr **RAMOUL Hichem** Doctor from the University of Abbas Leghrour- Khenchela and Mr. **ELLAGGOUNE Fateh** Professor from 8 Mai 1945 university- Guelma, to have arranged their time to be able to examine my thesis and be part of the jury. Please accept the expression of my gratitude.

Finally, I would like to thank anyone who has contributed closely or far to the completion of this thesis.

Thank you all ...

Dedication

To everyone who is overwhelmed with joy for my joy. To my dear parents,to all my family.

To the companions of my first step and to the companions of this step.

Abstract

This thesis contain a results of the existence of positive solutions of certain nonlinear elliptic and parabolic systems, involving the (p, q)-Laplace and the $p(x)$-Laplace operators. The method used to obtain the results is that of sub and supersolution, which is based on the maximum principle and the comparison theorem.

Keywords: Sub and supersolutions; semipositon elliptic systems; $p(x)$-Kirchoff parabolic systems.

Résumé

Cette thèse comporte des résultats d'existence de solutions positives de certains systèmes elliptiques et paraboliques non linéaires, intervenant les opérateurs (p, q)-Laplace et le $p(x)$-Laplace. La méthode utilisé pour obtenir les résultats est celle de sous et sur-solution, qui est basé sur le principe de maximum et le théorème de comparaison.

Mots clés: Sous et sursolutions; systèms elliptiques semipositones; Systèmes paraboliques $p(x)$-Kirchoff.

الملخص

هذه الرسالة تحتوي على نتائج خاصة بوجود حلول موجبة لبعض الجمل الإهليلجية والقطعية المكافئة الغير خطية، والتي تتضمن المؤثرات(p,q)-لابلاص و $p(x)$-لابلاص. الطريقة المنتهجة للحصول على هذه النتائج هي طريقة الحلول الفوقية والتحتية والتي تعتمد على مبدأ الحد الأعضمي ونظرية المقارنة.

الكلمات المفتاحية: الحلول الفوقية والتحتية، الجمل الإهليجية الشبه موجبة و الرتيبة، الجمل القطعية المكافئة لكيرشوف.

Notations

Ω a bounded smooth domain in $\mathbb{R}^N$.

$\partial\Omega$ boundary of Ω.

∇u gradient u, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}\right)^t$.

$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}\right)$.

$\Delta_{p(x)} u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(|\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_i}\right)$.

$\mathcal{C}^\infty(\Omega)$ the space of indefinitely derivable functions on Ω.

$\mathcal{D}(\Omega)$ the space of functions of class $\mathcal{C}^\infty(\Omega)$, with compact support included in Ω.

$L^p(\Omega)$ p-Lebesgue integrable functions on Ω, $p \in [1, +\infty[$.

$L^\infty(\Omega)$ essentially bounded functions on Ω.

$W^{k,p}(\Omega)$ sobolev space; L^p-integrable functions, with weak derivatives up to order k in $L^p(\Omega)$.

$W_0^{1,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$.

$L^{p(x)}(\Omega)$ generalized Lebesgue space.

$W^{1,p(x)}(\Omega)$ generalized Sobolev space.

$u_n \rightharpoonup u$ the weak convergence of sequence $\{u_n\}_n$ to u.

$u_n \to u$ and strong convergence of sequence $\{u_n\}_n$ to u.

Introduction

The theory of partial differential equations has developed considerably in recent years. Notably the nonlinear elliptic problems with quasilinear homogeneous operators type such as the p-Laplace, these nonlinear elliptical problems are in general not integrable, which means that one cannot practically find explicit solutions, this gives a great importance to search for a weak solutions basing on the theory of Sobolev spaces, these solutions can be as critical points or fixed points of a functional or an appropriate operator, or via sub and supersolution concept,. . . etc.

Hence, in the case of nonhomogeneous $p(x)$-Laplace operators, it is necessary to introduce the appropriate spaces, in which we can study the problem with a variable exponent, such as the space $L^{p(x)}$ called a variable exponent Lebesgue spaces, which were appeared in the literature, for the first time in 1931 by W. Orlicz, then in $1950 - 1951$ Nakonov developed the theory of modular spaces by generalizing Orlicz spaces by giving an example, the generalized Lebesgue space with a variable exponent $L^{p(x)}(\Omega)$.

Afterwards, in the 70s and 80s, with more explicit version, the $L^{p(x)}$ spaces took over again from the Polish school (Hudzick and Museielak). The results and studies were followed during the 80s and 90s, when a major stage of these investigations was in 1991 by Kovácik and Rákosnik [25], which had provided the standard base reference of $L^{p(x)}\left(\mathbb{R}^N\right)$ spaces which are also called generalized Lebesgue spaces, their work had covered only basic properties such as reflexivity, separability, duality and the first results concerning inclusions and density of regular functions. These spaces are a generalization of the corresponding standard spaces, for which the $p(.)$ is a constant. And which are the functional framework in the resolution of the nonlinear partial differential equations involving the operator $p(x)$-Laplacian, who paved the way to more applications of these theories in partial differential equations.

This type of operators appeared after the development of numerous physical phenomena concerned with the characteristics of materials which are not homogeneous, like nonlinear elasticity, the fluids electrorheological (the interaction between fluids and electromagnetic fields) and termorheological, image processing, propagation through porous media and calculation of variations.

The objective of this thesis is the study of certain elliptic and parabolic problems, involving the p-Laplace and the $p(x)$-Laplace operators.

Our approach is based on the method of sub and super-solutions. The concepts of the latter were introduced by Nagumo [29] in 1937 who proved using also the shooting method, the existence of at least one solution for a class of nonlinear Sturm-Liouville problems. In fact, the premises of the sub and supersolution method can be traced back to Picard. He applied, in the early 1880s, the method of successive approximations to argue the existence of solutions for nonlinear elliptic equations that are suitable perturbations of uniquely solvable linear problems. This is the starting point of the use of sub and supersolutions in connection with monotone methods. Picards techniques were applied later by Poincaré [30] in connection with problems arising in astrophysics.

We draw the reader's attention to the references ([18, 26, 31, 35]) which are applied the method of sub and supersolution for nonlinear infinite semipositone elliptic problems. The elliptic problems considered in this study are an infinite semipositon problems, the positone problem expression means that the nonlinearity $F(u)$ is positive and monotone function. The semipositone problem expression means that the nonlinearity $F(u)$ is monotone and $F(0) = k < 0, k \in \mathbb{R}$, and the infinity semipositone expression means that $F(u)$ tends to $-\infty$ as u tends to 0.

In the first chapter, we start by giving some basic notions, which are concerning the functional framework necessary to support the existence of solutions for the studied problems.

The second chapter, is concerns the study of the existence of a weak solution of the following (p, q)-Laplace system

$$\begin{cases} -\Delta_p u = \lambda l(x) u^{p-1} - f_1(u, v) - a u^{-\alpha_1} v^{\beta_2} \text{ in } \Omega. \\ \\ -\Delta_q v = \mu k(x) v^{q-1} - f_2(u, v) - b u^{\alpha_2} v^{-\beta_2} \text{ in } \Omega. \\ \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

where $p, q > 1,\ 0 < \beta_1 < \alpha_1 < 1, 0 < \alpha_2 < \beta_2 < 1,\ \lambda, \mu, a, b > 0,\ f_i : [0, +\infty) \times [0, +\infty) \longrightarrow$

$\mathbb{R}, i = 1, 2$ are continuous functions, $l(x), k(x) \in C(\Omega)$, and

$$\frac{1-\alpha_1}{p^*} + \frac{\beta_1}{q^*} < 1 \text{ and } \frac{\alpha_2}{p^*} + \frac{1-\beta_2}{q^*} < 1.$$

In the third chapter, we study the following infinite semipositone elliptic system

$$\begin{cases} -\Delta_{pi} u_i = \mu_i u_i^{p_i - 1} - a_i \prod_{j=1}^{m} u_j^{\alpha_{ij}}, i = \overline{1, m}, \text{ in } \Omega. \\ \\ u_i = 0, i = \overline{1, m} \text{ on } \partial\Omega, \end{cases}$$

where

$$\frac{1+\alpha_{ii}}{p_i^*} + \sum_{i \neq j = 1}^{m} \frac{\alpha_{ij}}{p_j^*} < 1, \forall i = \overline{1, m}.$$

We prove the existence of a weak solution and we give an example of application.

The fourth chapter, we are interested in the $p(x)$-Kirchhoff parabolic systems of the form

$$\begin{cases} \frac{\partial u}{\partial t} - M(I_0(u)) \Delta_{p(x)} u = \lambda^{p(x)} [\lambda_1 f(v) + \mu_1 h(u)], \text{ in } Q_T = \Omega \times [0, T], \\ \\ \frac{\partial u}{\partial t} - M(I_0(v)) \Delta_{p(x)} v = \lambda^{p(x)} [\lambda_2 g(u) + \mu_2 \tau(v)], \text{ in } Q_T = \Omega \times [0, T], \\ \\ u = v = 0, \text{ on } \partial Q_T, \\ \\ u(x, 0) = \varphi(x). \end{cases}$$

With a suitable assumption, we prove the existence of a positive weak solutions of certain classes of parabolic systems intervening the $p(x)$–Kirchhoff operator.

During these studies Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The boundary conditions are the homogeneous Dirichlet conditions, by using the method of sub and supersolution.

Chapter 1

Preliminaries

Contents

We assume that the reader is familiar with the concept of a vector space over the real (or complex) scalar field and with the related notions of dimension, subspace and linear forms. We also assume familiarity with the basic concepts of general topology, Hausdorff topological spaces, continuous functions, topological product spaces, subspaces, and relative topology, Banach spaces and weaker, stronger convergent sequences and distributions and weak derivatives.

1.1 The space $L^p(\Omega)$

Let Ω be an open set of $\mathbb{R}^N$, equipped with the Lebesgue measure dx, and let p be a positive real number. We denote by $L^1(\Omega)$ the space of integrable functions on Ω with values on $\mathbb{R}$, it is provided with the norm

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)|\, dx.$$

Definition 1.1 *We define $L^p(\Omega)$ the space of the class of all measurable functions f, defined on Ω, for which*

$$\int_\Omega |f(x)|^p\, dx < +\infty,$$

equipped with the norm

$$\|f\|_{L^p(\Omega)} = (\int_\Omega |f(x)|^p\, dx)^{\frac{1}{p}}.$$

Definition 1.2 *We also define the space $L^\infty(\Omega)$ by*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R},\ f \ \textit{measurable},\ \exists c > 0,\ \textit{so that}\ |f(x)| < c \ \textit{a.e. on}\ \Omega\},$$

it will be equipped with the essential-sup norm

$$\|f\|_{L^\infty(\Omega)} = ess \sup_{x\in\Omega} |f(x)| = \inf\{c, |f(x)| < c \ \textit{a.e. on}\ \Omega\ \}.$$

We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $f 1_k \in L^p(\Omega)$ for any compact $k \subset \Omega$.

1.1.1 Hölder inequality and L^p completeness

If $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ where the real numbers p and p' satisfy $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we have Hölder inequality:

$$\int_\Omega |f(x)\, g(x)|\, dx \leq (\int_\Omega |f(x)|^p\, dx)^{\frac{1}{p}} (\int_\Omega |g(x)|^{p'}\, dx)^{\frac{1}{p'}}.$$

Theorem 1.1 *[10] The space $L^p(\Omega)$ is Banach spaces if $1 \leq p \leq \infty$ (complete normed space), separable space if $1 \leq p < \infty$, and $L^p(\Omega)$ is reflexive if and only if $1 < p < \infty$.*

1.1.2 Some convergence criteria

Theorem 1.2 (Monotony convergence) *[24] Let $(f_n)_{n\geq 1}$ be an increasing sequence of a positive measurable functions. By noting $f(x) = \lim_{n\to\infty} f_n(x) = \sup_{n\geq 1} f_n(x)$ we have*

$$\int_\Omega f(x)\, dx = \lim_{n\to+\infty} \int_\Omega f_n(x)\, dx.$$

Theorem 1.3 (Lebesgue's dominated convergence) *[24] Let $(f_n)_n$ be a sequence of functions of $L^1(\Omega)$ converging almost everywhere to a measurable function f. We suppose that there exists $g \in L^1(\Omega)$ such that for all $n \geq 1$, we have $|f_n| \leq g$ a.e. on Ω. Then $f \in L^1(\Omega)$ and*

$$\lim_{n\to+\infty} \|f_n - f\| = 0, \quad \int_\Omega f(x)\, dx = \lim_{n\to+\infty} \int_\Omega f_n(x)\, dx.$$

Lemma 1.1 (Brezis-Lieb) *[10] Let $1 \leq p < \infty$ and $(f_n)_n$ be a bounded sequence of functions from $L^p(\Omega)$ converging a.e. to f. Then $f \in L^p(\Omega)$ and*

$$\|f\|^p_{L^p(\Omega)} = \lim_{n\to+\infty} \left(\|f_n\|^p_{L^p(\Omega)} - \|f - f_n\|^p_{L^p(\Omega)} \right).$$

Lemma 1.2 *Let $1 < p < \infty$ and $(f_n)_n$ be a bounded sequence of $L^p(\Omega)$ converging a.e. to f. Then $f_n \rightharpoonup f$ in $L^p(\Omega)$.*

1.2 Sobolev space

In this section, we present a brief reminder on Sobolev spaces. We denote by $\mathcal{D}(\Omega)$ the space of functions of class $\mathcal{C}^{\infty}(\Omega)$, with compact support included in Ω, and by $\mathcal{D}'(\Omega)$ the topological dual of $\mathcal{D}(\Omega)$.

1.2.1 Weak derivative

Definition 1.3 *Let be an open set of $\mathbb{R}^N$; and $1 \leq i \leq N$. A function $f \in L^1_{loc}(\Omega)$ has an i^{th} weak derivative in $L^1_{loc}(\Omega)$ if there exists $f_i \in L^1_{loc}(\Omega)$ such that for all $\varphi \in \mathcal{C}^{\infty}_0(\Omega)$ we have*

$$\int_{\Omega} f(x)\,\partial_i\varphi(x)\,dx = -\int_{\Omega} f_i(x)\,\varphi(x)\,dx,$$

this leads to say that the i^{th} derivative within the meaning of distributions of f belongs to $L^1_{loc}(\Omega)$, we write

$$\partial_i f = \frac{\partial f}{\partial x_i} = f_i.$$

1.2.2 The space $W^{1,p}(\Omega)$

Definition 1.4 *[10] The space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega), \text{such that } \partial_i u \in L^p(\Omega), 1 \leq i \leq N\},$$

where ∂_i is the i^{th} weak derivative of u belongs $L^1_{loc}(\Omega)$.

The space $W^{1,p}(\Omega)$ is provided with the norm

$$\|u\|_{W^{1,p}(\Omega)} = (\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)})^{\frac{1}{p}}, u \in W^{1,p}(\Omega).$$

1.2.3 The space $W^{m,p}(\Omega)$

When $\alpha \in \mathbb{N}^n$, we denote by $|\alpha| = \alpha_1 + \alpha_2 + ...\alpha_n$ the length of α and we denote

$$\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} u,$$

in all that follows $\partial^\alpha u$ (or $D^\alpha u$) denotes the weak derivative of a function $u \in L^1_{loc}(\Omega)$.

Definition 1.5 *[10] We define the space $W^{m,p}(\Omega), m \geq 2$ as following*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega), \textit{such that } \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m,\ \partial^\alpha u \in L^p(\Omega), |\alpha| \leq m\},$$

equipped by the norm

$$\|u\|_{W^{m,p}(\Omega)} = (\sum_{|\alpha| \leq m} \|\partial^\alpha u\|^p_{L^p(\Omega)})^{\frac{1}{p}}.$$

Remark 1.1 *For $p = 2$, it is customary to replace the notation $W^{m,2}(\Omega)$ by $H^m(\Omega)$.*

Proposition 1.1 *[24] The space $W^{m,p}(\Omega)$ provided with the norm defined by*

$$\|u\|_{W^{m,p}(\Omega)} = \begin{cases} (\sum\limits_{|\alpha| \leq m} \|\partial^\alpha u\|^p_{L^p(\Omega)})^{\frac{1}{p}}, 1 \leq p < +\infty \\ \max\limits_{|\alpha| \leq m} \|\partial^\alpha u\|^p_{L^p(\Omega)}, p = +\infty, \end{cases}$$

is a Banach space, and for $p \in]1, \infty[$, this space is convex, so it is a reflexive space. The space $H^m(\Omega)$,

endowed with the scalar product

$$(u, v) = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)},$$

is a Hilbert space.

1.2.4 The space $W_0^{1,p}(\Omega)$

Definition 1.6 *For $1 \leq p < +\infty$ we define the space $W_0^{1,p}(\Omega)$ as being the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$, and we write*

$$W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}}.$$

Proposition 1.2 (Continuous embeddings) *[10] Let Ω be an open set of $\mathbb{R}^N$. We suppose that Ω is a bounded with Lipschitzian border or that $\Omega = \mathbb{R}^N$.*

1- If $1 \leq p < N$, then

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega), \text{ with } p^* = \frac{pN}{N-p},$$

and the embedding is continuous, i.e there is $C \in \mathbb{R}_+$ such that $\forall u \in W^{1,p}(\Omega)$,

$$\|u\|_{L^{p^*}} \leq C \|u\|_{W^{1,p}(\Omega)},$$

we note this

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega),$$

2- If $p = N$,

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \forall q \in [p, +\infty[.$$

3- If $p > N$, then

$$W^{1,p}(\Omega) \subset L^\infty(\Omega).$$

Proposition 1.3 (Compact embeddings) *1- If $1 \leq p < N$, then*

$$W^{1,p}(\Omega) \subset L^q(\Omega), \ \forall q \in [1, p^*[\text{ with } p^* = \frac{pN}{N-p}.$$

2- If $p = N$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \forall q \in [1, +\infty[$.

3- If $p > N$, then

$$W^{1,p}(\Omega) \subset \mathcal{C}(\overline{\Omega}).$$

Corollaire 1.1 (Poincaré inequality) *Let Ω be an open and bounded set of $\mathbb{R}^N$, then there exists a constant C $(C(\Omega,p))$ such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \forall u \in W^{1,p}(\Omega), 1 \leq p < +\infty.$$

We need to recall some basic properties on spaces $L^{P(x)}(\Omega)$ and $W^{P(x)}(\Omega)$.

1.3 Lebesgue and Sobolev spaces with variable exponents

Let $p : \Omega \to [1, +\infty[$ a measurable and bounded function, we denote by p^- and p^+ respectively the essential inf $\left(\inf_{\Omega} p(x)\right)$ and the essential sup $\left(\sup_{\Omega} p(x)\right)$ of the function p, we assume $p^- \leq p^+ < +\infty$.

We also introduce the space

$$L^\infty_+(\Omega) = \left\{p \in L^\infty(\Omega), p^- \geq 1\right\}.$$

Definition 1.7 *[17], [25] Let $u : \Omega \to \mathbb{R}$ a measurable function,we define the modulus of u by the quantity*

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Such that ρ verifies the following properties

a) $\rho_{p(.)}(u) = 0 \Longleftrightarrow u = 0.$

b) $\rho_{p(.)}(-u) = \rho_{p(.)}(u).$

c) The map $\lambda \to \rho_{p(.)}(\lambda u)$ is convex, continuous and even. In addition, she is strictly increasing over $[0, +\infty[$.

d)

$$\rho_{p(.)}(\alpha u + \beta v) \leq \alpha \rho_{p(.)}(u) + \beta \rho_{p(.)}(v), \alpha + \beta = 1.$$

Definition 1.8 *Let p be a measurable function of $[1, +\infty[$ in $\mathbb{R}^*_+$. We define and denote $L^{p(x)}(\Omega)$*

the Lebesgue space of variable exponent p

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R},\ \textit{measurable};\ \rho_{p(x)}(u) < +\infty \right\}.$$

We define on $L^{p(.)}(\Omega)$ the so-called Luxembourg norm by

$$\|u\|_{\rho} = \inf \left\{ \lambda > 0, \int\limits_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Proposition 1.4 *The space $L^{p(x)}(\Omega)$ is a separable Banach space. If $p^{-} > 1$, $L^{p(x)}(\Omega)$ is uniformly convex and reflexive.*

The following two results show the relationship between the Luxembourg norm and module $\rho_{p(x)}$

Proposition 1.5 *Let $p \in L_{+}^{\infty}(\Omega)$*

(i) If $u \in L^{p(x)}(\Omega)$, then $\|u\|_{L^{p(x)}(\Omega)} = a \Longleftrightarrow \rho\left(\frac{u}{a}\right) = 1$.

(ii) $\|u\|_{L^{p(x)}(\Omega)} < 1\,(= 1, > 1) \Longleftrightarrow \rho_{p(x)} < 1\,(= 1, > 1)$.

(iii) If $\|u\|_{L^{p(x)}(\Omega)} > 1$ then $\|u\|_{L^{p(x)}(\Omega)}^{p-} \leq \rho_{p(x)} \leq \|u\|_{L^{p(x)}(\Omega)}^{p+}$.

(iiii) If $\|u\|_{L^{p(x)}(\Omega)} < 1$ then $\|u\|_{L^{p(x)}(\Omega)}^{p+} \leq \rho_{p(x)} \leq \|u\|_{L^{p(x)}(\Omega)}^{p-}$.

Proposition 1.6 *If $p \in L_{+}^{\infty}(\Omega), (u_n) \subset L^{p(x)}(\Omega)$ and $u \in L^{p(x)}(\Omega)$.*

The following statements are equivalent

(i) $\lim\limits_{n \to +\infty} \|u - u_n\|_{L^{p(x)}(\Omega)} = 0$.

(ii) $\lim\limits_{n \to +\infty} \rho_{p(x)}(u - u_n) = 0$.

Remark 1.2 *Let $p \in L_{+}^{\infty}(\Omega), (u_n) \subset L^{p(x)}(\Omega)$ and $u \in L^{p(x)}(\Omega)$. If*

$$\lim_{n \to +\infty} \|u - u_n\|_{L^{p(x)}(\Omega)} = 0,$$

then there exists a subsequence $\left(u_{n_j}\right) \subset (u_n)$, and a function $g \in L^{p(x)}(\Omega)$ such that

(i) $u_{n_j} \to u$ a.e in Ω.

(ii) $\left|u_{n_j}\right| \leq g(x)$ a.e in Ω.

Theorem 1.4 (Interpolation in the spaces $L^{p(x)}(\Omega)$) *Let $p,q,r \in L_+^\infty(\Omega), u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ such that*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{r(x)} \ a.e \ in \ \Omega,$$

then

$$\|uv\|_{L^{r(x)}(\Omega)} \leq \left(\frac{1}{(p/r)^-} + \frac{1}{(q/r)^-}\right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)}.$$

Remark 1.3 *Let $p \in L_+^\infty(\Omega)$ and $p' : \Omega \to [1,+\infty[$ the conjugate of p such that*

$$p'(x) = \begin{cases} \frac{p(x)}{p(x)-1}, & if\ p(x) > 1 \\ \infty, & if\ p(x) = 1. \end{cases}$$

For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, there exists a constant C_p such that

$$\int_\Omega |u(x)v(x)|\,dx \leq C_p \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)}.$$

For the results of injections we draw the reader to Kovacik and Rákosnik [25] and Fan and Zhao [17].

Proposition 1.7 *Let Ω be a bounded open of $\mathbb{R}^N$ and $p,q \in L_+^\infty(\Omega)$. if $p(x) \leq q(x)$ a, e in Ω, then*

$$L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega),$$

(i.e $L^{q(x)}(\Omega)$ continuously injects into $L^{p(x)}(\Omega)$).

Definition 1.9 *For all $p \in L_+^\infty(\Omega)$, and $m \in \mathbb{N}^*$ we define the generalized Sobolev space (or Sobolev space with variable exponent) by*

$$W^{m,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega), D^\alpha u \in L^{p(x)}(\Omega),\ for\ all\ |\alpha| \leq m\right\},$$

providing him with the norm

$$\|u\|_{W^{m,p(x)}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^{p(x)}(\Omega)}.$$

The space $W^{m,p(x)}(\Omega)$ provided with the norm $\|u\|_{W^{m,p(x)}(\Omega)}$, is a separable and reflexive Banach space for $p^- > 1$.

We define the subspace $W_0^{m,p(x)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$

$$W_0^{m,p(x)}(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{W^{m,p(x)}(\Omega)}.$$

Now let us generalize the well known Sobolev imbedding.

Proposition 1.8 *Let $m \in \mathbb{N}^*$ and $p, q \in L_+^\infty(\Omega)$. If $p(x) \leq q(x)$ a.e in Ω, then*

$$W^{m,q(x)}(\Omega) \hookrightarrow W^{m,p(x)}(\Omega),$$

is continuous.

The continuity of the injection of the space $W^{m,q(x)}(\Omega)$ into $L^{p^*(x)}(\Omega)$ was obtained by Edmunds and Rákosnik [21], where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, p(x) < N \\ \infty, p(x) \geq N. \end{cases}$$

Theorem 1.5 *[11] Let $p, q \in \mathcal{C}\left(\overline{\Omega}\right)$ and $p, q \in L_+^\infty(\Omega)$. Assume that*

$$mp(x) < N, 1 < q(x) < p^*(x), x \in \overline{\Omega}.$$

Then there is a continuous and compact imbedding

$$W^{m,q(x)}(\Omega) \to L^{q(x)}(\Omega).$$

For more information concerning this section we refer the reader to [17] and [25]

1.4 Maximum principle

Let Ω be an open connected set in $\mathbb{R}^N$ with boundary $\partial\Omega = \overline{\Omega} \cap \left(\mathbb{R}^N \diagdown \Omega\right)$. Let $\mathcal{L}$ be the second order differential operator

$$\mathcal{L} = \sum_{i,j=1}^{N} a_{ij}(x) D_{ij} + \sum_{i=1}^{N} b_i D_i + c(x),$$

with $a_{ij} \in L^{\infty}_{loc}(\Omega)\,, and\ b_i, c(x) \in L^{\infty}(\Omega)$. Here we have used $D_i = \frac{\partial}{\partial x_i}$ and $D_{ij} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial xj}$. Without loss of generality one assumes $a_{ij} = a_{ji}$.

Definition 1.10 *We will fix the following notions*

- *The operator $\mathcal{L}$ is called elliptic on Ω if for every $x \in \Omega$ there is $\lambda(x) > 0$, such that*

$$\sum_{i,j=1}^{N} a_{ij}(x)\,\xi_i\xi_j \geq \lambda(x)\,|\xi|^2\,, \forall \xi \in \mathbb{R}^N.$$

- *The operator $\mathcal{L}$ is called strictly elliptic on Ω if there is $\lambda > 0$, such that*

$$\sum_{i,j=1}^{N} a_{ij}(x)\,\xi_i\xi_j \geq \lambda\,|\xi|^2\,, \forall \xi \in \mathbb{R}^N, x \in \Omega.$$

- *The operator $\mathcal{L}$ is called uniformly elliptic on Ω if there are Λ, $\lambda > 0$ such that*

$$\lambda\,|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\,\xi_i\xi_j \geq \Lambda\,|\xi|^2\,, \forall \xi \in \mathbb{R}^N, x \in \Omega.$$

1.4.1 Strong maximum principle

Theorem 1.6 *Suppose that $\mathcal{L}$ is strictly elliptic with $c \leq 0$, if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ and $\mathcal{L}(u) \geq 0$ in Ω, then either $u \equiv \sup_{\Omega} u$ or u does not attain a nonnegative maximum in Ω.*

1.4.2 Weak maximum principle

Theorem 1.7 *Suppose that* Ω *is bounded and that* $\mathcal{L}$ *is strictly elliptic with* $c \leq 0$, *if* $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ *and* $\mathcal{L}(u) \geq 0$ *in* Ω, *then a nonnegative maximum is attained at the boundary.*

1.4.3 Comparison principle

Lemma 1.3 *Let* $u, v \in W_0^{1,p}(\Omega) \left(\in W_0^{1,p(x)}(\Omega)\right)$ *such that* $A(u) - A(v) \geq 0$ *in* $\left(W_0^{1,p}(\Omega)\right)^*$, $\left(in \left(W_0^{1,p(x)}(\Omega)\right)^*\right), \varphi(x) = \min\{u(x) - v(x), 0\}$, *if* $\varphi(x) \in W_0^{1,p}(\Omega), \left(\in W_0^{1,p(x)}(\Omega)\right)$ *then* $u \geq v$ *in* Ω.

1.5 Properties of the p-Laplacian operator

For $1 < p < \infty$, the $p-$laplacian of a function f on an open bounded domain Ω is defined by

$$\Delta_p f = div(|\nabla f|^{p-2} \nabla f).$$

Lemma 1.4 *Let* V *be a closed subspace of* $W^{1,p}(\Omega)$ *and* $W_0^{1,p}(\Omega) \subseteq V \subseteq W^{1,p}(\Omega)$.

Then it holds

(i) $-\Delta_p : V \to V^*$ *is continuous, bounded and has the* $(S+)$*-property, ie, if every sequence* $\{u_n\}_n$ *in* V *such that* $u_n \rightharpoonup u$ *and*

$\lim_{n\to+\infty} sup \langle -\Delta_p u_n, u_n - u\rangle \leq 0$ *has a convergent subsequence* $\{u_{n_k}\}_k$ *such that* $u_{n_k} \to u$.

(ii) $-\Delta_p : W^{1,p}(\Omega) \to W^{-1q}(\Omega)$ *is*

a) strictly monotone if $1 < p < \infty$

b) strongly monotone if $p = 2$ *(which is the well-known Laplace operator)*

c) uniformly monotone if $2 < p < \infty$.

1.5.1 Eigenvalue problem

We consider the following eigenvalue problem :

$$\Delta_p u(x) + \lambda |u(x)|^{p-2} u(x) = 0, \text{in } \Omega. \tag{1.1}$$

Where we impose the Dirichlet boundary conditions. We say that λ is an eigenvalue of $-\Delta_p$ if (1.1) has a nontrivial weak solution $u_\lambda \in W_0^{1,p}(\Omega)$. That is, for any $v \in \mathcal{C}_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u_\lambda(x)|^{p-2} \nabla u_\lambda(x) \nabla v(x)\, dx - \lambda \int_\Omega |u_\lambda(x)|^{p-2} u_\lambda(x) v(x)\, dx = 0.$$

The function u_λ is then called an eigenfunction of $-\Delta_p$ associated to the eigenvalue λ. Note that if $p = 2$, the $p-$laplacian corresponds to the usual laplacian.

The first eigenvalue of the Dirichlet eigenvalue problem of the $p-$Laplace operator, denoted by $\lambda_{1,p}$, is characterized as,

$$\lambda_{1,p} = \min_{0 \neq u_\lambda \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u_\lambda(x)|^p\, dx}{\int_\Omega |u_\lambda(x)|^p\, dx}.$$

The infimum is attained for a function $\varphi_{1,p} \in W_0^{1,p}(\Omega)$. In addition, $\lambda_{1,p}$ is simple and isolated. Moreover, the eigenfunction $\varphi_{1,p}$ associated to $\lambda_{1,p}$ does not change sign, and it is only such eigenfunction, with $\|\varphi_{1,p}\|_{W_0^{1,p}(\Omega)} = 1$, and there are $m_p, \varepsilon, \upsilon > 0$, such that

$$|\nabla \varphi_{1,p}| \geq m_p, \text{ on } \bar{\Omega}_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) \leq \varepsilon\},$$

$$\varphi_{1,p} \geq \upsilon > 0, \text{ on } \Omega \backslash \Omega_\varepsilon.$$

1.6 Properties of the $p(x)$-Laplacian operator

We consider the separable and reflexive Banach space $V = W_0^{1p(x)}(\Omega)$

$$-\Delta_{p(x)}u : V \to V^*,$$

defined by

$$\left\langle -\Delta_{p(x)}u, v \right\rangle = \int_\Omega |\nabla u|^{p(x)} \nabla u.\nabla v dx, u, v \in V.$$

Lemma 1.5 *(i) $-\Delta_{p(x)} : V \to V^*$ is a homeomorphism from V into V^*.*

(ii) $-\Delta_{p(x)} : V \to V^*$ is a strictly monotone operator, that means

$$\left\langle -\Delta_{p(x)}u - \left(-\Delta_{p(x)}\right) v, u - v \right\rangle > 0, u \neq v \in V.$$

(iii) $-\Delta_{p(x)} : V \to V^*$ is a mapping of type $(S+)$.

Remark 1.4 *Since the structure of the $p(x)$-Laplace is more complicated than that of the p-Laplace operator,such as it is nonhomogeneous, the extension from p-Laplace operator to $p(x)$-Laplace operator will not be well-worn. Furthermore, many concepts for p-Laplacian are not true for the $p(x)$-Laplacian, for instance ,if Ω is bounded, then the Rayleigh quotient*

$$\lambda_{1,p(x)} = \min_{0 \neq u \in W_0^{1,p(x)}(\Omega)} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx}{\int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} dx},$$

is a zero in general, so the first eigenvalue and the first eigenfunction of the $p(x)-$Laplacian may not be existing.

Chapter 2

Infinite semipositone elliptic systems

Contents

The results of this chapter have been accepted to publish in " *Discontinuity, Nonlinearity, and Complexity*" journal, 2020.

2.1 Introduction

In recent years, there has been a considerable progress on the study of semipositione problems, (see [21,34]). It is well documented in the literature that studying positive solutions to such semipositone problems is mathematically challenging. Even more challenging infinite semipositone problem has been studied by [3, 26, 32], for example in [26] E.K. Lee, R. Shivaji and J. Ye, have studied the singular problem when

$$F(u) = au - f(u) - \frac{c}{u^{\alpha}},$$

under the following assumptions

$$\begin{cases} \exists A > 0, p > 1 : f(u) \leq Au^{p}, \\ \\ \exists M > 0 : f(u) \geq alu - M, \end{cases}$$

and the result has been extended to the system

$$\begin{cases} -\Delta u & = a_1 u - f_1(u) - \frac{c_1}{v^{\alpha}} \text{ in } \Omega, \\ \\ -\Delta v & = a_2 v - f_2(v) - \frac{c_2}{u^{\alpha}} \text{ in } \Omega, \\ \\ u = v & = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$.

There are many results concerning the p-Laplace problems see [22] or (p, q)-Laplace as like as [21]. But the fact of that the problem is an infinite semipositone make things more difficult, in [4] K. Akrout, has studied the infinite semipositone problems involving the p-Laplace system.

In this work, motivated by the ideas introduced in [4], we used Sub-supersolution method with comparison principle to prove the existence of positive solution of our p, q-system, depending on the parameters λ and μ.

We study the following infinite semipositone system

$$\begin{cases} -\Delta_p u = \lambda l(x) u^{p-1} - f_1(u,v) - a u^{-\alpha_1} v^{\beta_1}, \text{ in } \Omega, \\ \\ -\Delta_q v = \mu k(x) v^{q-1} - f_2(u,v) - b u^{\alpha_2} v^{-\beta_2}, \text{ in } \Omega, \\ \\ u = v = 0, \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $p, q > 1,\ 0 < \beta_1 < \alpha_1 < 1, 0 < \alpha_2 < \beta_2 < 1,\ \lambda, \mu, a, b > 0$ are real constants, Δ_s is a s-Laplace operator. The weight functions l and k satisfy $l(x),\ k(x) \in C(\Omega)$ and $l(x) > l_0 > 0,\ k(x) > k_0 > 0$ for all $x \in \Omega$ also $\|l\|_\infty = l_1 < +\infty,\ \|k\|_\infty = k_1 < +\infty$ and $f_i : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbb{R}, i = 1, 2$ are continuous functions.

We used first eigenfunction of p and q-Laplace operator for constructing the subsolution, the supersolution is a solution of a well defined problem, while respecting the comparison principle between them (sub supersolution) by controlling the constants.

Now, to go on with the problem (2.1), we add the following assumptions

$$\frac{1-\alpha_1}{p^*} + \frac{\beta_1}{q^*} < 1 \text{ and } \frac{\alpha_2}{p^*} + \frac{1-\beta_2}{q^*} < 1, \tag{2.2}$$

$$\exists A_1, A_2 > 0, \zeta_1, \zeta_2, \eta_1, \eta_2 > 1 : \begin{cases} f_1(u,v) \leq A_1 u^{\zeta_1(p-1)} v^{\eta_1(p-1)}, \\ \\ f_2(u,v) \leq A_2 u^{\zeta_2(q-1)} v^{\eta_2(q-1)}, \end{cases} \tag{2.3}$$

with

$$\tfrac{\zeta_1(p-1)+1}{p^*} + \tfrac{\eta_1(p-1)}{q^*} < 1 \text{ and } \tfrac{\zeta_2(q-1)}{p^*} + \tfrac{\eta_2(q-1)+1}{q^*} < 1,$$

$$\exists\ M_1, M_2 > 0 : \begin{cases} f_1(u,v) \geq \lambda l_1 u^{p-1} - M_1, \\ \\ f_2(u,v) \geq \mu k_1 v^{q-1} - M_2. \end{cases} \tag{2.4}$$

Let

$$F(x,u,v) = \lambda l(x) u^{p-1} - f_1(u,v) - a u^{-\alpha_1} v^{\beta_1},$$

$$G(x,u,v) = \mu k(x) v^{q-1} - f_2(u,v) - b u^{\alpha_2} v^{-\beta_2}.$$

Then

$$\lim_{(u,v)\to(0,0)} F(x,u,v) = \lim_{(u,v)\to(0,0)} G(x,u,v) = -\infty.$$

Hence, we refer to (2.1) as an infinite semipositone system, such as F and G are increasing functions.

We introduced here some definitions and important Lemmas for proof

Definition 2.1 *We called a weak positive subsolution* (ψ_1, ψ_2) *and supersolution* (z_1, z_2) *in* $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ *of (2.1) such that they satisfy* $\psi_i \leq z_i$, $i = 1, 2$, $(\psi_1, \psi_2) = (0,0) = (z_1, z_2)$ *on* $\partial\Omega$ *and*

$$\begin{cases} \int\limits_\Omega |\nabla\psi_1|^{p-2} \nabla\psi_1.\nabla w_1 dx \leq \lambda \int\limits_\Omega l(x) \psi_1^{p-1} w_1 dx - \int\limits_\Omega f_1(\psi_1,\psi_2) w_1 dx - a \int\limits_\Omega \psi_1^{-\alpha_1} \psi_2^{\beta_1} w_1 dx, \\ \\ \int\limits_\Omega |\nabla\psi_2|^{q-2} \nabla\psi_2.\nabla w_2 dx \leq \mu \int\limits_\Omega k(x) \psi_2^{q-1} w_2 dx - \int\limits_\Omega f_2(\psi_1,\psi_2) w_2 dx - b \int\limits_\Omega \psi_1^{\alpha_2} \psi_2^{-\beta_2} w_2 dx. \end{cases}$$

And

$$\begin{cases} \int\limits_\Omega |\nabla z_1|^{p-2} \nabla z_1.\nabla w_1 dx \geq \lambda \int\limits_\Omega l(x) z_1^{p-1} w_1 dx - \int\limits_\Omega f_1(z_1,z_2) w_1 dx - a \int\limits_\Omega z_1^{-\alpha 1} z_2^{\beta 1} w_1 dx, \\ \\ \int\limits_\Omega |\nabla z_2|^{q-2} \nabla z_2.\nabla w_2 dx \geq \mu \int\limits_\Omega k(x) z_2^{q-1} w_2 dx - \int\limits_\Omega f_2(z_1,z_2) w_2 dx - b \int\limits_\Omega z_1^{\alpha_2} z_2^{-\beta_2} w_2 dx, \end{cases}$$

for all test functions $w_1 \in W_0^{1,p}(\Omega)$, $w_2 \in W_0^{1,q}(\Omega)$ *and* $w_1, w_2 \geq 0$,*in* Ω.

Lemma 2.1 *[10] The following problem*

$$\begin{cases} -\Delta_p u = f(x), in \ \Omega, \\ u = 0, on \ \partial\Omega, \end{cases}$$

has a unique positive solution $u \in W_0^{1,p}(\Omega)$ *if* $,f \in L^{p'}(\Omega)$.

Lemma 2.2 *[2] Let* $u, v \in W_0^{1,p}(\Omega)$ *satisfy*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla w dx \leq \int_\Omega |\nabla v|^{p-2} \nabla v.\nabla w dx,$$

for all $w \in W_0^{1,p}(\Omega), w \geq 0$ *then* $u \leq v$ *a.e in* Ω.

Lemma 2.3 *If there exist sub-supersolution* (ψ_1, ψ_2) *and* (z_1, z_2), *respectively, such that* $\psi_i \leq z_i$, $i = 1, 2$ *in* Ω, *then (2.1) has at least a positive solution satisfying* $\psi_1 \leq u \leq z_1$ *and* $\psi_2 \leq v \leq z_2$ *in* Ω.

Let λ_p (resp. λ_q) be the first eigenvalue of $-\Delta_p$ (resp. $-\Delta_q$) with Dirichlet boundary conditions and φ_p $\left(\text{resp. } \varphi_q\right)$ the corresponding positive eigenfunction with $\|\varphi_j\| = 1$, $j = p, q$, and there are $m_p, m_q, \varepsilon, \upsilon > 0$, such that

$$\begin{aligned} &\left|\nabla \varphi_j\right| \geq m_j, \ j = p, q \text{ on } \bar{\Omega}_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) \leq \varepsilon\}, \\ &\varphi_j \geq \upsilon > 0, \ j = p, q \text{ on } \Omega \backslash \Omega_\varepsilon. \end{aligned} \tag{2.5}$$

There exists positive constants h_1 and h_2 such that

$$h_1 \varphi_p \leq \varphi_q \leq h_2 \varphi_p \text{ for all } x \in \Omega. \tag{2.6}$$

We denote by

$$\begin{cases} \theta_1 = \frac{p(\beta_2+q-1)+q\beta_1}{\delta}, \\ \\ \theta_2 = \frac{q(\alpha_1+p-1)+p\alpha_2}{\delta}, \end{cases}$$

$$\text{with } \delta \;=\; (\alpha_1+p-1)(\beta_2+q-1)-\alpha_2\beta_1 \neq 0.$$

$$C_1 = \left(\left(\frac{m_p}{a} h_2^{-\theta_2\beta_1} \theta_1^{p-1} (\theta_1-1)(p-1) \right)^{-(\beta_2+q-1)} \left(\frac{m_q}{b} h_1^{\theta_1\alpha_2} \theta_2(\theta_2-1)(q-1) \right)^{-\beta_1} \right)^{\frac{1}{\delta}},$$

$$C_2 = \left(\left(\frac{m_p}{a} h_2^{-\theta_2\beta_1} \theta_1^{p-1} (\theta_1-1)(p-1) \right)^{-\alpha_2} \left(\frac{m_q}{b} \theta_2^{q-1} (\theta_2-1)(q-1) h_1^{\theta_1\alpha_2} \right)^{-(\alpha_1+p-1)} \right)^{\frac{1}{\delta}},$$

and

$$\begin{aligned} \lambda^* &= \frac{1}{l_0} \left(\theta_1^{p-1}\lambda_p + A_1 h_2^{\eta_1\theta_2(p-1)} C_1^{(\zeta_1-1)(p-1)} C_2^{\eta_2(p-1)} \right) \\ \mu^* &= \frac{1}{k_0} \left(\theta_2^{q-1}\lambda_q + A_2 h_1^{-\zeta_2\theta_1(q-1)} C_1^{\zeta_1(q-1)} C_2^{(\eta_2-1)(q-1)} \right). \end{aligned} \tag{2.7}$$

Remark 2.1 *It's easy to verify that $\theta_1, \theta_2 > 1$.*

2.2 Main results

In this section we proved the main result of this paper, and we will be deeply based on the sub-supersolution method to prove it.

The following main result hold.

Theorem 2.1 *Assume that we have all hypotheses (2.2) – (2.7). Then for*

$$\begin{cases} \lambda \geq \lambda^* + \theta_1^{p-1}(\theta_1-1)(p-1)\left(\frac{m_p}{l_0 \upsilon^p}\right), \\ \\ \mu \geq \mu^* + \theta_2^{q-1}(\theta_2-1)(q-1)\left(\frac{m_q}{k_0 \upsilon^q}\right), \end{cases} \tag{2.8}$$

the system (2.1) has a positive solution (u,v) *.*

Proof First, we construct a positive subsolution of (2.1)

Let

$$\psi_1 = C_1\varphi_p^{\theta_1} \text{ and } \psi_2 = C_2\varphi_q^{\theta_2}.$$

A calculation shows that

$$-\Delta_p\psi_1 = C_1^{p-1}\theta_1^{p-1}\lambda_p\varphi_p^{\theta_1(p-1)} - C_1^{p-1}\theta_1^{p-1}(\theta_1-1)(p-1)\varphi_p^{\theta_1(p-1)-p}\left|\nabla\varphi_p\right|^p,$$

$$-\Delta_q\psi_2 = C_2^{q-1}\theta_2^{q-1}\lambda_q\varphi_q^{\theta_2(q-1)} - C_2^{q-1}\theta_2^{q-1}(\theta_2-1)(q-1)\varphi_p^{\theta_2(q-1)-q}\left|\nabla\varphi_q\right|^q.$$

In $\bar{\Omega}_\varepsilon$, by using (2.5) , we have

$$\begin{aligned}-\Delta_p\psi_1 &\leq C_1^{p-1}\theta_1^{p-1}\lambda_p\varphi_p^{\theta_1(p-1)} - m_pC_1^{p-1}\theta_1^{p-1}(\theta_1-1)(p-1)\varphi_p^{\theta_1(p-1)-p} \\ &= \left(\theta_1^{p-1}\lambda_p + A_1h_2^{\eta_1\theta_2(p-1)}C_1^{(\zeta_1-1)(p-1)}C_2^{\eta_2(p-1)}\right)C_1^{p-1}\varphi_p^{\theta_1(p-1)} \\ &\quad -A_1h_2^{\eta_1\theta_2(p-1)}C_1^{\zeta_1(p-1)}C_2^{\eta_1(p-1)}\varphi_p^{\theta_1(p-1)} - m_pC_1^{p-1}\theta_1^{p-1\prime}(\theta_1-1)(p-1)\varphi_p^{\theta_1(p-1)-p},\end{aligned}$$

and

$$\begin{aligned}-\Delta_q\psi_2 &\leq \left(\theta_2^{q-1}\lambda_q + A_2h_1^{-\zeta_2\theta_1(q-1)}C_1^{(\zeta_2-1)(q-1)}C_2^{(\eta_2-1)(q-1)}\right)C_2^{q-1}\varphi_q^{\theta_2(q-1)} \\ &\quad -A_2h_1^{-\zeta_2\theta_1(q-1)}C_1^{\zeta_2(q-1)}C_2^{\eta_2(q-1)}\varphi_q^{\theta_2(q-1)} - m_qC_2^{q-1}\theta_2^{q-1}(\theta_2-1)(q-1)\varphi_q^{\theta_2(q-1)-q}.\end{aligned}$$

A calculation show that

$$\theta_1(p-1)-p = -\alpha_1\theta_1 + \beta_1\theta_2,$$

$$\theta_2(q-1)-q = \alpha_2\theta_1 - \beta_2\theta_2,$$

$$
\begin{aligned}
aC_1^{-\alpha_1}C_2^{\beta_1} &= m_p h_2^{-\theta_2\beta_1} C_1^{p-1}\theta_1^{p-1}(\theta_1-1)(p-1), \\
bC_1^{\alpha_2}C_2^{-\beta_2} &= m_q h_1^{\theta_1\alpha_2} C_2^{q-1}\theta_2^{q-1}(\theta_2-1)(q-1).
\end{aligned}
\tag{2.9}
$$

It follow that

$$
\begin{aligned}
-m_p C_1^{p-1}\theta_1^{p-1}(\theta_1-1)(p-1)\varphi_p^{\theta_1(p-1)-p} &= -\frac{a h_2^{\theta_2\beta_1}}{C_1^{\alpha_1}C_2^{-\beta_1}}\varphi_p^{-\alpha_1\theta_1+\beta_1\theta_2} \\
&= -\frac{a h_2^{\theta_2\beta_1}\left(C_2^{\beta_1}\varphi_p^{\beta_1\theta_2}\right)}{\left(C_1^{\alpha_1}\varphi_p^{\alpha_1\theta_1}\right)}, \\
-m_q C_2^{q-1}\theta_2^{q-1}(\theta_2-1)(q-1)\varphi_q^{\theta_2(q-1)-q} &= -\frac{b h_1^{-\theta_1\alpha_2}}{C_1^{-\alpha_2}C_2^{\beta_2}}\varphi_q^{\alpha_2\theta_1-\beta_2\theta_2} \\
&= -\frac{b h_1^{-\theta_1\alpha_2}\left(C_1^{\alpha_2}\varphi_q^{\alpha_2\theta_1}\right)}{\left(C_2^{\beta_2}\varphi_q^{\beta_2\theta_2}\right)}.
\end{aligned}
\tag{2.10}
$$

Moreover, we have

$$
\begin{aligned}
-m_p C_1^{p-1}\theta_1^{p-1}(\theta_1-1)(p-1)\varphi_p^{\theta_1(p-1)-p} &\le -a\psi_1^{-\alpha_1}\psi_2^{\beta_1}, \\
-m_q C_2^{q-1}\theta_2^{q-1}(\theta_2-1)(q-1)\varphi_q^{\theta_2(q-1)-q} &\le -b\psi_1^{\alpha_2}\psi_2^{-\beta_2}.
\end{aligned}
\tag{2.11}
$$

In the other hand

$$\begin{aligned}
-A_1 h_2^{\eta_1\theta_2(p-1)} C_1^{\zeta_1(p-1)} C_2^{\eta_1(p-1)} \varphi_p^{\theta_1(p-1)} &\leq -A_1 h_2^{\eta_1\theta_2(p-1)} C_1^{\zeta_1(p-1)} C_2^{\eta_1(p-1)} \varphi_p^{(p-1)(\zeta_1\theta_1+\eta_1\theta_2)} \\
&= -A_1 \left(C_1^{\zeta_1(p-1)} \varphi_p^{\zeta_1\theta_1(p-1)} \right) \left(C_2^{\eta_1} h_2^{\eta_1\theta_2(p-1)} \varphi_p^{\eta_1\theta_2(p-1)} \right) \\
&\leq -A_1 \psi_1^{\zeta_1(p-1)} \psi_2^{\eta_1(p-1)} \leq -f_1(\psi_1, \psi_2), \\
-A_2 h_1^{-\theta_1\zeta_2(q-1)} C_1^{\zeta_2(q-1)} C_2^{\eta_2(q-1)} \varphi_q^{\theta_2(q-1)} &\leq -A_2 h_1^{-\theta_1\zeta_2(q-1)} C_1^{\zeta_2(q-1)} C_2^{\eta_2(q-1)} \varphi_q^{(q-1)(\zeta_2\theta_1+\eta_2\theta_2)} \\
&= -A_2 (C_1^{\zeta_2(q-1)} h_1^{-\theta_1\zeta_2(q-1)} \varphi_q^{\zeta_2\theta_1(q-1)})(C_2^{\eta_2(q-1)} \varphi_q^{\eta_2\theta_2(q-1)}) \\
&\leq -A_2 \psi_1^{\zeta_2(q-1)} \psi_2^{\eta_2(q-1)} \leq -f_2(\psi_1, \psi_2),
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
\left(\theta_1^{p-1} \lambda_p + A_1 h_2^{\eta_1\theta_2(p-1)} C_1^{(\zeta_1-1)(p-1)} C_2^{\eta_1(p-1)} \right) C_1^{p-1} \varphi_p^{\theta_1(p-1)} &\leq \lambda^* l_0 \left(C_1 \varphi_p^{\theta_1} \right)^{p-1} \\
&\leq \lambda l(x) \psi_1^{p-1}. \\
\left(\theta_2^{q-1} \lambda_q + A_2 h_1^{-\zeta_2\theta_1(q-1)} C_1^{\zeta_2(q-1)} C_2^{(\eta_2-1)(q-1)} \right) C_2^{q-1} \varphi_q^{\theta_2(q-1)} &\leq \mu^* k_0 \left(C_2 \varphi_q^{\theta_2} \right)^{q-1} \\
&\leq \mu k(x) \psi_2^{q-1}.
\end{aligned} \tag{2.13}$$

Formulas (2.11) − (2.13) imply that

$$\begin{cases} -\Delta_p \psi_1 \leq \lambda l(x) \psi_1^{p-1} - f_1(\psi_1, \psi_2) - a\psi_1^{-\alpha_1} \psi_2^{\beta_1}, \\ -\Delta_q \psi_2 \leq \mu k(x) \psi_2^{q-1} - f_2(\psi_1, \psi_2) - b\psi_1^{\alpha_2} \psi_2^{-\beta_2}. \end{cases}$$

Now, in $\Omega \backslash \Omega_\varepsilon$ we have $\varphi_j \geq \upsilon > 0$, $j = p, q$, and

$$
\begin{aligned}
-\Delta_p\psi_1 &= C_1^{p-1}\theta_1^{p-1}\lambda_p\varphi_p^{\theta_1(p-1)} - C_1^{p-1}\theta_1^{p-1}(\theta_1-1)(p-1)\varphi_p^{\theta_1(p-1)-p}\left|\nabla\varphi_p\right|^p \\
&\le C_1^{p-1}\theta_1^{p-1}\lambda_p\varphi_p^{\theta_1(p-1)} \\
&\le \left(C_1^{p-1}\theta_1^{p-1}\lambda_p + A_1 h_2^{\eta_1\theta_2(p-1)}C_1^{\zeta_1(p-1)}C_2^{\eta_1(p-1)} + \tfrac{a h_2^{\theta_2\beta_1}}{\upsilon^p C_1^{\alpha_1}C_2^{-\beta_1}}\right)\varphi_p^{\theta_1(p-1)} \\
&\quad -A_1 h_2^{\eta_1\theta_2(p-1)}C_1^{\zeta_1(p-1)}C_2^{\eta_1(p-1)}\varphi_p^{\theta_1(p-1)} - \tfrac{a h_2^{\theta_2\beta_1}\varphi^p}{\upsilon^p C_1^{\alpha_1}C_2^{-\beta_1}}\varphi_p^{\theta_1(p-1)-p} \\
&\le \left(C_1^{p-1}\theta_1^{p-1}\lambda_p + A_1 h_2^{\eta_1\theta_2(p-1)}C_1^{\zeta_1(p-1)}C_2^{\eta_1(p-1)} + \tfrac{a h_2^{\theta_2\beta_1}}{\upsilon^p C_1^{\alpha_1}C_2^{-\beta_1}}\right)\varphi_p^{\theta_1(p-1)} \\
&\quad -A_1 h_2^{\eta_1\theta_2(p-1)}C_1^{\zeta_1(p-1)}C_2^{\eta_1(p-1)}\varphi_p^{(p-1)(\zeta_1\theta_1+\eta_1\theta_2)} - \tfrac{a}{C_1^{\alpha_1}C_2^{-\beta_1}\varphi_p^{(\alpha_1\theta_1-\beta_1\theta_2)}}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
-\Delta_p\psi_1 &\le \left(\lambda^* + \theta_1^{p-1}(\theta_1-1)(p-1)\left(\tfrac{m_p}{\upsilon^p l_0}\right)\right) l_0 C_1^{p-1}\varphi_p^{\theta_1(p-1)} \\
&\quad -A_1\psi_1^{\zeta_1(p-1)}\psi_2^{\eta_1(p-1)} - a\psi_1^{-\alpha_1}\psi_2^{\beta_1}.
\end{aligned}
$$

By using (2.8) and (2.9), we obtain

$$
-\Delta_p\psi_1 \le \lambda l\left(x\right)\psi_1^{p-1} - f_1\left(\psi_1,\psi_2\right) - a\psi_1^{-\alpha_1}\psi_2^{\beta_1}.
$$

By the same manner, we get

$$
\begin{aligned}
-\Delta_q\psi_2 &\le \left(\mu^* + \theta_2^{q-1}(\theta_2-1)(q-1)\left(\tfrac{m_q}{\upsilon^q k_0}\right)\right) k_0 C_2^{q-1}\varphi_q^{\theta_2(q-1)} \\
&\quad -A_2\psi_1^{\zeta_2(q-1)}\psi_2^{\eta_2(q-1)} - b\psi_1^{\alpha_2}\psi_2^{-\beta_2} \\
&\le \mu k\left(x\right)\psi_2^{q-1} - f_2\left(\psi_1,\psi_2\right) - b\psi_1^{\alpha_2}\psi_2^{-\beta_2}.
\end{aligned}
$$

Hence

$$\begin{aligned}\int_{\Omega} |\nabla \psi_1|^{p-2} \nabla \psi_1 . \nabla w_1 dx &\leq \lambda \int_{\Omega} l(x) \psi_1^{p-1} w_1 dx - \int_{\Omega} f_1(\psi_1, \psi_2) w_1 dx - a \int_{\Omega} \psi_1^{-\alpha_1} \psi_2^{\beta_1} w_1 dx, \\ \int_{\Omega} |\nabla \psi_2|^{q-2} \nabla \psi_1 . \nabla w_2 dx &\leq \mu \int_{\Omega} k(x) \psi_1^{q-1} w_2 dx - \int_{\Omega} f_2(\psi_1, \psi_2) w_2 dx - b \int_{\Omega} \psi_1^{\alpha_2} \psi_2^{-\beta_2} w_2 dx,\end{aligned}$$

then (ψ_1, ψ_2) is a subsolution of (2.1).

Now as a second step of the proof, we will construct a supersolution of (2.1), for this, we let

$$\begin{cases} -\Delta_p e_1 = 1 \text{ in } \Omega, \\ \\ e_1 = 0 \text{ on } \partial\Omega, \end{cases} \text{ and } \begin{cases} -\Delta_q e_2 = 1 \text{ in } \Omega, \\ \\ e_2 = 0 \text{ on } \partial\Omega. \end{cases}$$

Let

$$\begin{aligned} &z_1 = (M')^{\frac{1}{p-1}} e_1 \text{ and } z_2 = (M'')^{\frac{1}{q-1}} e_2, \\ &\text{where } M' = \max\left(M_1, \tfrac{\psi_1}{e_1}\right) \text{ and } M'' = \max\left(M_2, \tfrac{\psi_2}{e_2}\right). \end{aligned} \tag{2.14}$$

Then

$$\begin{cases} -\Delta_p z_1 &= M' \Delta_p e_1 = M' \geq \lambda l(x) z_1^{p-1} - f_1(z_1, z_2) - a z_1^{-\alpha_1} z_2^{\beta_1}, \\ \\ -\Delta_q z_2 &= M'' \Delta_q e_2 = M'' \geq \mu k(x) z_2^{q-1} - f_2(z_1, z_2) - b z_1^{\alpha_2} z_2^{-\beta_2}. \end{cases}$$

Hence

$$\int_{\Omega} |\nabla z_1|^{p-2} \nabla z_1 . \nabla w_1 dx \geq \lambda \int_{\Omega} l(x) z_1^{p-1} w_1 dx - \int_{\Omega} f_1(z_1, z_2) w_1 dx - a \int_{\Omega} z_1^{-\alpha 1} z_2^{\beta 1} w_1 dx,$$

$$\int_{\Omega} |\nabla z_2|^{q-2} \nabla z_2 . \nabla w_2 dx \geq \mu \int_{\Omega} k(x) z_2^{q-1} w_2 dx - \int_{\Omega} f_2(z_1, z_2) w_2 dx - b \int_{\Omega} z_1^{\alpha_2} z_2^{-\beta_2} w_2 dx.$$

Therefore (z_1, z_2) is a supersolution of (2.1). The formula (2.14) implies that $\psi_1 \leq z_1$ and $\psi_2 \leq z_2$.

To preside the proof of the theorem, as a third step, and in order to obtain a weak solution of the problem (2.1) we define the sequence

$$\{(u_n, v_n)\} \subset E = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega),$$

as follows: $(u_0, v_0) = (z_1, z_2) \in E$ and (u_n, v_n) is the unique solution of the system

$$\begin{cases} -\Delta_p u_n = \lambda l(x)\, u_{n-1}^{p-1} - f_1(u_{n-1}, v_{n-1}) - a u_{n-1}^{-\alpha_1} v_{n-1}^{\beta_1}, \text{ in } \Omega, \\ -\Delta_q v_n = \mu k(x)\, v_{n-1}^{q-1} - f_2(u_{n-1}, v_{n-1}) - b u_{n-1}^{\alpha_1} v_{n-1}^{-\beta_2}, \text{ in } \Omega, \\ u_n = v_n = 0, \text{ on } \partial\Omega. \end{cases} \tag{2.15}$$

From the 2.1 the system (2.15) for $n = 1$ has a unique positive solution $(u_1, v_1) \in E$. Observing that

$$\begin{cases} -\Delta_p u_1 = \lambda l(x)\, u_0^{p-1} - f_1(u_0, v_0) - a u_0^{-\alpha_1} v_0^{\beta_1}, \text{ in } \Omega, \\ -\Delta_q v_1 = \mu k(x)\, v_0^{q-1} - f_2(u_0, v_0) - b u_0^{\alpha_1} v_0^{-\beta_2}, \text{ in } \Omega, \\ u_1 = v_1 = 0, \text{ on } \partial\Omega, \end{cases}$$

and recall that (u_0, v_0) is a weak supersolution to (2.1), the following hold

$$\begin{cases} -\Delta_p u_0 \geq \lambda l(x)\, u_0^{p-1} - f_1(u_0, v_0) - a u_0^{-\alpha_1} v_0^{\beta_1} = -\Delta_p u_1, \\ -\Delta_q v_0 \geq \mu k(x)\, v_0^{q-1} - f_2(u_0, v_0) - b u_0^{\alpha_1} v_0^{-\beta_2} = -\Delta_q v_1. \end{cases}$$

Moreover, the fact that $u_0 \geq \psi_1$, $v_0 \geq \psi_2$ and F and G are monotonous functions, we can get that

$$\begin{cases} -\Delta_p u_1 = \lambda l(x) u_0^{p-1} - f_1(u_0, v_0) - a u_0^{-\alpha_1} v_0^{\beta_1} \geq \lambda l(x) \psi_1^{p-1} - f_1(\psi_1, \psi_2) - a \psi_1^{-\alpha_1} \psi_2^{\beta_1}, \\ \\ -\Delta_q v_1 = \mu k(x) v_0^{q-1} - f_2(u_0, v_0) - b u_0^{\alpha_1} v_0^{-\beta_2} \geq \mu k(x) \psi_2^{q-1} - f_1(\psi_1, \psi_2) - b \psi_1^{\alpha_2} \psi_2^{-\beta_2}. \end{cases}$$

Combining the above inequalities with 2.2, one obtain that

$$u_1 \geq \psi_1 \text{ and } v_1 \geq \psi_2.$$

Similarly, for u_1, v_1 we get that

$$u_1 \geq u_2, v_1 \geq v_2 \text{ and } u_2 \geq \psi_1, v_2 \geq \psi_2.$$

Repeating this argument we get a bounded strictly decreasing sequence $\{(u_n, v_n)\} \subset E$ such that

$$\begin{aligned} z_1 &= u_0 \geq u_1 \geq u_2 ... \geq \psi_1 \geq 0, \\ z_2 &= v_0 \geq v_1 \geq v_2 ... \geq \psi_2 \geq 0, \end{aligned}$$

consequently, we deduce that the sequence $\{(u_n, v_n)\}$ converge punctually, for all x in Ω.

By going to the limit in the first equation of (2.15), we will obtain

$$\int_\Omega |\nabla u_n|^p \, dx \to \int_\Omega |\nabla u|^p \, dx,$$

which implies the strong convergence in $W^{1,p}(\Omega)$, (see [14], for a closely idea).

Indeed, by extraction of subsequence we have

$$\nabla u_n \to \nabla u \ a.e.$$

with using the dominated convergence theorem we deduce

$$\int_{\Omega} F(x, u_{n-1}, v_{n-1})\, u_n dx \to \int_{\Omega} \lambda l(x)\, u^p dx - \int_{\Omega} f_1(u,v)\, u dx - a \int_{\Omega} u^{1-\alpha_1} v^{\beta_1} dx,$$

so

$$\int_{\Omega} |\nabla u_n|^p\, dx \to \int_{\Omega} \lambda l(x)\, u^p dx - \int_{\Omega} f_1(u,v)\, u dx - a \int_{\Omega} u^{1-\alpha_1} v^{\beta_1} dx,$$

and

$$\int_{\Omega} |\nabla u|^p\, dx = \int_{\Omega} \lambda l(x)\, u^p dx - \int_{\Omega} f_1(u,v)\, u dx - a \int_{\Omega} u^{1-\alpha_1} v^{\beta_1} dx.$$

Now, one can easily deduce that, by using Hölder inequality

$$\begin{aligned} \int_{\Omega} |\nabla u|^p\, dx &= \overline{\lim} \int_{\Omega} |\nabla u_n|^p\, dx \\ &\leq \overline{\lim}\, \|\nabla u_n\|_p^{p-1} \|\nabla u_n\|_p\,. \end{aligned}$$

We divide by $\overline{\lim} \int_{\Omega} |\nabla u_n|^p\, dx$, we have immediately

$$\overline{\lim} \left(\int_{\Omega} |\nabla u_n|^p\, dx \right)^{\frac{1}{p}} \leq \|\nabla u\|_p\,,$$

which leads to the result, since by semi lower continuity for the weak topology of $L^p(\Omega)$ we already have

$$\|\nabla u\|_p \leq \underline{\lim}\, \|\nabla u_n\|_p\,.$$

To achieve the proof of this theorem we applied the same approaches to the second equation of (2.15), then we get the convergence of the sequence $\{(u_n, v_n)\}$ to the solution (u, v) of the problem (2.1). ■

Chapter 3

Infinite semipositone elliptic systems with m-equations

Contents

3.1 Introduction

In this chapter, we study the existence of positive solution to infinite semipositone systems of the form

$$\begin{cases} -\Delta_{pi} u_i = \mu_i u_i^{p_i - 1} - a_i \prod_{j=1}^{m} u_j^{\alpha_{ij}}, i = \overline{1, m}, \text{ in } \Omega \\ \\ u_i = 0, \forall i = \overline{1, m} \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ_{p_i} is the p_i-Laplace operator, and $p_i > 1, a_i, \mu_i > 0, i = \overline{1, m}$,

$-1 < \alpha_{ii} < 0$, $0 < \alpha_{ij} < 1, \forall i, j = \overline{1, m}, i \neq j$ are a positive constants.

Recently, there has appeared many important results on the study of semipositone problems (see [5, 11, 18, 19, 23]), and there are several result concerning an infinite semi positone problems have been reported, in [31] the authors have study the existence of positive solution of the equation

$$-\Delta u = \lambda \left[f(u) - u^{-\alpha} \right],$$

with Dirichlet boundary conditions where

$$\alpha \in (0, 1), f(0) \geq 0, f' > 0, \lim_{s \to \infty} \tfrac{f(s)}{s} = 0, \ g(u) = f(u) - u^{-\alpha} \text{ and } \lim_{u \to 0} g(u) = -\infty.$$

The case $g(u) = au - f(u)$ has been treated in [26] with a conditions on f, in [33] S.H. Rasouli,Z. Firouzjahi have discussed the problem

$$\begin{cases} -\Delta_p u = \lambda g(x) \left[f(u) - u^{-\alpha} \right], x \in \Omega, \\ u = 0, x \in \partial\Omega. \end{cases}$$

This equation have a positive solution where f, g verify certain conditions, we refer to [4] for corresponding result of a system intervening the p-Laplace operator, for $F(u) = \lambda l(x) u^{p-1} - f(u, v) - \frac{a}{u^{\alpha}.v^{\gamma}}$, $G(u) = \mu k(x) v^{p-1} - g(u, v) - \frac{b}{u^{\delta}.v^{\beta}}$, where $0 < \alpha, \gamma, \beta, \delta < 1$ and a, b, λ, μ are a positive constants. With a suitables conditions on f and g he proves the existence of a positive solution, we refer [35] for a result concerns the p,q-Laplace system.

Our approach is based on the method of sub and supersolution were the first eingenfunction is used to construct the sub solution of the problem 3.1.

3.2 Preliminaries

Let $\lambda_i, i = 1, m$ be the first eigenvalue of $-\Delta_{pi}, i = \overline{1,m}$ with Dirichlet boundary conditions and φ_i its corresponding positive eigenfunction.

Lemma 3.1 *There exists positive constants h_{ij} $i,j = \overline{1,m}$ such that*

$$h_{ji}^{-1}\varphi_j \leq \varphi_i \leq h_{ij}\varphi_j,\ i,j = \overline{1,m} \text{ for all } x \in \Omega, \tag{3.2}$$

with $\|\varphi_i\| = 1$, $i = \overline{1,m}$, and $M_i, \varepsilon, \upsilon_i > 0$, such that

$$\begin{aligned} &|\nabla\varphi_i| \geq M_i,\ \ i = \overline{1,m} \text{ in } \bar{\Omega} = \{x \in \Omega : d(x, \partial\Omega) \leq \varepsilon\}, \\ &\varphi_i \geq \upsilon_i > 0,\ \ i = \overline{1,m} \text{ in } \Omega\backslash\Omega_\varepsilon. \end{aligned} \tag{3.3}$$

We denote by

$$\begin{cases} A = \left(\hat{\alpha}_{ij}\right)_{i,j=\overline{1,m}}, \hat{\alpha}_{ij} = -\alpha_{ij} + \delta_{ij}(p_i - 1), \delta_{ij} = \begin{cases} 1,\ i = j. \\ 0,\ i \neq j. \end{cases}. \\ \\ \circleddash = (\theta_i)_{ij=\overline{1,m}}\ . \\ \\ P = (p_i)_{i=\overline{1,m}}\ . \end{cases}$$

Knowing that $\circleddash$ is the solution of the algebraic system

$$A\circleddash = P,\ \text{ with } \det(A) \neq 0.$$

Let

$$C=\max_{1\leq i\leq m}\left(\frac{M_i}{a_i}\left(\prod_{i\neq j=1}^{m}h_{ji}^{-\alpha_{ij}\theta_j}\right)\theta_i^{p_i-1}\left(\theta_i-1\right)\left(p_i-1\right)\right)^{-\left(\sum\limits_{j=1}^{m}\overset{\wedge}{\alpha_{ij}}\right)^{-1}}, \tag{3.4}$$

and

$$\mu_i^*=\theta_i^{p_i-1}\left(\lambda_i+\frac{M_i\left(\theta_i-1\right)\left(p_i-1\right)}{\upsilon_i}\right), i=\overline{1,m}. \tag{3.5}$$

Definition 3.1 *We called a weak positive solution $(u_i)_{i=\overline{1,m}}$ in $E=(\prod\limits_{i=1}^{m}W^{1,p_i}\left(\Omega\right))$ of (3.1) satisfy*

$$\begin{cases}\int\limits_{\Omega}\left|\nabla u_i\right|^{p_i-2}\nabla u_i.\nabla v_i dx=\mu_i\int\limits_{\Omega}u_i^{p_i-1}v_i dx-a_i\int\limits_{\Omega}\prod\limits_{j=1}^{m}u_j^{\alpha_{ij}}v_i dx, \text{ in } \Omega\\ \\ u_i=0, \text{ on } \partial\Omega,\end{cases}$$

for all functions $v_i\in W_0^{1,p_i}\left(\Omega\right)$, and $v_i\geq 0$, in Ω.

Definition 3.2 *We called a weak positive subsolution $(\psi_i)_{i=\overline{1,m}}$ and supersolution $(z_i)_{i=\overline{1,m}}$ in$\left(\prod\limits_{i=1}^{m}W^{1,p_i}\left(\Omega\right)\right)$ of (3.1) such that they satisfy $\psi_i\leq z_i$ $i=\overline{1,m}$ and*

$$\begin{cases}\int\limits_{\Omega}\left|\nabla\psi_i\right|^{p_i-2}\nabla\psi_i.\nabla w_i dx\leq\mu_i\int\limits_{\Omega}\psi_i^{p_i-1}w_i dx-a_i\int\limits_{\Omega}\prod\limits_{j=1}^{m}\psi_j^{\alpha_{ij}}w_i dx, \text{in } \Omega\\ \\ \psi_i=0, i=\overline{1,m} \text{ on } \partial\Omega,\end{cases}$$

and

$$\begin{cases}\int\limits_{\Omega}\left|\nabla z_i\right|^{p_i-2}\nabla z_i.\nabla w_i dx\geq\mu_i\int\limits_{\Omega}z_i^{p_i-1}w_i dx-a_i\int\limits_{\Omega}\prod\limits_{j=1}^{m}z_j^{\alpha_{ij}}w_i dx, \text{in } \Omega\\ \\ z_i=0, i=\overline{1,m}, \text{ on } \partial\Omega.\end{cases}$$

for all functions $w_i\in W_0^{1,p_i}\left(\Omega\right)$, and $w_i\geq 0$, in Ω.

3.3 Main results

We consider the system (3.1) under the following assumptions

$$\sum_{j=1}^{m} \alpha_{ij} < 0, \forall i = \overline{1, m}. \tag{3.6}$$

$$\det(A) \neq 0. \tag{3.7}$$

$$\theta_i > 1, \forall i = \overline{1, m}. \tag{3.8}$$

Remark 3.1 *The assumption (3.6) guarantees that*

$$\frac{1+\alpha_{ii}}{p_i^*} + \sum_{i \neq j=1}^{m} \frac{\alpha_{ij}}{p_j^*} < 1, \forall i = \overline{1, m},$$

in order that the system (3.1) to be well defined.

Let

$$F(x, U) = \quad \mu_i u_i^{p_i - 1} - a_i \prod_{j=1}^{m} u_j^{\alpha_{ij}}, i = \overline{1, m}, \text{where } U = (u_1, u_2, ..., u_m),$$

so

$$\lim_{U \to 0} F(x, U) = -\infty,$$

which referred the system as a infinite semiposition problem.

Theorem 3.1 *Let (3.3) − (3.8) hold. Then for a positive constant K such that*

$$\mu_i^* \leq \mu_i \leq a_i K^{-\sum_{j=1}^{m} \hat{\alpha}_{ij}}, \ i = \overline{1, m}, \tag{3.9}$$

the system (3.1) has a positive solution.

Proof First we construct a positive subsolution of (3.1)

Let

$$\psi_i = C \varphi_i^{\theta_i}, i = \overline{1, m}.$$

A calculation shows that

$$-\Delta_{p_i}\psi_i \;= C^{p_i-1}\theta_i^{p_i-1}\lambda_i\varphi_i^{\theta_i(p_i-1)} - C^{p_i-1}\theta_i^{p_i-1}(\theta_i-1)(p_i-1)\varphi_i^{\theta_i(p_i-1)-p_i}\left|\nabla\varphi_i\right|^{p_i}, i=\overline{1,m},$$

in $\overline{\Omega_\varepsilon}$. By using (3.3) , we have

$$-\Delta_{p_i}\psi_i \;\leq C^{p_i-1}\theta_i^{p_i-1}\lambda_i\varphi_i^{\theta_i(p_i-1)} - M_iC^{p_i-1}\theta_i^{p_i-1}(\theta_i-1)(p_i-1)\varphi_i^{\theta_i(p_i-1)-p_i}.i=\overline{1,m}, \tag{3.10}$$

and a calculation show that

$$\theta_i(p_i-1)-p_i=\sum_{j=1}^{m}\alpha_{ij}\theta_j, i=\overline{1,m}.$$

It follow that

$$-M_iC^{p_i-1}\theta_i^{p_i-1}(\theta_i-1)(p_i-1)\varphi_i^{\theta_i(p_i-1)-p_i}$$

$$\leq -a_iC^{\sum\limits_{j=1}^{m}\alpha_{ij}}\left(\prod_{i\neq j=1}^{m}h_{ji}^{\alpha_{ij}\theta_j}\right)\varphi_i^{\sum\limits_{j=1}^{m}\alpha_{ij}}$$

$$\leq -a_iC^{\sum\limits_{j=1}^{m}\alpha_{ij}}\left(\prod_{i\neq j=1}^{m}h_{ji}^{\alpha_{ij}\theta_j}\right)\prod_{j=1}^{m}\varphi_i^{\alpha_{ij}\theta_j}$$

$$\leq -a_iC^{\alpha_{ii}}\varphi_i^{\alpha_{ii}\theta_i}\left(\prod_{i\neq j=1}^{m}C^{\alpha_{ij}}\right)\left(\prod_{i\neq j=1}^{m}h_{ij}^{\alpha_{ij}\theta_j}\right)\prod_{i\neq j=1}^{m}\varphi_i^{\alpha_{ij}\theta_j}$$

$$\leq -a_iC^{\alpha_{ii}}\varphi_i^{\alpha_{ii}\theta_i}\prod_{i\neq j=1}^{m}C^{\alpha_{ij}}\left(h_{ji}\varphi_i\right)^{\alpha_{ij}\theta_j}$$

$$\leq -a_iC^{\alpha_{ii}}\varphi_i^{\alpha_{ii}\theta_i}\prod_{i\neq j=1}^{m}C^{\alpha_{ij}}\varphi_j^{\alpha_{ij}\theta_j}.$$

Then, we have

$$-M_iC_i^{p_i-1}\theta_i^{p_i-1}(\theta_i-1)(p_i-1)\varphi_i^{\theta_i(p_i-1)-p_i} \quad \leq -a_i\prod_{j=1}^{m}C^{\alpha_{ij}}\varphi_j^{\alpha_{ij}\theta_j}, i=\overline{1,m}, \tag{3.11}$$

and $(3.10), (3.11)$ imply that

$$-\Delta_{p_i}\psi_i \leq \mu_i\psi_i^{p_i-1} - a_i\prod_{j=1}^{m}\psi_j^{\alpha_{ij}}, i=\overline{1,m}.$$

Next, in $\Omega\backslash\Omega_\varepsilon$ we have $\varphi_i \geq \upsilon_i > 0$, $i = 1, m$,

and

$$\begin{aligned}
-\Delta_{p_i}\psi_i \quad &= C^{p_i-1}\theta_i^{p_i-1}\lambda_i\varphi_i^{\theta_i(p_i-1)} \\
&-C^{p_i-1}\theta_i^{p_i-1}(\theta_i-1)(p_i-1)\varphi_i^{\theta_i(p_i-1)-p_i}\left|\nabla\varphi_i\right|^{p_i}, i=\overline{1,m} \\
&\leq C^{p_i-1}\theta_i^{p_i-1}\lambda_i\varphi_i^{\theta_i(p_i-1)} \\
&\leq \left(C^{p_i-1}\theta_i^{p_i-1}\lambda_i + \tfrac{a_i}{\upsilon_i^{p_i}}\left(\prod_{j=1}^{m}C^{\alpha_{ij}}\right)\left(\prod_{i\neq j=1}^{m}h_{ji}^{\alpha_{ij}\theta_j}\right)\right)\varphi_i^{\theta_i(p_i-1)} \\
&-\tfrac{a_i\varphi_i^{p_i}}{\upsilon_i^{p_i}}\left(\prod_{j=1}^{m}C^{\alpha_{ij}}\right)\prod_{i\neq j=1}^{m}h_{ji}^{\alpha_{ij}\theta_j}\varphi_i^{\theta_i(p_i-1)-p_i} \\
&\leq \left(C^{p_i-1}\theta_i^{p_i-1}\lambda_i + \tfrac{M_i}{\upsilon_i}C^{p_i-1}\theta_i^{p_i-1}(\theta_i-1)(p_i-1)\right)\varphi_i^{\theta_1(p-1)} \\
&-a_i\left(\prod_{j=1}^{m}C^{\alpha_{ij}}\right)\prod_{i\neq j=1}^{m}h_{ij}^{\alpha_{ij}\theta_j}\varphi_i^{\sum_{j=1}^{m}\alpha_{ij}\theta_j} \\
&\leq \left(\lambda_i + \tfrac{M_i(\theta_i-1)(p_i-1)}{\upsilon_i}\right)\theta_i^{p_i-1}C^{p_i-1}\varphi_i^{\theta_i(p-1)} - a_i\prod_{j=1}^{m}C^{\alpha_{ij}}\varphi_j^{\alpha_{ij}\theta_j}, i=\overline{1,m} \\
&\leq \mu_i^*C^{p_i-1}\varphi_i^{\theta_i(p-1)} - a_i\prod_{j=1}^{m}C^{\alpha_{ij}}\varphi_j^{\alpha_{ij}\theta_j}, i=\overline{1,m}.
\end{aligned}$$

So we have

$$-\Delta_{p_i}\psi_i \quad \leq \mu_i\psi_i^{p_i-1} - a_i\prod_{j=1}^{m}\psi_j^{\alpha_{ij}}, i=\overline{1,m},$$

Hence

$$\int_{\Omega} |\nabla\psi_i|^{p_i-2} \nabla\psi_i.\nabla v_i dx \leq \mu_i \int_{\Omega} \psi_i^{p-1} v_i dx - a_i \int_{\Omega} \prod_{j=1}^{m} \psi_j^{\alpha_{ij}} v_i dx, \text{ for all } v_i \in W_0^{1,p}(\Omega), i = \overline{1,m},$$

then $(\psi_i)_{i=1,m}$ is a subsolution of (3.1).

Now we will construct a supersolution of (3.1). Let

$$\begin{aligned} & z_i = K, i = \overline{1,m}. \\ & \text{with } K \geq \max_{1\leq i\leq m} \left\{C, \tfrac{\mu_i^*}{a_i}\right\}.i = 1, m. \end{aligned} \tag{3.12}$$

We have

$$\begin{aligned} -\Delta_{p_i} z_i = \quad & 0 \geq \left(\mu_i - a_i K^{-\sum_{j=1}^{m}\hat{\alpha}_{ij}}\right) K^{p_i-1} \\ & \geq \mu_i K^{p_i-1} - a_i K^{\sum_{j=1}^{m}\alpha_{ij}} \\ & = \mu_i K^{p_i-1} - a_i \prod_{j=1}^{m} K^{\alpha_{ij}} \\ & = \mu_i z_i^{p_i-1} - a_i \prod_{j=1}^{m} z_j^{\alpha_{ij}}, \end{aligned}$$

and we have

$$\int_{\Omega} |\nabla z_i|^{p_i-2} \nabla z_i.\nabla w_i dx \leq \mu_i \int_{\Omega} z_i^{p-1} w_i dx - a_i \int_{\Omega} \prod_{j=1}^{m} z_j^{\alpha_{ij}} w_i dx, \text{ for all } w_i \in W_0^{1,p}(\Omega), i = \overline{1,m}.$$

Therefore $(z_i)_{i=\overline{1,m}}$ is a supersolution of (3.1), while the condition 3.12 guarantees that $\psi_i \leq z_i, \forall i = \overline{1,m}$.

As done in the third step of the section 3 on the previous chapter. We adapt the same procedure to complete the proof by using the convergence criteria of subsequences.

This finalizes the proof of the theorem. ■

3.4 Application

We consider the following system

$$\begin{cases} -\Delta u_1 = \mu_1 u_1 - a_1 u_1^{-\frac{1}{2}} u_2^{\frac{1}{4}} u_3^{\frac{1}{8}}, in\ \Omega \\ -\Delta u_2 = \mu_2 u_2 - a_2 u_1^{\frac{1}{4}} u_2^{-\frac{1}{2}} u_3^{\frac{1}{8}}, in\ \Omega \\ -\Delta u_3 = \mu_3 u_3 - a_3 u_1^{\frac{1}{8}} u_2^{\frac{1}{8}} u_3^{-\frac{1}{2}}, in\ \Omega \\ u_i = 0, i = 1, 3, on\ \partial\Omega. \end{cases} \tag{3.13}$$

According to the previous notations we have

$$F(x, U) = \begin{pmatrix} \mu_1 u_1 - a_1 u_1^{-\frac{1}{2}} u_2^{\frac{1}{4}} u_3^{\frac{1}{8}} \\ \mu_2 u_2 - a_2 u_1^{\frac{1}{4}} u_2^{-\frac{1}{2}} u_3^{\frac{1}{8}} \\ \mu_3 u_3 - a_3 u_1^{\frac{1}{8}} u_2^{\frac{1}{8}} u_3^{-\frac{1}{2}} \end{pmatrix},$$

$$A = \begin{pmatrix} 3/2 & -1/4 & -1/8 \\ -1/4 & 3/2 & -1/8 \\ -1/8 & -1/8 & 3/2 \end{pmatrix},$$

$\det A = \frac{413}{128}$.

A calculation show that

$$\begin{cases} \theta_1 = \frac{104}{59} \\ \\ \theta_2 = \frac{104}{59} \\ \\ \theta_3 = \frac{96}{59}. \end{cases}$$

$$C = \left(\frac{1}{M} \left(\frac{3481}{3552} \right)^{\frac{1}{10}} \max_{1 \le i \le 3} \left\{ a_1^{\frac{1}{9}}, a_2^{\frac{1}{9}}, a_3^{\frac{1}{10}} \right\} \right)^8,$$

and

$$\mu_1^* = \tfrac{104}{59} \left(\lambda_1 + \tfrac{45}{59} \tfrac{M}{\upsilon} \right).$$

$$\mu_2^* = \tfrac{104}{59} \left(\lambda_1 + \tfrac{45}{59} \tfrac{M}{\upsilon} \right).$$

$$\mu_3^* = \tfrac{96}{59} \left(\lambda_1 + \tfrac{37}{59} \tfrac{M}{\upsilon} \right).$$

Theorem 3.2 *For a positive constant K such that*

$$\mu_i^* \le \mu_i \le a_i K^{-\sum_{j=1}^{3} \overset{\wedge}{\alpha}_{ij}}, i = \overline{1,3},$$

the system (3.13) has a positive solution.

Proof Indeed, the assumptions $(3.6), (3.7), (3.8)$ and (3.9) are all satisfied

$$\sum_{j=1}^{3} \alpha_{ij} < 0.i = 1, 3,$$

$$\lim_{x \to 0} F(x, U) = -\infty,$$

$$\det A = \frac{413}{128} \ne 0,$$

and

$$\theta_1, \theta_2 \text{ and } \theta_3 > 1.$$

Then, for a some constant K such that

$$K > \max\left(C, \frac{\mu_i^*}{a_i}\right), i = 1, 3,$$

and according to the **3.1**, (3.13) has a positive solution. ■

Chapter 4

Nonlocal $p(x)$-Kirchhoff parabolic systems

Contents

The results of this chapter have been published, " **Zediri, S., Guefaifia, R., and Boulaaras**, **S**. Existence of positive solutions of a new class of nonlocal $p(x)$-kirchhoff parabolic systems via sub-super-solutions concept. Journal of Applied Analysis 26, 1 (2020), 49–58." [36]

4.1 Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids and other applications (see [1,7,8,39]). Many existence results have been obtained on this kind of problems, see for example, ([6,15,37,38]. In [13,16,17]), the regularity of solutions for differential equations with nonstandard $p(x)$-growth conditions were studied.

In this chapter, we are interested in the $p(x)$-Kirchhoff parabolic systems of the form

$$\begin{cases} \frac{\partial u}{\partial t} - M\left(I_0\left(u\right)\right)\Delta_{p(x)}u = \lambda^{p(x)}\left[\lambda_1 f\left(v\right) + \mu_1 h\left(u\right)\right], \text{ in } Q_T = \Omega\times[0,T], \\ \frac{\partial u}{\partial t} - M\left(I_0\left(v\right)\right)\Delta_{p(x)}v = \lambda^{p(x)}\left[\lambda_2 g\left(u\right) + \mu_2 \tau\left(v\right)\right], \text{ in } Q_T = \Omega\times[0,T], \\ u = v = 0, \text{ on } \partial Q_T, \\ u\left(x,0\right) = \varphi\left(x\right). \end{cases} \tag{4.1}$$

Here $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, $1 < p\left(x\right)\in C^1\left(\overline{\Omega}\right)$ is a function with

$$1 < p^- = \inf_{\Omega} p\left(x\right) \le p^* = p^- = \sup_{\Omega} p\left(x\right) < \infty,$$

the operator

$$\Delta_{p(x)}u = \operatorname{div}\left(\left|\nabla u\right|^{p(x)-2}\nabla u\right),$$

is called $p(x)$-Laplacian, λ, λ_1, λ_2, μ_1 and μ_2 are positive parameters,

$$I_0(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx,$$

and $M(t)$ is a continuous function.

Problem (4.1) is a generalization of a model introduced by Kirchhoff [11]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right| dx \right) \frac{\partial^2 u}{\partial x^2}, \tag{4.2}$$

where ρ, P_0, h, E and L are constants,which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. In recent years, problems involving Kirchhoff-type operators have been studied in many papers (see [2, 9, 12, 20]) in which the authors have used variational and topological methods to get the existence of solutions.

In this chapter, motivated by the ideas introduced in [27] and the properties of Kirchhoff-type operators in [27], we study the existence of positive solutions for system (4.1) by using the sub and super-solutions technique.

This is a new research topic for nonlocal problems. The following is organized as follows. In Section 4.2, we present some preliminary results on the variable exponent Sobolev space $W_0^{1p(x)}(\Omega)$, properties of the p(x) −Kirchhoff-Laplacian operator and the method of sub and super-solutions. Section 4. 3 is devoted to state and prove the main result.

4.2 Preliminary results

In order to discuss problem (4.1), we need some theories on $W_0^{1p(x)}(\Omega)$, which we will call variable exponent Sobolev spaces. Firstly we state some basic properties of the spaces $W_0^{1p(x)}(\Omega)$ which will be used later (for details, see [28]).

Let us define

$$L^{p(x)}(\Omega) = \left\{ u, u \text{ is a measurable real-valued function such that } \int_{\Omega} |u|^{p(x)}\, dx < \infty \right\}.$$

We introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u(x)|_{p(x)} = \inf \left\{ \lambda > 0, \int_{\Omega} \left| \frac{u(x)}{\lambda} \right| dx \leq 1 \right\},$$

and we introduce

$$W_0^{1p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)} \text{ for all } u \in W_0^{1p(x)}(\Omega).$$

We denote by $W_0^{1p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1p(x)}(\Omega)$.

Proposition 4.1 *The spaces $L^{p(x)}(\Omega)$, $W^{1p(x)}(\Omega)$ and $W_0^{1p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Now we mention some properties of the $p(x)-$Kirchhoff$-$Laplace operator. For each $u \in X = W^{1p(x)}(\Omega)$, define

$$\phi(u) = \widehat{M}\left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right),$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$ and M is a continuous and increasing function on R_+ and its values are completely positive.denote by $u_n \rightharpoonup u$ and $u_n \to u$ the weak convergence and strong convergence of sequence $\{u_n\}$ in X, respectively. the Gâteaux derivative at the point $u \in X$ of ϕ is the functional $\phi'(u) \in X^*$, given by

$$\langle \phi'(u), v \rangle = M\left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) . \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v(x) dx,$$

$\langle .,. \rangle$ is the duality pairing between X and X^*.

Lemma 4.1 *(i) $\phi^{'} : X \to X^*$ is a continuous, bounded and strictly monotone operator.*

(ii) $\phi^{'}$ is a mapping of type (S_+), i.e. if $u_n \rightharpoonup u$ in X and

$\lim\limits_{n\to+\infty} \langle \phi^{'}(u_n) - \phi^{'}(u), u_n - u \rangle \leq 0$, then $u_n \to u$ in X.

(iii) $\phi^{'}(u) : X \to X^*$ is a homeomorphism.

We using the Euler time scheme of problem (4.1), we obtain the following problems:

$$\begin{cases} u_k - \tau' M\left(I_0\left(u_k\right)\right) \Delta_{p(x)} u_k = \tau' \lambda^{p(x)} \left[\lambda_1 f\left(v\right) + \mu_1 h\left(u_k\right)\right] + u_{k-1}, \text{ in } \Omega, \\ \\ u_k - \tau' M\left(I_0\left(v\right)\right) \Delta_{p(x)} v = \tau' \lambda^{p(x)} \left[\lambda_2 g\left(u_k\right) + \mu_2 \tau\left(v\right)\right] + u_{k-1}, \text{ in } \Omega, \\ \\ u_k = v = 0, \text{ on } \partial\Omega, \\ \\ u_0 = \varphi_0\left(x\right) \text{ in } \Omega, \end{cases} \tag{4.3}$$

for $1 \leq k \leq N$ and where $N\tau' = T$, $0 < \tau' < 1$.

Throughout the paper, we will assume the following conditions:

$(H1)$ $M : [0, +\infty) \to [m_0, m_\infty]$ is a continuous and increasing function with $m_0 > 0$.

$(H2)$ $p \in \mathcal{C}^1\left(\overline{\Omega}\right)$, $1 < p^- \leq p^+$.

$(H3)$ f, g, h and τ are monotone $\mathcal{C}^1$ functions such that

$$\lim_{u\to\infty} f\left(u_k\right) = +\infty,$$

$$\lim_{u\to\infty} g\left(u_k\right) = +\infty,$$

$$\lim_{u\to\infty} h\left(u_k\right) = +\infty,$$

$$\lim_{u\to\infty} \tau\left(u_k\right) = +\infty,$$

$(H4)$ One has

$$\lim_{u\to+\infty}\frac{f\left(Lg\left(u_k\right)^{1/(p^--1)}\right)}{u_k^{p^--1}}=0 \text{ for all } L>0.$$

$(H5)$ One has

$$\lim_{u\to+\infty}\frac{h\left(u_k\right)}{u_k^{p^--1}}=0.$$

$(H6)$ One has

$$\lim_{u\to+\infty}\frac{\tau\left(u_k\right)}{u_k^{p^--1}}=0.$$

Definition 4.1 *If $u_k, v\in W_0^{1p(x)}\left(\Omega\right)$, we say that*

$$-M\left(I_0\left(u_k\right)\right)\Delta_{p(x)}u_k\leq -M\left(I_0\left(v\right)\right)\Delta_{p(x)}v.$$

If for all $\varphi\in W_0^{1p(x)}\left(\Omega\right)$ with $\varphi\geq 0$ one has

$$M\left(I_0\left(u_k\right)\right)\int_\Omega\left|\nabla u_k\right|^{p(x)-2}\nabla u_k.\nabla\varphi dx\leq M\left(I_0\left(v\right)\right)\int_\Omega\left|\nabla v\right|^{p(x)-2}\nabla v.\nabla\varphi dx,$$

where

$$I_0\left(u_k\right)=\int_\Omega\frac{1}{p\left(x\right)}\left|\nabla u_k\right|^{p(x)}dx.$$

Definition 4.2 *(i) If $u_k, v\in W_0^{1p(x)}\left(\Omega\right)$, then (u_k, v) is called a weak solution of (4.1) if it satisfies*

$$M\left(I_0\left(u_k\right)\right)\int_\Omega\left|\nabla u_k\right|^{p(x)-2}\nabla u_k.\nabla\varphi dx=\int_\Omega\left[\lambda^{p(x)}\left[\lambda_1 f\left(v\right)+\mu_1 h\left(u_k\right)\right]-\tfrac{u_k-u_{k-1}}{\tau'}\right]\varphi dx,$$

$$M\left(I_0\left(v\right)\right)\int_\Omega\left|\nabla v\right|^{p(x)-2}\nabla v.\nabla\varphi dx=\int_\Omega\left[\lambda^{p(x)}\left[\lambda_2 g\left(u_k\right)+\mu_2\tau\left(v\right)\right]-\tfrac{u_k-u_{k-1}}{\tau'}\right]\varphi dx,$$

for all $\varphi\in W_0^{1p(x)}\left(\Omega\right)$ with $\varphi\geq 0$.

(ii) We say that (u_k, v) is called a sub-solution (resp. a super-solution) of (4.1) if

$$M\left(I_0\left(u_k\right)\right) \int\limits_{\Omega} \left|\nabla u_k\right|^{p(x)-2} \nabla u_k . \nabla \varphi dx \leq \int\limits_{\Omega} \left[\lambda^{p(x)}\left[\lambda_1 f\left(v\right)+\mu_1 h\left(u_k\right)\right]-\frac{u_k-u_{k-1}}{\tau'}\right] \varphi dx, \text{ (resp. } \geq \text{),}$$

$$M\left(I_0\left(v\right)\right) \int\limits_{\Omega} \left|\nabla v\right|^{p(x)-2} \nabla v . \nabla \varphi dx \leq \int\limits_{\Omega} \left[\lambda^{p(x)}\left[\lambda_2 g\left(u_k\right)+\mu_2 \tau\left(v\right)\right]-\frac{u_k-u_{k-1}}{\tau'}\right] \varphi dx, \text{ (resp. } \geq \text{).}$$

Lemma 4.2 *(Comparison principle [28]). Let u_k , $v \in W_0^{1p(x)}(\Omega)$ and let (H1) hold. If*

$$-M\left(I_0\left(u_k\right)\right) \Delta_{p(x)} u_k \leq -M\left(I_0(v)\right) \Delta_{p(x)} v,$$

and $(u_k-v)^+ \in W_0^{1p(x)}(\Omega)$, then $u_k \leq v$ in Ω.

Lemma 4.3 *(see [28]) Let (H1) hold let $\mu > 0$ and let u_k be the unique solution of the problem*

$$\begin{cases} -M\left(I_0\left(u_k\right)\right) \operatorname{div}\left|\nabla u_k\right|^{p(x)-2} \nabla u_k=\mu \text{ in } \Omega, \\ u_k=0, \text{ on } \partial \Omega. \end{cases}$$

Set

$$h=\frac{m_0 p^{-}}{2|\Omega|^{\frac{1}{N}} C_0}.$$

When $\mu \geq h$, we have

$$\left|u_k\right|_{\infty} \leq C^* \mu^{\frac{1}{p^{-}-1}},$$

and when $\mu < h$, we have

$$\left|u_k\right|_{\infty} \leq C_* \mu^{\frac{1}{p^{+}-1}},$$

where C^* and C_* are positive constants depending on $p^-, p^+, N, |\Omega|, C_0$ and m_0.

Here and hereafter, we use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$. Set $d(x) = d(x, \partial\Omega)$ and

$$\partial \Omega_{\varepsilon}=\{x \in \Omega: d(x, \partial \Omega)<\varepsilon\}.$$

Since $\partial\Omega$ is C^2 regularly, there exists a constant $\delta \in (0, 1)$ such that $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$ and

$|\nabla d(x)| = 1$.

Set

$$v_1 = \begin{cases} \gamma d(x), & d(x) < \delta, \\ \gamma\delta + \int\limits_{\delta}^{d(x)} \gamma \left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} (\lambda_1 + \mu_1)^{\frac{2}{p^- - 1}} dt, & \delta \le d(x) < 2\delta, \\ \gamma\delta + \int\limits_{\delta}^{2\delta} \gamma \left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} (\lambda_1 + \mu_1)^{\frac{2}{p^- - 1}} dt, & 2\delta \le d(x), \end{cases}$$

$$v_2 = \begin{cases} \gamma d(x), & d(x) < \delta, \\ \gamma\delta + \int\limits_{\delta}^{d(x)} \gamma \left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} (\lambda_2 + \mu_2)^{\frac{2}{p^- - 1}} dt, & \delta \le d(x) < 2\delta, \\ \gamma\delta + \int\limits_{\delta}^{2\delta} \gamma \left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} (\lambda_2 + \mu_2)^{\frac{2}{p^- - 1}} dt, & 2\delta \le d(x). \end{cases}$$

0bviously, $0 \le v_1(x), v_2(x) \in C^1\left(\overline{\Omega}\right)$. Considering

$$\begin{aligned} & -M\Big(\int\limits_{\Omega} \frac{1}{p(x)} |\nabla \omega|^{p(x)} dx\Big)\Delta_{p(x)}\omega(x) = \eta, \text{ in } \Omega, \\ & \omega = 0, \text{ on } \partial\Omega, \end{aligned} \tag{4.4}$$

we have the following result

Lemma 4.4 *(see [27]) If the positive parameter η is large enough and ω is the unique solution of (4.4), then we have the following assertions*

(i) For any $\theta \in (0,1)$ there exists a positive constant C_1 such that

$$C_1 \eta^{\frac{1}{p^+ - 1 + \theta}} \le \max_{x \in \overline{\Omega}} \omega(x).$$

(ii) There exists a positive constant C_2 such that

$$\max_{x \in \overline{\Omega}} \omega(x) \le C_2 \eta^{\frac{1}{p^- - 1}}.$$

4.3 Main result

In the following, when there is no misunderstanding, we always use C_i to denote positive constants.

Theorem 4.1 *Assume that conditions (H_1)-(H_6) are satisfied. Then problem (4.3) has a positive solution when λ is large enough.*

Proof We shall establish **Theorem** 4.1 by constructing a positive sub-solution (ϕ_k, ϕ_1) and a super-solution (z_k, z_1) of (4.1) such that $\phi_k \leq z_k$ and $\phi_1 \leq z_1$, that is (ϕ_k, ϕ_1) and (z_k, z_1) satisfy

$$\begin{cases} M\left(I_0\left(\phi_k\right)\right) \int\limits_\Omega \left|\nabla \phi_k\right|^{p(x)-2} \nabla \phi_k . \nabla q dx \leq \int\limits_\Omega \left[\lambda^{p(x)}\left[\lambda_1 f\left(\phi_1\right)+\mu_1 h\left(\phi_k\right)\right]-\frac{\phi_k-\phi_{k-1}}{\tau'}\right] q dx, \\ M\left(I_0\left(\phi_1\right)\right) \int\limits_\Omega \left|\nabla \phi_1\right|^{p(x)-2} \nabla \phi_1 . \nabla q dx \leq \int\limits_\Omega \left[\lambda^{p(x)}\left[\lambda_2 g\left(\phi_k\right)+\mu_2 \tau\left(\phi_1\right)\right]-\frac{\phi_k-\phi_{k-1}}{\tau'}\right] q dx, \end{cases}$$

and

$$\begin{cases} M\left(I_0\left(z_k\right)\right) \int\limits_\Omega \left|\nabla z_k\right|^{p(x)-2} \nabla z_k . \nabla q dx \geq \int\limits_\Omega \left[\lambda^{p(x)}\left[\lambda_1 f\left(z_1\right)+\mu_1 h\left(z_k\right)\right]-\frac{z_k-z_{k-1}}{\tau'}\right] q dx, \\ M\left(I_0\left(z_1\right)\right) \int\limits_\Omega \left|\nabla z_1\right|^{p(x)-2} \nabla z_1 . \nabla q dx \geq \int\limits_\Omega \left[\lambda^{p(x)}\left[\lambda_2 g\left(z_k\right)+\mu_2 \tau\left(z_1\right)\right]-\frac{z_k-z_{k-1}}{\tau'}\right] q dx, \end{cases}$$

for all $q \in W^{1,p(x)}(\Omega)$ with $q \geq 0$. According to the sub-super-solution method for $p(x)-$ Kirchhoff-type equations (see [28]), we obtain that (4.1) has a positive solution.

Step1: We will construct a sub-solution of (4.1). Let $\sigma \in (0, \delta)$ be small enough.

Set

$$\phi_k(x) = \begin{cases} e^{k'd(x)} - 1, & d(x) < \sigma, \\ e^{k'\sigma} - 1 + \int_{\sigma}^{d(x)} k' e^{k'\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} (\lambda_1 + \mu_1)^{\frac{2}{p^- - 1}} \, dt, & \sigma \leq d(x) < 2\delta, \\ e^{k'\sigma} - 1 + \int_{\sigma}^{2\delta} k' e^{k'\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} (\lambda_1 + \mu_2)^{\frac{2}{p^- - 1}} \, dt, & 2\delta \leq d(x), \end{cases}$$

and

$$\phi_1 = \begin{cases} e^{k'd(x)} - 1, & d(x) < \sigma, \\ e^{k'\sigma} - 1 + \int_{\sigma}^{d(x)} k' e^{k'\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} (\lambda_2 + \mu_2)^{\frac{2}{p^- - 1}} \, dt, & \sigma \leq d(x) < 2\delta, \\ e^{k'\sigma} - 1 + \int_{\sigma}^{2\delta} k' e^{k'\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} (\lambda_2 + \mu_2)^{\frac{2}{p^- - 1}} \, dt, & 2\delta \leq d(x). \end{cases}$$

It is easy to see that $\phi_k, \phi_1 \in \mathcal{C}^1\left(\overline{\Omega}\right)$. Set

$$\alpha = \min\left[\frac{\inf p(x) - 1}{4(\sup|\nabla p(x)| + 1)}, 1\right],$$

$$\zeta = \min\left\{\lambda_1 f(0) + \mu_1 h(0), \lambda_2 g(0) + \mu_2 \tau(0), -1\right\}.$$

By some a computations, we can obtain

$$-\Delta_{p(x)}\phi_k = \begin{cases} -k'\left(k'e^{k'd(x)}\right)^{p(x)-1}\left[(p(x)-1)+(d(x)+\frac{\ln k'}{k'})\nabla p\nabla d+\frac{\Delta d}{k'}\right], & d(x)<\sigma, \\ \frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^{-}-1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln k'e^{k'\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}\nabla p\nabla d+\Delta d\right] & \\ \times\left(ke^{k\sigma}\right)^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^{-}-1}-1}(\lambda_1+\mu_1), & \sigma\le d(x)<2\delta, \\ 0, & 2\delta\le d(x), \end{cases}$$

and

$$-\Delta_{p(x)}\phi_1 = \begin{cases} -k'\left(k'e^{k'd(x)}\right)^{p(x)-1}\left[(p(x)-1)+(d(x)+\frac{\ln k'}{k'})\nabla p\nabla d+\frac{\Delta d}{k'}\right], & d(x)<\sigma, \\ \frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^{-}-1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln k'e^{k'\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}\nabla p\nabla d+\Delta d\right] & \\ \times\left(ke^{k\sigma}\right)^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^{-}-1}-1}(\lambda_2+\mu_2), & \sigma\le d(x)<2\delta, \\ 0, & 2\delta\le d(x). \end{cases}$$

By $(H4)$, there exists a positive constant $L>1$ such that

$$f(L-1)\ge 1,$$

$$g(L-1)\ge 1,$$

$$h(L-1)\ge 1,$$

$$\tau(L-1)\ge 1.$$

Let $\sigma=\frac{1}{k'}\ln L$. Then

$$\sigma k' = \ln L. \tag{4.5}$$

If k' is sufficiently large, form (4.5) we have

$$-\Delta_{p(x)}\phi_1 \leq -k'\alpha, d\left(x\right) < \sigma. \tag{4.6}$$

Let $\frac{\lambda\zeta}{m_\infty} = k'\alpha$.Then

$$k'^{p(x)}\alpha \geq -\lambda^{p(x)}\frac{\zeta}{m_\infty}.$$

From (4.6) we have

$$\begin{aligned}
-M(I_0(\phi_k))\Delta_{p(x)}\phi_k &\leq M(I_0(\phi_1))\lambda^{p(x)}\tfrac{\zeta}{m_\infty} \\
&\leq \lambda^{p(x)}\zeta \\
&\leq \lambda^{p(x)}(\lambda_1 f(0) + \mu_1 h(0)) \\
&\lambda^{p(x)}(\lambda_1 f(\phi_1) + \mu_1 h(\phi_k)) - \tfrac{\phi_k - \phi_{k-1}}{\tau'}, d(x) < \sigma.
\end{aligned}$$

Since $d\left(x\right) \in C^2\left(\overline{\partial\Omega_{3\delta}}\right)$, there exists a positive constant C_3 such that

$$\begin{aligned}
-M(I_0(\phi_k))\Delta_{p(x)}\phi_k &\leq m_\infty (ke^{k\sigma})^{p(x)-1}\left(\tfrac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^- -1}-1}(\lambda_1+\mu_1) \\
&\times\left|\tfrac{1}{2\delta-1}\tfrac{2(p(x)-1)}{p^- -1} - \left(\tfrac{2\delta-d}{2\delta-\sigma}\right)\left[(\ln k' e^{k'\sigma})\left(\tfrac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^- -1}}\nabla p\nabla d + \Delta d\right]\right| \\
&\leq C_3 m_\infty (ke^{k\sigma})^{p(x)-1}(\lambda_1 a_1 + \mu_1 c_1)\ln k', \sigma \leq d\left(x\right) < 2\delta.
\end{aligned}$$

If k' is sufficiently large, letting $\frac{\lambda\zeta}{m_\infty} = k'\alpha$, then we have

$$C_3 m_\infty (ke^{k\sigma})^{p(x)-1}(\lambda_1 a_1 + \mu_1 c_1)\ln k' \quad = C_3 m_\infty (kL)^{p(x)-1}(\lambda_1 + \mu_1)\ln k'$$

$$\leq \lambda^{p(x)}\left(\lambda_1 + \mu_1\right) - \tfrac{\phi_k - \phi_{k-1}}{\tau'}.$$

Then

$$-M(I_0(\phi_k))\Delta_{p(x)}\phi_k \leq \lambda^{p(x)}\left(\lambda_1 + \mu_1\right) - \tfrac{\phi_k - \phi_{k-1}}{\tau'}, \sigma \leq d\left(x\right) < 2\delta. \tag{4.7}$$

Since $\phi_k\left(x\right), \phi_1\left(x\right), f$ and h are monotone, when λ is large enough, we have

$$-M\left(\int_\Omega \tfrac{1}{p(x)}\left|\nabla\phi_k\right|^{p(x)} dx\right)\Delta_{p(x)}\phi_k\left(x\right) \leq \lambda^{p(x)}\left(\lambda_1 f\left(\phi_1\right) + \mu_1 h\left(\phi_k\right)\right) - \tfrac{\phi_k - \phi_{k-1}}{\tau'}, \sigma \leq d\left(x\right) < 2\delta,$$

and

$$-M(I_0(\phi_k))\Delta_{p(x)}\phi_k = 0 \quad \leq \lambda^{p(x)}\left(\lambda_1 f\left(\phi_1\right) + \mu_1 h\left(\phi_k\right)\right) - \tfrac{\phi_k - \phi_{k-1}}{\tau'}, 2\delta \leq d\left(x\right). \tag{4.8}$$

Combining (4.7) and (4.8),we can conclude that

$$-M(I_0(\phi_k))\Delta_{p(x)}\phi_k \leq \lambda^{p(x)}\left(\lambda_1 f\left(\phi_1\right) + \mu_1 h\left(\phi_k\right)\right) - \tfrac{\phi_k - \phi_{k-1}}{\tau'} \text{ a.e in } \Omega. \tag{4.9}$$

Similarly,

$$-M(I_0(\phi_1))\Delta_{p(x)}\phi_1 \leq \lambda^{p(x)}\left(\lambda^{p(x)}\left[\lambda_2 g\left(\phi_k\right) + \mu_2 \tau\left(\phi_1\right)\right]\right) - \tfrac{\phi_k - \phi_{k-1}}{\tau'} \text{ a.e in } \Omega. \tag{4.10}$$

From (4.9) and (4.10) we can see that (ϕ_k, ϕ_1) is a sub-solution of problem (4.3).

Step 2: We will construct a super-solution of problem (4.3). We consider

$$\begin{cases} -M\left(I_0\left(z_k\right)\right)\Delta_{p(x)}z_k & = \frac{\lambda^{p^+}}{m_0}(\lambda_1+\mu_1)\mu - \frac{z_k - z_{k-1}}{\tau'}, \text{ in } \Omega \\ M\left(I_0\left(z_1\right)\right)\Delta_{p(x)}z_1 & = \frac{\lambda^{p^+}}{m_0}(\lambda_2+\mu_2)g\left(\beta\left(\lambda^{p^+}(\lambda_1+\mu_1)\mu\right)\right) - \frac{z_k - z_{k-1}}{\tau'}, \text{ in } \Omega, \\ z_k = z_1 & = 0, \text{on } \partial\Omega, \end{cases}$$

where

$$\beta = \beta\left(\lambda^{p^+}(\lambda_1+\mu_1)\mu\right) = \max_{x\in\overline{\Omega}} z_k\left(x\right).$$

We shall prove that (z_k, z_1) is a super-solution of problem (4.3).

For $q \in W_0^{1,p(x)}(\Omega)$with $q \geq 0$, it easy to see that

$$\begin{aligned} M\left(I_0\left(z_1\right)\right)\int_\Omega \left|\nabla z_1\right|^{p(x)-2}\nabla z_1.\nabla q dx & = \frac{1}{m_0}M\left(I_0\left(z_1\right)\right)\int_\Omega \lambda^{p^+}(\lambda_2+\mu_2)g\left(\beta\left(\lambda^{p^+}(\lambda_1+\mu_1)\mu\right)\right)q dx \\ & \geq \int_\Omega \lambda^{p^+}(\lambda_2 b\left(x\right)g\left(z_k\right)q dx + \int_\Omega \lambda^{p^+}\mu_2 d\left(x\right)g\left(\beta\left(\lambda^{p^+}(\lambda_1+\mu_1)\mu\right)\right)q dx \end{aligned}$$

By (H6), for μ large enough and using **Lemma 4.4,** we have

$$g\left(\beta\left(\lambda^{p^+}(\lambda_1+\mu_1)\mu\right)\right) \geq \tau(C_2\left[\lambda^{p^+}(\lambda_2+\mu_2)g(\beta\left(\lambda^{p^+}(\lambda_1+\mu_1)\mu\right)\right]^{\frac{1}{p^- -1}}) \geq \tau\left(z_1\right).$$

Hence,

$$\begin{aligned} & M\left(I_0\left(z_1\right)\right)\int_\Omega \left|\nabla z_1\right|^{p(x)-2}\nabla z_1.\nabla q dx \\ & \geq \int_\Omega \lambda^{p^+}(\lambda_2 g\left(z_k\right)q dx + \int_\Omega \lambda^{p^+}\mu_2 \tau\left(z_1\right)q dx - \int_\Omega \frac{z_k - z_{k-1}}{\tau'}q dx. \end{aligned} \tag{4.11}$$

Also,

$$M\left(I_0\left(z_k\right)\right)\int\limits_{\Omega}\left|\nabla z_k\right|^{p(x)-2}\nabla z_k.\nabla q dx \;=\frac{1}{m_0}M\left(I_0\left(z_k\right)\right)\int\limits_{\Omega}\lambda^{p^+}(\lambda_1+\mu_1)\mu q dx$$

$$\geq\int\limits_{\Omega}\lambda^{p^+}(\lambda_1+\mu_1)\mu q dx.$$

By (H4), (H5) and **Lemma 4.4**, when μ is sufficiently large, we have

$$(\lambda_1+\mu_1)\mu \;\geq\frac{1}{\lambda^{p^+}}\left[\frac{1}{C_2}\beta(\lambda^{p^+}(\lambda_1+\mu_1)\mu)\right]^{p^--1}$$

$$\geq\mu_1 h(\beta(\lambda^{p^+}(\lambda_1+\mu_1)\mu))+\lambda_1 f(C_2\left[\lambda^{p^+}(\lambda_2+\mu_2)g(\beta\left(\lambda^{p^+}(\lambda_1+\mu_1)\mu\right)\right)\right]^{\frac{1}{p^--1}}).$$

Then

$$M\left(I_0\left(z_k\right)\right)\int\limits_{\Omega}\left|\nabla z_k\right|^{p(x)-2}\nabla z_k.\nabla q dx\geq\int\limits_{\Omega}\lambda^{p^+}\lambda_1 f(z_1)q dx+\int\limits_{\Omega}\lambda^{p^+}\mu_1 h(z_k)q dx-\int\limits_{\Omega}\frac{z_k-z_{k-1}}{\tau'}q dx. \tag{4.12}$$

According to (4.11) and (4.12), we can conclude that (z_k, z_1) is a super-solution of problem (4.3). It only remains to prove that

$$\phi_k\leq z_k \text{ and } \phi_1\leq z_1.$$

In the definition of $v_1\left(x\right)$, let

$$\gamma=\frac{2}{\delta}(\max_{\overline{\Omega}}\phi_k\left(x\right)+\max_{\overline{\Omega}}\left|\nabla\phi_k\right|\left(x\right)).$$

We claim that

$$\phi_k(x)\leq v_1\left(x\right) \text{ for all } x\in\Omega. \tag{4.13}$$

From the definition of v_1 it is easy to see that for $d(x)=\delta$, we have

$$\phi_k(x)\leq 2\max_{\overline{\Omega}}\phi_k\left(x\right)\leq v_1\left(x\right),$$

for $d(x) \geq \delta$ we have

$$\phi_k(x) \leq 2\max_{\overline{\Omega}} \phi_k(x) \leq v_1(x),$$

and for $d(x) < \delta$ we have

$$\phi_k(x) \leq v_1(x).$$

Since $v_1 - \phi_k \in \mathcal{C}^1\left(\overline{\partial\Omega_\delta}\right)$, there exists a point $x_0 \in \overline{\partial\Omega_\delta}$ such that

$$v_1(x_0) - \phi_k(x_0) = \min_{x \in \overline{\partial\Omega_\delta}} (v_1(x) - \phi_k(x)).$$

If $v_1(x_0) - \phi_k(x_0) < 0$, it easy to see that $0 < d(x_0) < \delta$, and then

$$\nabla v_1(x_0) - \nabla \phi_k(x_0) = 0.$$

From the definition of v_1 we have

$$|\nabla v_1(x_0)| = \gamma = \frac{2}{\delta}(\max_{\overline{\Omega}} \phi_k(x) + \max_{\overline{\Omega}} |\nabla \phi_k|(x)) > |\nabla \phi_k|(x_0).$$

This is contradiction to

$$\nabla v_1(x_0) - \nabla \phi_k(x_0) = 0.$$

Thus (4.13) is valid.

Obviously, there exists a positive constant C_3 such that

$$\gamma \leq C_3 \lambda.$$

Since $d(x) \in \mathcal{C}^1\left(\overline{\partial\Omega_\delta}\right)$. According to the proof of **Lemma 4.4**, there exists a positive constant C_4 such that

$$M(I_0(v_1))\,\Delta_{p(x)} v_1(x) \leq C_* \gamma^{p(x)-1} \leq C_4 \lambda^{p(x)-1+\theta} \text{ a.e in } \Omega,$$

where $\theta \in (0,1)$. When $\eta \geq \lambda^{p^+}$ is large enough, we have

$$-\Delta_{p(x)} v_1(x) \leq \eta.$$

According to the comparison principle, we have

$$v_1(x) \leq \omega(x) \text{ for all } x \in \Omega. \tag{4.14}$$

From (4.13) and (4.14), when $\eta \geq \lambda^{p^+}$ and $\lambda \geq 1$ is sufficiently large, we have

$$\phi_k(x) \leq v_1(x) \leq \omega(x), \text{ for all } x \in \Omega. \tag{4.15}$$

According to the comparison principle, when η is large enough, we have

$$v_1(x) \leq \omega(x) \leq z_k(x), \text{ for all } x \in \Omega.$$

Combining the definition of $v_1(x)$ and (4.15) it easy to see that

$$\phi_k(x) \leq v_1(x) \leq \omega(x) \leq z_k(x) \text{ for all } x \in \Omega.$$

When $\mu \geq 1$ and λ is large enough, from **Lemma 4.4** we can see that $\beta\left(\lambda^{p^+}(\lambda_1+\mu_1)\mu\right)$ is large enough. Then

$$\frac{\lambda^{p^+}}{m_0}(\lambda_2+\mu_2)g\left(\beta\left(\lambda^{p^+}(\lambda_1+\mu_1)\mu\right)\right),$$

is large enough. Similarly, we have $\phi_1 \leq z_1$. This completes the proof. ■

Conclusion

The method of sub and supersolution deals with the question of existence of positive solutions of nonvariational problems with different types of nonlinearity.

The results obtained in this work can be generalized in fractional elliptic problems, and we have aspirations to apply the fibering map approch for some elliptic problems invilving the p and $p(x)-$Laplace operators.

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