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Theme

Study of The Existence and Blow-up For a Class of Nonlinear Damped Wave Equation

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Dedication

To my lovely Father, may God bless his soul,

To my beloved Mother; I dedicate this work.

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I would like to thank ALLAH above all for giving me the strength, courage, patience, and will to carry out this work. Prayers and peace be upon our Prophet Muhammad, may God bless him and grant him peace.

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Contents

Abstract

This dissertation is devoted to study the well-posedness and the blow-up of solutions of some nonlinear hyperbolic problems involving non-classical nonlinearities. We proved under suitable assumptions on the exponents of nonlinearity the local, global existence and establish the results of blow-up of some wave equations.

It seems that the source term inhibits the global existence (in time) of the solution of the problem is to say that the energy of the problem (or solution) tends to infinity for the norm of space when t tends to a finite time T. Obviously, the damping term stabilizes the solution of the problem, and it is clear that in the absence of source terms, if the solution exists locally, we can always expand it into a global solution. This interaction between source and damping terms has been a target in many studies and is stills -It is important also to know which term is dominant to the other-.

We can say that our research is an expansion of some results done by previous researchers. Mainly, by making appropriate modifications, we extended some known results of some nonlinear wave equations with constant and variable-exponent nonlinearities studied by Messaoudi, and exploit ideas by Georgiev and Todorova.

Keywords and Phrases: Blow-up, global existence, source term, wave equation, viscosity, negative initial energy, variable exponents, positive initial energy, existence and uniqueness, Faedo-Galerkin.

الملخـــــــــص

هـذه الرسـالة مخصصـة لدراسـة وجـود و انفجـار الحلـول لـبعض المسـائل غيـر الخطيـة الزائديـة التـي تتضـمن حـدود غيـر خطيـة غيـر كلاسـيكية ، حيـث أثبتنـا فـي ظـل فرضـيات مناسـبة على الأسس اللاخطية الوجود المحلي، الكلي و نتائج انفجار بعض معادلات الموجات.

يبـدو أن حـد المصـدر يثـبط الوجـود الكلـي (فـي الوقـت المناسـب) لحـل المسـألة وهـو القول بـأن طاقـة المسـألة (أو الحـل) تـؤول إلـى اللانهايـة لنظـيم الفضـاء عنـدما تـؤول t إلـى وقـت محدود T.

ومـن الواضـح أيضـا أن حـد التخامـد يعمـل علـى اسـتقرار حـل المسـألة حيـث نـرى أنـه فـي حالـة عـدم وجـود حـدود المصـدر، إذا كـان الحـل موجـود محليـا ، فيمكننـا دائمـا توسـيعه إلـى حـل كلـي. كـان هـذا التفاعـل بـين حـدود المصـدر والتخامـد هـدفا فـي العديـد مـن الدراسـات ولايـزال كذلك – من المهم معرفة الحد المسيطر على الآخر-.

يمكننـا القـول أن بحثنـا هـو توسـيع لـبعض النتـائج التـي قـام بهـا مؤلفـون سـابقون. علـى وجـه الخصـوص ، مـن خـلال فـرض التعـديلات المناسـبة، قمنـا بتوسـيع بعـض النتـائج المعروفـة لـبعض معـادلات الموجـات اللاخطيـة مـع اللاخطيـة ذات الأس الثابـت أو المتغيـر التـي درسـها مسـعودي واستغلال طريقة جورجييف وتودوروفا.

الكلمات المفتاحية والعبارات: الإنفجار، الوجود الكلي ، حد القوة الخارجية، معادلة الموجات، اللزوجة ، الطاقة الأولية السلبية ، الأسس المتغيرة ، الطاقة الابتدائية الموجبة ، الوجود والوحدانية ، فايدوغالركين.

.

Résumé

Cette thèse consacrée à l'étude d'existence et l'explosion des solutions de quelques problèmes hyperboliques non linéaires impliquant des non-linéarités non classiques. Nous avons prouvé sous des hypothèses appropriées sur les exposants de la non-linéarité l'existence locale, globale et établissons les résultats d'explosion de certaines équations des ondes.

Il semble que le terme source inhibe l'existence globale (en temps) de la solution du problème c'est-à-dire que l'énergie du problème (ou de la solution) tend vers l'infini pour la norme de l'espace lorsque t tend vers un temps finis T. Évidemment, le terme d'amortissement stabilise la solution du problème, et il est clair qu'en l'absence de termes sources, si la solution existe localement, on peut toujours l'étendre en une solution globale. Cette interaction entre les termes source et amortissement a été un but dans de nombreuses études et elle l'est toujours -Il est également important de savoir quel terme est dominant par rapport à l'autre- .

Nous pouvons dire que notre recherche est une expansion de certains résultats réalisés par des auteurs antérieurs. En particulier, en imposant des modifications appropriées, nous avons développé certains résultats connus de certaines équations d'onde non linéaires avec des non-linéarités à exposant constant et variable étudiés par Messaoudi, et exploité les idées de Georgiev et Todorova.

Mots-Clés et Phrases: Explosion, existence globale, terme source, équation d'onde, viscosité, énergie initiale négative, exposants variables, énergie initiale positive, existence et unicité, Faedo-Galerkin.

Notations

Throughout this dissertation, we will use the following conventions:

Ω : denotes a bounded domain in $\mathbb{R}^N$.

We denote by $\mathbb{R}^N$ the $n-$dimensional Euclidean space, and $n \in \mathbb{N}$ always stands for the dimension of the space.

$\overline{\Omega}$: The adhesion of Ω.

$\partial\Omega$: Smooth boundary.

$x = (x_1, x_2, \cdots, x_N)$: Generic point of $\mathbb{R}^N$.

∇u : Gradient of u.

Δu : Laplacian of u.

$u : u(x,t)$.

$v_j : v_j(x)$.

$\dfrac{\partial u}{\partial \eta}$: The normal derivative of u over $\partial\Omega$.

$\dfrac{\partial u}{\partial t}$: j The partial derivative of u with respect to t.

$\rightarrow$: Strong convergence.

$\rightharpoonup$: Weak convergence.

$\rightharpoonup *$: Weak star convergence.

$a.e$: Almost everywhere.

p': Conjugate of p, i.e $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

$D(\Omega)$: Space of differentiable functions with compact support in Ω.

$D'(\Omega)$: The dual of $D(\Omega)$: The space of distributions on Ω.

For $k \geq 1$ integer, $C^k(\Omega)$ is the space of functions u which are k times differentiable and whose derivative of order k is continuous on Ω.

$C_c^k(\Omega)$: Space of functions of $C^k(\Omega)$ whose support is compact and contained in Ω.

$C_0(\Omega)$: Space of continuous functions null board in Ω.

$L^p(\Omega)$: Space of functions p-th power integrated on Ω with a measure of dx.

$\|f\|_p = \left(\int_\Omega |f(x)|^p \right)^{\frac{1}{p}}$.

$W^{1,p}(\Omega) = \{u \in L^p(\Omega), \nabla u \in L^p(\Omega)\}$.

$W_0^{1,p}(\Omega)$: The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.

$W_0^{1,p(.)}(\Omega)$: The closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$.

$W^{-1,p'}(\Omega)$: The dual space of $W_0^{1,p}(\Omega)$.

$W^{k,p}([0,T], X)$: Sobolev space.

H : Hilbert space.

$H_0^1 = W_0^{1,2}(\Omega)$.

$H_0^m(\Omega) = W_0^{m,2}(\Omega)$: The adhesion of $D(\Omega)$ in $H^m(\Omega)$.

$C^k([0,T], X)$: Space of functions $k-$times continuously differentiable for $[0,T] \to X$.

$L^{p(.)}(\Omega)$: Lebesgue space with variable exponent $p(.)$.

$E(t)$: Energie.

T^* : Explosion time.

$\varrho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} dx$.

$\|.\|_X$: The norm of X.

D^α : The derivative of order α in the sense of distributions.

$D([0,T], X)$: The space of functions continuously differentiable of $[0,T] \to X$ with compact support in $[0,T]$.

$D'([0,T], X)$: The distribution space.

$C(\Omega) = \{u : u \text{ continuous in } \Omega\}$.

$supp\ u = \overline{\{x \in \Omega : u(x) \neq 0\}}$ = The support of u.

$C_0(\Omega) = \{u \in C(\Omega) : supp\ u \text{ is a compact subset of } \Omega\}$.

$C^k(\Omega) = \{u \in C(\Omega) : u \text{ is } k \text{ times continuously differentiable}\}$.

$C_0^k(\Omega) = C^k(\Omega) \cap C_0(\Omega)$.

$C^\infty(\Omega) = \bigcap_{k=1}^{\infty} C^k(\Omega)$ = smooth functions.

$C_0^\infty(\Omega) = C^\infty(\Omega) \cap C_0(\Omega)$ = compactly supported smooth functions = test functions

General Introduction

Variable Exponent Spaces: Brief History

The topic of variable exponent spaces has undergone extensive evolution in the past few years. However, the main reference is still the paper [40] by **O. Kováčik** and **J. Rákosník (1991)**. This work covers only basic characteristics, like reflexivity, separability, duality, and first results in connection with embeddings and density of smooth functions. Particularly, **L. Diening** in **2002** demonstrated the boundedness of the maximal operator, and its consequences are absent. Of course, progress on more advanced properties is dispersed in a great number of papers.

To familiarize students and colleagues more to the main results led around **2005** to the publication of some short survey articles. Furthermore, **L. Diening** gave in **2005** lectures at the University of Freiburg and **M. Růžička** gave in **2006** a course at the Spring School NAFSA 8 in Prague.

In the summer of **2006**, **L. Diening et al** decided to write a book consisting of basic and advanced properties, with amended assumptions. Two additional lecture sessions were given by **P. Hästö** (**2008** in Oulu and **2009** at the Spring School in Paseky); another synopsis, is the habilitation thesis of **L. Diening's** in **2007.**

In the last few years, the domain of variable exponent function spaces has seen tremendous growth. For example, a search for "variable exponent" in Mathematical Reviews yields **15** articles before **2000**, **31** articles between **2000** and **2004**, and **267** articles between **2005** & **2010**. This measure is crude with some errors in rating, but nonetheless quite expressive.

Lebesgue spaces for variable exponents was presented for the first time in **1931** by **W. Orlicz** in his article [68]. The question posed in this article is to search for necessary and sufficient conditions on (y_i) in which $\sum_i x_i y_i$ to converge ? for (x_i) and (p_i) (with $p_i > 1$) be sequences of real numbers such that $\sum_i x_i^{p_i}$ converges. Then it became clear that the answer is that $\sum_i (\lambda y_i)^{p'_i}$ should converge for some $\lambda > 0$ and $p'_i = \dfrac{p_i}{p_i - 1}$. Also he considered the variable exponent function space $L^{p(\cdot)}$ on the real line, and proved the Hölder inequality in this setting.

Thereafter, function spaces theory received great interest from **Orlicz**, which bears his name now (see [65]). In the theory of **Orlicz** spaces, the space L^{φ} is contained of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\varrho(\lambda u) = \int_{\Omega} \varphi(\lambda |u(x)|)\, dx < \infty,$$

for some $\lambda > 0$ [φ is a function of real-value that may depend on x and satisfies certain conditions].

H. Nakano [66, 67] was the first who studied a more general class of so-called modular function spaces, called modular spaces by putting certain properties of ϱ. After **Nakano**'s work, several people investigated the modular spaces, most importantly by groups at **Sapporo (Japan)**, **Voronezh (U.S.S.R.)**, and **Leiden (the Netherlands)**. Later, Polish mathematicians investigated a more explicit version of modular function spaces, for example, **H. Hudzik**, **A. Kamińska**, and **J. Musielak.**

The variable-exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as the Orlicz space $L^{\varphi_{p(\cdot)}}(\Omega)$ where

$$\varphi_{p(.)}(t) = t^{p(.)} \text{ or } \varphi_{p(.)}(t) = \frac{t^{p(.)}}{p(.)},$$

i.e.,

$$L^{\varphi_{p(.)}}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \varrho(\lambda u) = \int_{\Omega} \varphi_{p(x)}(\lambda |u(x)|)\, dx < +\infty \right\},$$

for some $\lambda > 0$ equipped with the Luxemburg norm

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0 \text{ such that } \int_{\Omega} \varphi_{p(x)}\left(\left| \frac{u(x)}{\lambda} \right| \right) dx \leq 1 \right\}.$$

The Russian researchers have been independently developing the variable exponent Lebesgue spaces on the real line. These investigations originated in a paper written by **Tsenov** [75] (**1961**). **I. Sharapudinov** presented in [70] the Luxemburg norm for the Lebesgue space and showed that this space is Banach if the exponent satisfies $1 < ess \inf p \leq ess \sup p < \infty$. In the mid-**80**s, **V. Zhikov** [78] started a new line of investigation of variable-exponent spaces, by considering variational integrals with non-standard growth conditions.

The early '**90**s was the next main step in the fulfillment of variable-exponent spaces by **Kováčik** and **Rákosník**'s article [40], in their work they established many essential properties of Lebesgue and Sobolev spaces in $\mathbb{R}^n$.

At the beginning of the new millennium, great progress has been made for a more precise study of variable-exponent spaces. Particularly, the connection was made between variable exponent spaces and variational integrals with non-standard growth and coercivity conditions.

The motivation for the recent systematic study of PDEs with variable exponents has been the description of several relevant models in electrorheological fluids or fluids with temperature-dependent viscosity, thermorheological fluids, nonlinear viscoelasticity, filtration processes through a porous media and image processing, or robotics. These models include hyperbolic, parabolic or elliptic equations that are nonlinear in a gradient of the unknown solution and with variable exponents of nonlinearity. In this regard, Chen, Levine, Rao [20], gave an example that concerns application to image restoration.

Generally, partial differential equations are of great importance in the modeling and description of a wide range of phenomena such as fluid dynamics, quantum physics, sound, heat, static electricity, diffusion, gravity, chemistry, biology, plane simulation, calculator diagrams, and time prediction.

Literature Review

During the past years, the linear and nonlinear wave equations with constant and variable-exponent nonlinearities have undergone considerable and great studies. Here, our goal is to introduce an overview of the current results and provide others.

Blow up in the Case of Constant and Variable Exponents Nonlinearities

The work of Levine [43] and Ball [5] in the following equation was the first study of finite time blow up of solutions of hyperbolic partial differential equations

$$u_{tt} - \Delta u = f(u).$$

Later, Levine [43, 44] was treated the interaction between the damping and the source terms for the following equation

$$u_{tt} - \Delta u + au_t = f(u),$$

and used the concavity method for proving blow-up of solutions at a finite time with negative initial energy.

To extend Levine's results, Georgiev and Todorova [31] considered a different method (when $m > 2$ (the nonlinear damping case)) to the nonlinear damped equation

$$u_{tt} - \Delta u + a\left|u_t\right|^m u_t = b\left|u\right|^p u \text{ in } \left(\Omega \times (0,\infty)\right),$$

and showed that solutions continue to exist globally "in time" with any initial data if $m \geq p$, and blow up in a finite time when the initial energy is sufficiently negative if $p > m$.

Recently, Levine and Serrin [47], Levine, Park, and Serrin [46], Levine and Park [45], and Messaoudi [52, 53] generalized this result to an abstract setting and unbounded domains. They proved that if $p > m$, no solution with negative energy can be continued to the whole $[0,\infty)$; they also demonstrated some non-continuation theorems. This generalization permitted them to use their result in quasilinear situations, a special case is apparent in the problem in reference [52].

In [52], Messaoudi extended the blow-up result of [31] to solutions with only negative initial energy, without imposing the condition that deems the initial energy sufficiently negative.

Vitillaro [76] expanded the results which were obtained in [47, 31] where the solution has a positive initial energy and the damping is non-linear. Messaoudi [51] expanded the result of [52] to the viscoelastic wave equation:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + au_t\left|u_t\right|^{m-2} = bu\left|u\right|^{p-2}, \ x \in \Omega, \quad t > 0,$$

and showed by imposing appropriate conditions on g, that solutions blow up in finite time if $p > m$ with negative initial energy and continue to exist globally if $m \geq p$ for arbitrary initial data. In [34] Kafini and Messaoudi proved the blowup result for the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = bu\left|u\right|^{p-2}, \text{ in } \mathbb{R}^n \times (0,\infty)$$

In [19], Cavalcanti et al. have treated the following related problem in a bounded domain:

$$\left|u_t\right|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma\Delta u_t = 0, x \in \Omega, \quad t > 0,$$

where $\rho > 0$. They achieved an exponential decay result for $\gamma > 0$, and global existence for $\gamma \geq 0$. Kafini and Messaoudi in [33] pushed the same result [34] to a system of the form

$$\begin{aligned} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau &= f_1(u,v), \text{ in } \mathbb{R}^n \times (0,\infty) \\ v_{tt} - \Delta v + \int_0^t h(t-\tau)\Delta v(\tau)d\tau &= f_2(u,v), \text{ in } \mathbb{R}^n \times (0,\infty). \end{aligned}$$

In [55], Messaoudi and Said-Houari proved the result of the global existence of certain solutions with positive initial energy for the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + |u_t|^{m-1} u_t = f_1(u,v), \text{ in } \Omega \times (0,\infty), \\ v_{tt} - \Delta v + \int_0^t h(t-\tau)\Delta u(\tau)d\tau + |v_t|^{m-1} v_t = f_2(u,v), \text{ in } \Omega \times (0,\infty), \\ u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. In the paper of Chen et al [21], they looked into the nonlinear $p-$Laplacian wave equation:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) - \Delta u_t + q(x,u) = f(x),$$

when $2 \leq p < n$ and f, q are given functions. Under suitable conditions on the initial data and the functions f, q, they realized global existence, uniqueness and also discussed the long-time behavior of the solution. Benaissa and Mokeddem in [10] considered:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) - \sigma(t) \operatorname{div}\left(|\nabla u_t|^{m-2} \nabla u_t\right) = 0.$$

They achieved an energy-decay estimate for the solutions where $p, m \geq 2, \sigma$ is a positive function, and expanded Yang [77] and Messaoudi [54] results. Recently, Mokeddem and Mansour [64] added some modification in the problem of Benaissa and Mokeddem [10] and established the same decay result.

Messaoudi and Houari [56] studied the nonlinear wave equation:

$$u_{tt} - \Delta u_t - \operatorname{div}\left(|\nabla u|^{\alpha-2} \nabla u\right) - \operatorname{div}\left(|\nabla u_t|^{\beta-2} \nabla u_t\right) + a|u_t|^{m-2} u_t = b|u|^{p-2} u,$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, $a, b, c > 0$ and $\alpha, \beta, m, p > 2$. They investigated with appropriate conditions imposed on $\alpha, \beta, m, p > 2$, a global nonexistence result for solutions assocciated with negative initial energy.

In the paper of Mohammad Kafini and Salim Messaoudi [36] the authors are concerned with a problem of a logarithmic nonlinear wave equation with delay and established the local existence result by using the semigroup theory. Also, they proved the result of a blow-up at a finite time for negative initial energy. In [35] the same previous authors treated a nonlinear wave equation with delay term and proved, under appropriate hypotheses on the initial data, that the energy of solutions explodes in a finite time. For more results, see the previous studies [9, 26, 30, 69].

There are several and great studies concerned with the study of nonlinear models of parabolic, elliptic, and hyperbolic equations in the case of variable exponents of nonlinearity. For example, some models from physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, image processing, and filtration processes through porous media, give rise to such problems.

Now, let us mention some problems in this direction. Antontsev [2] looked into the problem:

$$\partial_{tt}u - \operatorname{div}\left(a\left(x,t\right)\left|\nabla u\right|^{p(x,t)-2}\nabla u\right) - \alpha\Delta u_t = b\left(x,t\right)u\left|u\right|^{\sigma(x,t)-2},$$

when α is a nonnegative constant a, b, p, σ are given functions. He discussed the case when $\alpha = 0$ and $\alpha > 0$, and demonstrated a blow-up result under a particular hypothesis on a, b, p, σ. Thereafter, Antontsev in [1] considered the same equation and established a local, global existence of weak solutions for specific conditions on a, b, p, σ, and realized blow-up results for solutions with non-positive initial energy.

In [32] Guo and Gao considered the same problem of Antontsev [1], they picked the constant $\sigma(x,t) = r > 2$ and realized a blowup result in finite time, also they alleged without any proof the same blow-up result for $\sigma(x,t) = r(x)$. Sun et al in [71] studied the blow-up result for solutions with positive initial energy for the following equation:

$$u_{tt} - \operatorname{div}\left(a\left(x,t\right)\nabla u\right) + c\left(x,t\right)u_t\left|u_t\right|^{q(x,t)-1} = b\left(x,t\right)u\left|u\right|^{p(x,t)-1}.$$

They also gave lower and upper bounds for the blow-up time and provided numerical illustrations for their result. Lately, Messaoudi and Talahmeh [57] looked into

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{m(x)-2}\nabla u\right) + \mu u_t = |u|^{p(x)-2}u,$$

where $\mu \geq 0$. They proved a blow-up result for certain solutions with arbitrary positive initial energy. This result was generalized by the same authors in [58] to an equation of the form

$$u_{tt} - \operatorname{div}(|\nabla u|^{r(\cdot)-2}\nabla u) + a|u_t|^{m(\cdot)-2}u_t = b|u|^{p(\cdot)-2}u,$$

where the exponents of nonlinearity m, p and r are given functions and $a, b > 0$ are constants. They demonstrated a finite-time blowup result for the solutions with negative initial energy and for certain solutions with positive energy.

At the end of **2017**, Messaoudi et al. [60] studied the following class of nonlinear wave equation:

$$u_{tt} - \Delta u + au_t|u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2},$$

where the existence of a unique weak solution is established under suitable assumptions on the variable exponents m and p by using the Faedo–Galerkin method. Also, they proved the finite-time blow-up of solutions and gave a two-dimension numerical example to clarify the result of the blow-up. In [29] Yunzhu Gao and Wenjie Gao treated a nonlinear viscoelastic equation with variable exponents and achieved the existence of weak solutions under suitable assumptions by using the Faedo–Galerkin method.

For more information in the study of the phenomenon of explosion in hyperbolic equations, we guide the reader to Antontsev and Ferreira [3], Galaktionov [28] and the book by Antontsev and Shmarev [4].

Plan Work

Our purpose in this dissertation is to prove the well-posedness and the blow-up of solutions of several nonlinear hyperbolic problems involving nonclassical nonlinearities. Otherwise, we treated some problems and found under some appropriate assumptions the results of blowup.

This study generalizes and expands some results. In detail, we expanded the result of blow-up of several nonlinear wave equations with variable and constant exponent nonlinearities, studied by Messaoudi [51, 58, 60], by using different techniques.

This dissertation is consists of four principal chapters in addition to the general introduction, conclusion, and suggestions. The general introduction contains, in particular, a brief history of variable exponent spaces and a literature review on blow up in the case of constant and variable exponents nonlinearly, and it is ended by a third section devoted to the plan work of this dissertation.

The first chapter is devoted to some background and basic concepts needed. Especially, we reminded some basic results, notations, prerequisites, preliminaries, elementary properties, and proof of some principal inequalities used in the proof of lemmas and theorems in this dissertation, also we recalled the definition of Variable-exponent Lebesgue and Sobolev spaces, which will be useful to us later. We ended this chapter with the concept of blow-up, where we have specifically introduced what the authors mean by this notion.

We start our contributions from the second chapter (this chapter essentially corresponds to the paper [72]. Z. Tebba, S. Boulaaras, H. Degaichia and A. Allahem, Existence and blow-up of a new class of nonlinear damped wave equation, Journal of Intelligent and Fuzzy Systems, 38 (3) (2020), 2649-2660.), where we demonstrate the existence, uniqueness, and blow-up of solutions of the following nonlinear wave equation with variable exponents

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + a u_t \left|u_t\right|^{m(.)-2} = b u \left|u\right|^{p(.)-2}, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \quad \Omega, \end{cases} \tag{1}$$

where, Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$, $a, b \geq 0$ are constants and the exponents $m(.)$ and $p(.)$ are given log-Hölder[1] continuous functions on Ω verified:

$$2 \leq m_1 \leq m(x) \leq m_2 \leq \frac{2n}{n-2}, \; n \geq 3, \tag{2}$$

and

$$2 \leq p_1 \leq p(x) \leq p_2 \leq 2\frac{n-1}{n-2}, \; n \geq 3. \tag{3}$$

[1] Otto Ludwig Hölder (22/12/1859 - 29/08/1937) is a German mathematician born in Stuttgart, capital of the kingdom of Württemberg.

In 1877, he entered the University of Berlin, and he obtained his doctorate in 1882 at the University of Tübingen. The title of his doctoral dissertation is Beiträge zur Potentialtheorie (Contributions to the theory of potential). He taught at the University of Leipzig from 1899 until his emeritus in 1929.

Here, we use the famous Faedo-Galerkin method and fixed point theorem to show the existence and uniqueness of solutions under some suitable data. Also, we investigate the blow-up phenomena of solutions of problem (1), particularly we try to answer the question: under which conditions on the parameters p and m, the solution does not exist globally in time ?. And the obtained results are proved by using a different method.

The following chapter is number three (this chapter essentially corresponds to the paper [74]. Z. Tebba, H. Degaichia and H. Messaoudene, Global existence and finite time blow-up in a new class of non-linear viscoelastic wave equation, Journal of Discontinuity, Nonlinearity, and Complexity, 11 (2) (2022), 275-284.), and it is devoted to studying the global existence and finite time blow-up of the following new class of non-linear viscoelastic wave equation

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-\tau)\Delta u(\tau)d\tau + cu_t\left|u_t\right|^{m-2} = du\left|u\right|^{p-2}, x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{4}$$

where Ω be an open bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geq 1)$, with a Lipschitz-countinuous boundary $\partial\Omega, p > 2, m \geq 1$, and c, d are strictly positive constants. We show that solutions with arbitrary data continue to exist globally if $m \geq p$ and blow-up in finite with negative initial energy if $m < p$.

The next chapter is number four (this chapter present a very recent published work [73]. Z. Tebba, H. Degaichia, M. Abdalla, B. B. Cherif and I. Mekawy, Blow-Up of Solutions for a Class Quasilinear Wave Equation with Nonlinearity Variable Exponents, Journal of Function Spaces, 2021 (2021).), it contains four sections, and it is consecrated to study the finite-time blow-up of solutions of the following new category of a quasilinear wave equation with variable exponents nonlinearities

$$\begin{cases} u_{tt} - \operatorname{div}\left(\left|\nabla u\right|^{s(.)-2}\nabla u\right) - \Delta u_{tt} + \eta u_t\left|u_t\right|^{q(.)-2} = \mu u\left|u\right|^{p(.)-2}, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{5}$$

here $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, be a bounded domain with a smooth boundary $\partial\Omega$, $\eta, \mu > 0$ are constants, and the exponents $p(.), q(.)$ and $s(.)$ are given log-Hölder continuous functions on Ω such that:

$$2 \leq \max\{q_2, s_2\} < p_1 \leq p(x) \leq p_2 \leq s^*(x), \tag{6}$$

where

$$s^*(x) = \begin{cases} \frac{ns(x)}{ess\sup\limits_{x\in\Omega}(n-s(x))} & \text{if } \ s_2 < n \\ +\infty & \text{if } \ s_2 \geq n \end{cases},$$

and

$$ess\inf_{x\in\Omega}(s^*(x) - p(x)) > 0.$$

The first and second sections consist of basic assumptions, statements, and well-posedness of problem, in the third and fourth one, we achieve a finite time blow-up result for solutions with negative initial energy and certain solutions with positive energy.

We have finished this dissertation with a conclusion that contains some perspectives and proposals for open subjects. At the end of this work, there is an alphabetic list of the references used to prepare this dissertation under the title References.

Chapter 1

Background and Basic Concepts

1- Reminders and Prerequisites (Some Basic Results)

2- Variable Exponents Lebesgue and Sobolev Spaces

3- Notions of Blow-Up

Key Words and Phrases: Contraction mapping theorem, variable-exponent spaces, blowup, modular spaces.

This chapter contains some preliminaries and basic results used throughout this dissertation. After presenting some essential concepts, notations, and definitions which will be useful to us later. We will introduce some functional spaces, then we mention fundamental concepts used in this dissertation.

1.1 Reminders and Prerequisites (Some Basic Results)

In this section, we present some material and standard notations that we shall use in order to present our results.

- Let $x = (x_1, x_2, \cdots, x_n)$ denote the generic point of an open Ω of $\mathbb{R}^n$[1]. Let u be a defined function of Ω with values in $\mathbb{R}$, on indicated by $D^i u(x) = \dfrac{\partial u(x)}{\partial x_i}$ the partial derivative of the function u with respect to x_i.

- Also define the gradient and the Laplacian of u, respectively as follows

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\right)^T \text{ and } |\nabla u|^2 = \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2,$$

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2} = \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}\right)(x).$$

- We denote by $C(\Omega)$ the space of all continuously differentiable functions on Ω with values in $\mathbb{R}$.

- $C_0(\Omega) = \{u \in C(\Omega) : \textit{supp } u \text{ is a compact subset of } \Omega\}.$

- $(C(\Omega))^m$ is the space of continuous functions of Ω with values in $\mathbb{R}^m$.

- $C_b(\overline{\Omega})$ the space of continuous and bounded functions on $\overline{\Omega}$, we provide it with the standard $\|.\|_\infty$

$$\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)|.$$

- For $k \geq 1$ integer, $C^k(\Omega)$ is the space of functions u which are k times differentiable and whose derivative of order k is continuous on Ω.

- $C_c^k(\Omega)$ is the function space of $C^k(\Omega)$, whose support is compact and contained in Ω.

[1] By $\mathbb{R}^n$ we denote the $n-$dimensional Euclidean space, and $n \in \mathbb{N}$ always stands for the dimension of the space.

- $C_0^\infty(\Omega)$ or $D(\Omega)$, is the space of indefinitely differentiable functions (which is called space of test functions), with a compact supports contained in Ω, having continuous derivatives of all orders

$$D(\Omega) = C_0^\infty(\Omega) = \{u \in C^\infty(\Omega); \exists K \subset \Omega, K \text{ compact (closed, bounded)}; u = 0 \text{ on } K\}.$$

- The support of a continuous function f defined on Ω is the closure of the set of a point where $f(x)$ is nonzero. That is

$$supp(f) := \overline{\{x \in \Omega / f(x) \neq 0\}}.$$

- $D'(\Omega)$ is the Distribution space.
- We use throughout this dissertation the standard $L^2(\Omega)$ and $H^1(\Omega)$ spaces.
- The space $H^1(\Omega)^2$ is equipped with the norm

$$\|u\|^2_{H^1(\Omega)} = \|u\|_2^2 + \|\nabla u\|_2^2,$$

where $\|u\|_2^2 = \|u\|^2_{L^2(\Omega)}$.

- Also, we take advantage of space

$$\|u\|^2_{H_0^1(\Omega)} = \left\{u \in H^1(\Omega) : \exists \{u_m\}_{m=0}^\infty \subset C_0^1(\Omega), \text{ such that } u_m \to u \text{ in } H^1(\Omega)\right\},$$

equipped with the norm:

$$\|u\|^2_{H_0^1(\Omega)} = \|\nabla u\|_2^2,$$

if Ω is a bounded domain, where $H_0^1(\Omega)$ is a Hilbert[3] space.

- $u_t = \dfrac{\partial u}{\partial t}, u_{tt} = \dfrac{\partial^2 u}{\partial t^2}$.

[2] We set $H^1(\Omega) = W^{1,2}(\Omega)$.

[3] David Hilbert is a German mathematician born January 23, 1862, in KÄonigsberg in Prussia oriental and died on February 14, 1943, in GÄottingen in Germany. He is often considered one of the greatest mathematicians of the 20th century, just like Henri Poincaré. He created or developed a wide range of fundamental ideas, be it the theory of invariants, the axiomatization of geometry, or the foundations of functional analysis (with Hilbert spaces).

- $L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is a measurable function and } \int_\Omega |f|^p \, dx < \infty \right\}$, where $1 \leq p < \infty$.
- $L^\infty(\Omega) = \left\{ \begin{array}{c} f : \Omega \to \mathbb{R} \text{ is a measurable function and there is a constant } C \geq 0 \\ \text{such that } |f(x)| \leq C \text{ a.e. on } \Omega \end{array} \right\}$.
- $L^p_{loc}(\Omega) = \{ f : \Omega \to \mathbb{R},\ f \text{ is measurable function and } f \in L^p(K), \forall K \subset \Omega, K \text{ compact} \}$.
- $L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$.
- In particular, when $p = 2$, $L^2(\Omega)$ equipped with the inner product

  $$\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega u(x) \, v(x) \, dx,$$

  is a Hilbert[4] space.
- $L^p(\Omega)$ is a reflexive space for all $1 < p < \infty$.
- Let $T > 0$ be a real number and X be a real Banach space endowed with norm $\|.\|_X$. We consider the following definitions:

  The space $L^p(0,T;X)$[5] denotes the space of functions u which are L^p over $(0,T)$ with values in X, which are measurable and

  $$\|u\|_X \in L^p(0,T), L^p(0,T;X) = \left\{ u : (0,T) \to X \text{ is measurable; } \int_0^T |u(t)|_X^p \, dt < \infty \right\}.$$

  This space is a Banach space endowed with the norm

  $$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}} < +\infty,$$

  for $1 \leq p < \infty$.
- For $p = \infty$, $L^\infty(0,T;X)$ denotes the space of functions $\left\{ \begin{array}{c} u :]0,\ T[\to X \\ t \mapsto u(t) \end{array} \right.$ which are measurable and $\|u\|_X \in L^\infty(0,\ T)$,

  $$L^\infty(0,T;X) = \left\{ u : (0,T) \to X \text{ is measurable; } ess \sup_{0<t<T} |u(t)|_X^p < +\infty \right\}.$$

[4] A Hilbert space H is a vectorial space supplied with inner product $\langle u, v \rangle$ such that $\|u\| = \sqrt{\langle u, u \rangle}$ is the norm which let H complete.

[5] The space $L^p(0,T;X)$ is complete.

This space is a Banach space endowed with the norm:

$$\|u\|_{L^\infty(0,T;X)} := ess \sup_{0<t<T} \|u(t)\|_X < +\infty^6.$$

- We recall that if X and Y are two Banach spaces such that $X \hookrightarrow Y$ (continuous embedding), then

$$L^p(0,T;X) \hookrightarrow L^p(0,T;Y), 1 \leq p \leq \infty.$$

- The space $L^p_{loc}(0,T;X)$ consists of all measurable functions $u : (0,T) \to X$ with $u \in L^p([a,b];X)$ for every closed interval $[a,b] \subset (0,T)$.

- The space $C(0,T;X)$ consists of all continuous functions $u : [0;T] \to X$ with

$$\|u\|_{C(0,T;X)} := \max_{0\leq t\leq T} \|u\| < +\infty.$$

- The space $C^1(0,T;X)$ consists of all continuously differentiable functions $u : [0,T] \to X$ with

$$\|u\|_{C^1(0,T;X)} := \max_{0\leq t\leq T} \|u\| + \max_{0\leq t\leq T} \left\|\frac{du}{dt}\right\| < +\infty.$$

- $C^k(0,T;X)$ is the space of functions $k-$times continuously differentiable for $[0,T] \to X$.

1.2 Variable Exponents Lebesgue and Sobolev Spaces

In this section, we list briefly some definitions and well-known facts about generalized Lebesgue[7] spaces $L^{p(x)}(\Omega)$, and generalized Sobolev[8] spaces $W^{m,p(x)}(\Omega)$. These results provide the needful framework for studying variance problems.

[6] We use the symbol:= to define the left-hand side by the right-hand side.

[7] Henri-Léon Lebesgue (1875-1941), better known under the name of Henri Lebesgue, is one of the great French mathematicians of the first half of the 20th century. He is recognized for his theory of integration published initially in his dissertation Integral, length, area at the University of Nancy in 1902.

[8] Specialist in differential equations applied to the physical sciences, Sobolev introduces, from 1934, the notion of generalized function and derivative to better understand the phenomena physical where the concept of function was insufficient in the search for solutions of equations to partial derivatives. He is thus at the origin of the theory of distributions developed by his compatriot IsraÄel Guelfand and Frenchman Laurent Schwartz.

Most of the results are similar to those for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{m,p}(\Omega)$, but the Sobolev-like embedding theorem and result on density are new; they show the essential difference between $W^{m,p(x)}(\Omega)$ and $W^{m,p}(\Omega)$.

1.2.1 On the Spaces $L^{p(x)}(\Omega)$ (Variable Exponents Lebesgue Spaces)

Throughout this dissertation, Ω will be a non-empty, open, bounded subset in $\mathbb{R}^n, n \in \mathbb{N}$, and p will be a measurable function on Ω with values in $[1, \infty)$. By saying that Ω has a Lipschitz Boundary we mean that the boundary $\partial\Omega$ is locally described by Lipschitz-continuous functions.

We summarize in this subsection the most important basic properties of variable exponent Lebesgue spaces $L^{p(\cdot)}$ (see[38, 23 − 25]). They differ from classical L^p spaces in that the exponent p is not constant but a function from Ω to $[1, \infty)$, and we will give a brief description of their main properties.

Definition 1.1. *A function $\varrho : X \to [0, \infty)$ is said to be left-continuous if the mapping $\lambda \longmapsto \varrho(\lambda x)$ is left-continuous on $[0, \infty)$, for every $x \in X$ (in which X be a $\mathbb{k}$-vector space); that is,*

$$\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x), \forall x \in X.^9$$

Definition 1.2. *A function $\varrho : X \to [0, \infty)$ in which X be a $\mathbb{k}$-vector space (where $\mathbb{k}$ is either $\mathbb{R}$ or $\mathbb{C}$), is called a semi-modular on X if the following properties hold*

(a) $\varrho(0) = 0$.

(b) $\varrho(\lambda x) = \varrho(x)$, *for all $x \in X$ and $\lambda \in \mathbb{k}$, with $|\lambda| = 1$.*

(c) *ϱ is convex.*

(d) *ϱ is left-continuous.*

(e) $\varrho(\lambda x) = 0$, *for all $\lambda > 0$ implies $x = 0$.*

A semi-modular is called modular if

(f) $\varrho(x) = 0$ *implies $x = 0$.*

A semi-modular is named continuous if

(g) *the mapping $\lambda \longmapsto \varrho(\lambda x)$ is continuous on $[0, \infty)$ for all $x \in X$.*

[9] Here $a \to b^-$ means that a tends to b from below, i.e. $a < b$ and $a \to b$; $a \to b^+$ is defined analogously.

Example 1.1. *1) Let $L^0(\Omega)$ be the set of all Lebesgue-measurable functions defined on Ω. If $1 \leq p < +\infty$, then*

$$\varrho_p(f) := \int\limits_{\Omega} |f(x)|^p dx,$$

defines a continuous modular on $L^0(\Omega)$.

2) Let $\omega \in L^1_{loc}(\Omega)$ with $\omega > 0$ almost everywhere and $1 \leq p < \infty$. Then

$$\varrho(f) := \int\limits_{\Omega} |f(x)|^p \omega\,(x)\,dx,$$

defines a continuous modular on $L^0(\Omega)$.

3) Let $\varphi_\infty(t) := \infty \cdot \chi_{(1,\infty)}(t)$ for $t \geq 0$, i.e. $\varphi_\infty(t) = 0$ for $t \in [0,1]$ and $\varphi_\infty(t) = \infty$ for $t \in [0, \infty)$. Then

$$\varrho_\infty\,(f) := \int_\Omega \varphi_\infty\,(|f\,(x)|)\,dx,$$

defines a semi-modular on $L^0(\Omega)$ which is not continuous.

Theorem 1.1. [41] *Let ϱ be a semi-modular on X. Then, the mapping $\lambda \longmapsto \varrho(\lambda x)$ is non-decreasing on $[0, \infty)$ for every $x \in X$, by convexity and non-negativeness of ϱ and $\varrho(0) = 0$. Furthermore,*

$$\varrho(\lambda x) = \varrho(|\lambda| x) \leq |\lambda| \varrho(x) \qquad \text{for all} \quad |\lambda| \leq 1, \tag{1.1}$$

$$\varrho(\lambda x) = \varrho(|\lambda| x) \geq |\lambda| \varrho(x) \qquad \text{for all} \quad |\lambda| \geq 1.$$

Proof. - Assume that $0 \leq \lambda < \mu$, then $0 \leq \dfrac{\lambda}{\mu} < 1$. So for $x \in X$ we have

$$\varrho(\lambda x) = \varrho(\frac{\lambda}{\mu}\,(\mu x) + \left(1 - \frac{\lambda}{\mu}\right) \cdot 0) \leq \frac{\lambda}{\mu} \varrho\,(\mu x) + \left(1 - \frac{\lambda}{\mu}\right) \varrho\,(0) = \frac{\lambda}{\mu} \varrho\,(\mu x) \leq \varrho\,(\mu x)\,.$$

Hence for any $x \in X$, we have

$$\varrho(\lambda x) \leq \varrho\,(\mu x) \quad \text{for} \quad 0 \leq \lambda < \mu.$$

- For $\lambda \neq 0$, we have

$$\varrho(\lambda x) = \varrho(\frac{\lambda}{|\lambda|}\,|\lambda|\,x) = \varrho(|\lambda|\,x) \quad \left(\text{since } \left|\frac{\lambda}{|\lambda|}\right| = 1\right).$$

- For $|\lambda| \leq 1$, we have

$$\varrho(|\lambda|\, x) = \varrho(|\lambda|\, x + (1 - |\lambda|)\, 0) \leq |\lambda|\, \varrho\,(x) + (1 - |\lambda|)\, \varrho\,(0) = |\lambda|\, \varrho\,(x)\,.$$

thus,

$$\varrho(\lambda x) = \varrho(|\lambda|\, x) \leq |\lambda|\, \varrho(x) \qquad \forall x \in X \quad \text{and } |\lambda| \leq 1.$$

- For $|\lambda| \geq 1$, we have

$$\varrho(x) = \varrho(\frac{1}{|\lambda|}\, |\lambda|\, x + \left(1 - \frac{1}{|\lambda|}\right) 0) \leq \frac{1}{|\lambda|}\varrho(|\lambda|\, x) + \left(1 - \frac{1}{|\lambda|}\right) \varrho(0) = \frac{1}{|\lambda|}\varrho(|\lambda|\, x).$$

Therefore,

$$\varrho(\lambda x) = \varrho(|\lambda|\, x) \geq |\lambda|\, \varrho(x) \qquad \forall x \in X \quad \text{and } |\lambda| \geq 1.$$

□

Definition 1.3. *Let $(\Omega,\, \Sigma,\, \mu)$ be a $\sigma-$finite, complete measure space.*

Definition 1.4. *Let $\mathcal{P}(\Omega, \mu)$ be the set of all $\mu-$measurable functions $p : \Omega \to [1, \infty]$. The functions $p \in \mathcal{P}(\Omega, \mu)$ are named variable exponents on Ω. We introduce*

$$p_1 := ess \inf_{y\in\Omega} p(y) \qquad and \qquad p_2 := ess \sup_{y\in\Omega} p(y).$$

If $p_2 < +\infty$, then we call p a bounded variable exponent. If $p \in \mathcal{P}(\Omega, \mu)$, then $p' \in \mathcal{P}(\Omega, \mu)$ defined as follows

$$\frac{1}{p(y)} + \frac{1}{p'(y)} = 1, \;\; where \;\; \frac{1}{\infty} := 0.$$

The dual variable exponent of p is the function p'. Particularly when μ is the $n-$dimensional Lebesgue measure and Ω is an open subset of $\mathbb{R}^n$, we abbreviate $\mathcal{P}(\Omega) := \mathcal{P}(\Omega, \mu)$.

Definition 1.5. *Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$. We introduce the Lebesgue space with a variable exponent $p(\cdot)$ by*

$$L^{p(.)}\,(\Omega) := \left\{u : \Omega \to \mathbb{R};\; measurable\ in\ \Omega : \varrho_{p(.)}\,(\lambda u) < \infty,\; for\ some\ \lambda > 0\ \right\},$$

where

$$\varrho_{p(.)}\,(u) = \int\limits_{\Omega} |u(x)|^{p(x)}\, dx.$$

is a modular, endowed with the following Luxembourg-type norm

$$\|u\|_{p(.)} := \inf\left\{\lambda > 0 : \int\limits_{\Omega} \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$ [10],

$L^{p(.)}(\Omega)$ *is a Banach space.*

Remark 1.1. *The variable exponent-Lebesgue space is a special case of more general Orlics-Musielak spaces. For the constant function $p(x) = p$, the variable exponent-Lebesgue space coincides with classical Lebesgue space.*

Example 1.2. *Let $p(x) = x$ on $\Omega = (1, 2)$. Then, $\|1\|_{p(.)} = 1$. Indeed,*

$$\varrho_{p(.)}\left(\frac{1}{\lambda}\right) = \int\limits_{1}^{2} \lambda^{-x} dx = \frac{\lambda - 1}{\lambda^2 \ln \lambda}.$$

Since $\varrho_{p(.)}(1) = 1$, then, by definition of $\|1\|_{p(.)}$, we have $\|1\|_{p(.)} \leq 1$. Otherwise, it is easy to verify that $\varrho_{p(.)}\left(\frac{1}{\lambda}\right) > 1$, for $0 < \lambda < 1$. This gives $\|1\|_{p(.)} \geq 1$. Subsequently, we deduce that $\|1\|_{p(.)} = 1$.

Lemma 1.1. *If $p(x) \equiv p$, where p is constant. Then*

$$\|u\|_{p(\cdot)} = \lambda_0 = \left(\int\limits_{\Omega} |u|^p\right)^{\frac{1}{p}}. \tag{1.2}$$

Proof. Since $\varrho_{p(.)}\left(\frac{u}{\lambda_0}\right) = 1$, then

$$\|u\|_{p(\cdot)} \leq \lambda_0. \tag{1.3}$$

Next, by employing property of inf, then there exists a sequence $\{\lambda_j\}_{j=1}^{\infty} = 1$ such that $\lambda_j \geq \|u\|_{p(\cdot)}$, with

$$\varrho_{p(.)}\left(\frac{u}{\lambda_j}\right) \leq 1 \text{ and } \lambda_j \to \|u\|_{p(\cdot)}.$$

[10] *Proof.* see $Theorem$2.1.7. page 24 in reference [41]. □

Since,

$$\varrho_{p(.)}\left(\frac{u}{\lambda_j}\right) = \frac{1}{(\lambda_j)^p}\int_\Omega |u|^p \leq 1,$$

so we get

$$\lambda_0 \leq \|u\|_{p(\cdot)}. \tag{1.4}$$

Combining (1.3) and (1.4) gives (1.2). $\square$

Definition 1.6. *A function $\psi : \Omega \to \mathbb{R}$ is log-Hölder continuous on Ω, if there exist $A > 0$ and $0 < \delta < 1$ such that*

$$|\psi(x) - \psi(y)| \leq \frac{-A}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{with } |x-y| < \delta. \tag{1.5}$$

Lemma 1.2. *Let Ω be a domain of $\mathbb{R}^n$. If $p : \Omega \to \mathbb{R}$ is a Lipchitz function, then it is log-Hölder continuous on Ω.*

Proof. Let $x, y \in \Omega$, with $|x-y| < \delta$ and $0 < \delta < 1$. Then, since p is Lipchitz, there exists $L > 0$ such that

$$\begin{aligned} |p(x) - p(y)| &\leq L|x-y| \\ &\leq -\frac{L}{\log|x-y|}\left(-|x-y|\log|x-y|\right). \end{aligned} \tag{1.6}$$

Let $g(s) = -s\log s$. Then, g is continuous on $[0,1]$ and subsequently is bounded. So we get, $0 \leq -s\log s \leq M$. Thus, (1.6) becomes

$$|p(x) - p(y)| \leq \frac{-A}{\log|x-y|},$$

where $A = LM > 0$. Therefore, p is log-Hölder continuous. $\square$

Example 1.3. *Let $q(x) = x^2 + 2$ be defined on $\Omega = B(0,1)$. Then $q : \Omega \to \mathbb{R}$ is log-Hölder continuous on Ω . Indeed, let $(x,y), (x_0,y_0) \in \Omega$, with $|(x,y) - (x_0,y_0)| < \delta$ and $0 < \delta < 1$. Then,*

$$\begin{aligned} |q(x,y) - q(x_0,y_0)| &= \left|x^2 - x_0^2\right| \\ &= |x-x_0||x+x_0| \\ &\leq \frac{4\log\delta}{\log\delta} \\ &\leq -\frac{A}{\log|(x,y)-(x_0,y_0)|}, \end{aligned}$$

where $A = 4\log(1/\delta)$. *Subsequently,* q *is log-Hölder continuous.*

Lemma 1.3. (*Unit Ball Property*) [41] *Let* $p \in \mathcal{P}(\Omega, \mu)$ *and* $f \in L^p(\Omega, \mu)$ *be a measurable function on* Ω. *Then*

(i) $\|f\|_{p(.)} \leq 1$ *if and only if* $\varrho_{p(.)}(f) \leq 1$.

(ii) If $\|f\|_{p(.)} \leq 1$, *then* $\varrho_{p(.)}(f) \leq \|f\|_{p(.)}$.

(iii) If $\|f\|_{p(.)} \geq 1$, *then* $\|f\|_{p(.)} \leq \varrho_{p(.)}(f)$.

(iv) $\|f\|_{p(.)} \leq 1 + \varrho_{p(.)}(f)$.

Lemma 1.4. [41] *If* p *is a measurable function on* Ω *satisfying* $1 < p_1 \leq p(x) \leq p_2 < +\infty$, *then for a.e.* $x \in \Omega$, *we have*

$$\min\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\} \leq \varrho_{p(.)}(u) \leq \max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\},$$

for any $u \in L^{p(.)}(\Omega)$.

Theorem 1.2. [41] *If* $p \in \mathcal{P}(\Omega, \mu)$, *then* $L^{p(\cdot)}(\Omega, \mu)$ *is a Banach*[11] *space.*

Lemma 1.5. [41] *If* $p : \Omega \to [1, \infty)$ *is a measurable function with* $p_2 < \infty$, *then* $C_0^\infty(\Omega)$ *is dense in* $L^{p(\cdot)}(\Omega)$.

Some Useful Inequalities

We want here to recall some algebraic inequalities that we need later in this dissertation

Lemma 1.6. (*Cauchy Inequality*) *Let* Ω *be an open subset of* $\mathbb{R}^n$. *For all* $(a, b) \in \mathbb{R}^2$

$$|ab| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2.$$

Lemma 1.7. (*Cauchy Inequality with* ε ($\varepsilon-$*Inequality*)) *For all* $\varepsilon > 0$ *and* $(a, b) \in \mathbb{R}^2$, *we have:*

$$|ab| \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2.$$

[11] Stefan Banach: (30 March 1892 – 31 August 1945) was a Polish mathematician who is generally considered one of the world's most important and influential 20th-century mathematicians. He was the founder of modern functional analysis and an original member of the Lwów School of Mathematics. His major work was the 1932 book, Théorie des opérations linéaires (Theory of Linear Operations), the first monograph on the general theory of functional analysis.

Lemma 1.8. $\left(H\ddot{o}lder's\ Inequality\right)$ [41] *Let $p, q, s \in \mathcal{P}(\Omega, \mu)$ such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \text{for a.e. } y \in \Omega.$$

If $f \in L^{p(.)}(\Omega, \mu)$ and $g \in L^{q(.)}(\Omega, \mu)$, then $fg \in L^{s(.)}(\Omega, \mu)$ and

$$\| fg \|_{s(.)} \leq 2 \|f\|_{p(.)} \|g\|_{q(.)} .$$

*By taking $p = q = 2$, we have the **Cauchy**[12]–**Schwarz**[13] **inequality:** For all $u, v \in L^2(\Omega)$*

$$\left| \int_\Omega uvdx \right| \leq \int_\Omega |uv|\, dx \leq \left(\int_\Omega |u|^2 dx \right)^{1/2} \left(\int_\Omega |v|^2 dx \right)^{1/2},$$

that is to say

$$\|uv\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} .$$

Lemma 1.9. $(Young's\ Inequality)$ [41]

Let $p, q, s \in \mathcal{P}(\Omega, \mu)$ such that

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \text{for a.e. } y \in \Omega.$$

Then for all $a, b \geq 0$,

$$\frac{(ab)^{s(\cdot)}}{s(.)} \leq \frac{(a)^{p(\cdot)}}{p(.)} + \frac{(b)^{q(\cdot)}}{q(.)}. \tag{1.7}$$

By taking $s = 1$, and $1 < p, q < +\infty$ ($p, q,$ and s are constants), then we have for any $\varepsilon > 0$ the following Young's[14] *inequality with ε :*

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \forall a, b \geq 0,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $C_\varepsilon = \frac{1}{q(\varepsilon p)^{-\frac{q}{p}}}$.

[12] Augustin Louis, Baron Cauchy (August 21, 1789, in Paris - May 23, 1857, in Sceaux (Hauts-de-Seine)) is a French mathematician. He was one of the most prolific mathematicians, behind Euler, with almost 800 publications.

[13] Hermann Amandus Schwarz was born on January 25, 1843, in Poland and died on November 30, 1921, in Berlin. He is a famous mathematician whose work is marked by a strong interaction between analysis and geometry.

[14] William Henry Young (London, October 20, 1863 - Lausanne, July 7, 1942) is an English mathematician from Cambridge University who worked at the University of Liverpool and that of Lausanne.

For $p = q = 2$, we get other writing of Young's inequality with ε

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^{2 15},$$

or

$$|ab| \leq \frac{1}{p} \varepsilon \, |a|^p + \frac{p-1}{p} \left| \frac{b}{\varepsilon} \right|^{\frac{p}{p-1}}, \forall p > 1,$$

where ε is any positive constant.

Lemma 1.10. *($Gronwell's\ Inequality$) Let $T > 0, \varphi$ be a function such that $\varphi \in L^1(0,T), \varphi \geq 0$, almost everywhere and $\phi \in L^1(0,T), \phi \geq 0$, almost everywhere and $\varphi\phi \in L^1(0,T), C_1, C_2 \geq 0$. Suppose that*

$$\phi(t) \leq C_1 + C_2 \int_0^t \varphi(s)\, \phi(s)\, ds, \ \ a.e\ t \in \left]0, T\right[.$$

So we have

$$\phi(t) \leq C_1 e^{\left(C_2 \int_0^t \varphi(s) ds \right)}, \ \ a.e\ t \in \left]0, T\right[.$$

Lemma 1.11. *($Minkowski\ Inequality$) For $1 \leq p \leq \infty$, we have :*

$$\|u + v\|_{L^p} = \|u\|_{L^p} + \|v\|_{L^p}.$$

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Proof. Taking the well-known result

$$(2\varepsilon a - b)^2 \geq 0 \ \ for\ all\ a, b \in \mathbb{R}^n,$$

for all $\varepsilon > 0$, we have

$$4\varepsilon^2 a^2 + b^2 - 4\varepsilon ab \geq 0.$$

This implies

$$4\varepsilon ab \leq 4\varepsilon^2 a^2 + b^2,$$

consequently,

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

This ends the proof. □

Definition 1.7. (*Integration by Part*) *Let* $(u,v) \in H^1(\Omega)$, *for* $1 \leq i \leq n$ *so we have*

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = -\int_{\Omega} \frac{\partial v}{\partial x_i} u dx + \int_{\partial\Omega} u v \eta_i d\sigma,$$

where $\eta_i(x) = cos(\eta, x_i)$ *is the directing cosine of the angle between the normal outside* $\partial\Omega$ *at the point and the* x_i *axis.*

Lemma 1.12. (*Green's Formula*)[16] *For all* $u \in H^2(\Omega)$ *and* $v \in H^1(\Omega)$ *we have:*

$$-\int_{\Omega} \Delta u v dx = \int_{\Omega} \nabla u \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v ds,$$

where $\frac{\partial u}{\partial \eta}$ *is the normal derivative of* u *over* $\partial\Omega$.

Existence Method

Here, we state the fixed point theorem which is called the contraction mapping theorem. We use this theorem to prove the existence and the uniqueness of the solution of our nonlinear problem.

Definition 1.8. *Let* f *be a map of a metric space* E *to it self; i.e.* $f : E \to E$. *A point* $x \in X$ *is called a fixed point of* f *if*

$$f(u) = u.$$

Definition 1.9. *Let* (E, d_E) *and* (F, d_F) *be two metric spaces. The map* $\varphi : E \longrightarrow F$ *is called a contraction if there exists a positive constant* $C < 1$ *such that*

$$d_F(\varphi(u), \varphi(v)) \leq C d_E(u,v),$$

for all $x, y \in X$.

Theorem 1.3. (*Contraction Mapping Theorem*) *Let* (E, d) *be a complete metric space. If* $\varphi : E \longrightarrow E$ *is a contraction, then* φ *admits a unique fixed point.*

[16] George Green (July 1793 - 31 May 1841), physicien britannique.

1.2.2 On the Spaces $W^{m,p(x)}(\Omega)$ (Variable Exponents Sobolev Spaces)

In this subsection, we recall some preliminaries and definitions about Sobolev spaces with variable exponents and we study some functional analysis-type properties of these spaces.

Definition 1.10. *(Weak Derivative) Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $u \in L^1_{loc}(\Omega)$. Let $\alpha := (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ be a multi-index and let $|\alpha| = \alpha_1 + \cdots + \alpha_n$.*

If there exists $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u \frac{\partial^{|\alpha|}\psi}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n} dx = (-1)^{|\alpha|} \int_\Omega \psi g dx,$$

for all $\psi \in C_0^\infty(\Omega)$, then g is called a weak partial derivative of u of order α. The function g is denoted by $\partial^\alpha u$ or $\dfrac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n}$.

Definition 1.11. *Let $m \in \mathbb{N}$. The space $W^{m,p(.)}(\Omega)$ is defined as follows*

$$W^{m,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega) \text{ such that } \partial^{|\alpha|}u \in L^{p(.)}(\Omega), \ \forall |\alpha| \le m \right\}.$$

A semi-modular on $W^{m,p(.)}(\Omega)$ defined by

$$\varrho_{W^{m,p(\cdot)}(\Omega)}(u) = \sum_{0 \le |\alpha| \le m} \varrho_{L^{p(\cdot)}(\Omega)}(\partial_\alpha u).$$

This induces a norm [41] given by

$$\|u\|_{W^{m,p(\cdot)}(\Omega)} := \inf\left\{ \lambda > 0 : \varrho_{W^{m,p(\cdot)}(\Omega)}\left(\frac{u}{\lambda}\right) \le 1 \right\} := \sum_{0 \le |\alpha| \le m} \|\partial_\alpha u\|_{p(.)}.$$

For $m \in \mathbb{N}$, the space $W^{m,p(.)}(\Omega)$ is named Sobolev space and its elements are named Sobolev functions. Obviously $W^{0,p(.)}(\Omega) = L^{p(\cdot)}(\Omega)$ and

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{p(.)}(\Omega) \right\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}$.

We abbreviate $\|u\|_{W^{m,p(\cdot)}(\Omega)}$ to $\|u\|_{m,p(.)}$ and $\varrho_{W^{m,p(\cdot)}(\Omega)}$ to $\varrho_{m,p(.)}$. The Banach space $W_0^{1,p(.)}(\Omega)$ with $p(x) \in [p_1, p_2] \subset [1, \infty)$ is defined by

$$W_0^{1,p(.)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega), (|u|, |\nabla u|) \in L^{p(.)}(\Omega) \right\}.$$

An equivalent norm of $W_0^{1,p(.)}(\Omega)$ *is given by*

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(.)}.$$

If $p = 2$, *then* $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

Theorem 1.4. *Let* $p \in \mathcal{P}(\Omega)$. *The space* $W^{m,p(.)}(\Omega)$ *is a Banach space, which is reflexive if* $1 < p_1 \leq p_2 < +\infty$, *and separable if* p *is bounded* [17].

Definition 1.12. *Let* $p \in \mathcal{P}(\Omega)$ *and* $m \in \mathbb{N}$. *The Sobolev space* $W_0^{m,p(.)}(\Omega)$ *"with zero boundary trace" is the closure in* $W^{m,p(.)}(\Omega)$ *of the set of* $W^{m,p(.)}(\Omega)$*-functions with compact support, i. e.,*

$$W_0^{m,p(.)}(\Omega) = \overline{\{u \in W^{m,p(\cdot)}(\Omega) : u = u_{\chi K} \text{ for a compact } K \subset \Omega\}}.$$

Remark 1.2. [41] *Let* $p \in \mathcal{P}(\Omega)$ *and* $m \in \mathbb{N}$. *Then*

(i) The space $H_0^{m,p(\cdot)}(\Omega)$ *is defined as the closure of* $C_0^\infty(\Omega)$ *in* $W^{m,p(.)}(\Omega)$. *Furthermore, we set* $W_0^{1,p(.)}(\Omega)$ *to be the closure of* $C_0^\infty(\Omega)$ *in* $W^{1,p(.)}(\Omega)$. *Here we note that the space* $W^{1,p(.)}(\Omega)$ *is usually defined in a different way for the variable exponent case.*

(ii) $H_0^{m,p(\cdot)}(\Omega) \subset W_0^{m,p(\cdot)}(\Omega)$.

(iii) If p *is log-Hölder continuous on* Ω , *then* $W_0^{m,p(.)}(\Omega) = H_0^{m,p(\cdot)}(\Omega)$.

(iv) The dual of $W_0^{1,p(.)}(\Omega)$ *is defined as* $W_0^{-1,p'(.)}(\Omega)$, *in the same way as the usual (classical) Sobolev spaces, where* $\dfrac{1}{p(\cdot)} + \dfrac{1}{p'(\cdot)} = 1$.

Theorem 1.5. *Let* $p \in \mathcal{P}(\Omega)$. *The space* $W_0^{m,p(.)}(\Omega)$ *is a Banach space, which is separable if* p *is bounded, and reflexive if* $1 < p_1 \leq p_2 < +\infty$.

Lemma 1.13. *(Poincaré's Inequality)*[18] [41] *Let* Ω *be a bounded domain of* $\mathbb{R}^n$ *and* $p(.)$ *satisfies the Log-Hölder continuous property on* Ω, *then*

$$\|u\|_{p(.)} \leq C \|\nabla u\|_{p(.)}, \quad \text{for all } u \in W_0^{1,p(.)}(\Omega),$$

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Proof. See reference [41] page 249. □

18 Henri Poincaré (April 29, 1854, in Nancy - July 17, 1912, in Paris) is a mathematician, physicist and, a French philosopher. Theoretical engineer, his contributions to many fields of mathematics and physics have radically changed these two sciences.

where the positive constant C depends on Ω, p_1, p_2 only.

Remark 1.3. *Note that the following inequality*

$$\int_\Omega |u|^{p(x)}\,dx \leq C\int_\Omega |\nabla u|^{p(x)}\,dx,$$

does not in general hold.

Remark 1.4. *The log-Hölder continuity condition on $p(.)$ can be substituted by $p(.) \in C\left(\bar{\Omega}\right)$, if Ω is bounded.*

Remark 1.5. *Inversion of the constant-exponent case, the Poincaré inequality version for modular does not exist. The following example clarifies that the Poincaré inequality does not generally hold in a modular form.*

Example 1.4. [41] *Let $p : (-2, 2) \to [2, 3]$ be a Lipschitz continuous exponent defined by*

$$p(x) = \begin{cases} 3, & \text{if } x \in (-2,-1) \cup (1,2) \\ 2, & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -2x+1, & \text{if } x \in \left[-1, -\frac{1}{2}\right] \\ 2x+1, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Let u_μ be a Lipschitz function defined by

$$u_\mu(x) = \begin{cases} \mu x + 2\mu, & \text{if } x \in (-2,-1] \\ \mu, & \text{if } x \in (-1,1) \\ -\mu x + 2\mu, & \text{if } x \in [1,2). \end{cases}$$

Then

$$\frac{\varrho(u_\mu)}{\varrho\left(u'_\mu\right)} = \frac{\displaystyle\int_{-2}^{2} |u_\mu|^{p(x)}\,dx}{\displaystyle\int_{-2}^{2} \left|u'_\mu\right|^{p(x)}\,dx} \geq \frac{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \mu^2 dx}{2\displaystyle\int_{-2}^{-1} \mu^3 dx} = \frac{1}{2\mu} \to \infty,$$

as $\mu \to 0^+$.

Now, we recall some basic embedding results which are necessary for the proofs in this dissertation.

Lemma 1.14. [41] *Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Suppose that $p : \Omega \to [1, \infty)$ is a measurable function such that*

$$1 < p_1 \leq p(x) \leq p_2 < +\infty, \text{ for a.e.} x \in \Omega.$$

If $p(x), q(x) \in C(\overline{\Omega})$ and $q(x) < p^*(x)$ in $\overline{\Omega}$ with $p^*(x) = \begin{cases} \dfrac{np(x)}{n-p(x)}, & if \quad p_2 < n \\ \infty, & if \quad p_2 \geq n. \end{cases}$

Then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

As a special case, we have

Corollary 1.1. [41] *Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Suppose that $p(.) \in C(\overline{\Omega})$ is a continuous function such that*

$$2 \leq p_1 \leq p(x) \leq p_2 < \frac{2n}{n-2}, \qquad n \geq 3. \tag{1.8}$$

Then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.

1.2.3 Elementary Properties

We list here the most important properties of variable exponent Lebesgue and Sobolev spaces which hold without advanced conditions on the exponent. In another way, we collect properties that do not require any regularity of the exponent

For Any Measurable Exponent p

- $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ are Banach spaces.
- The modular $\varrho_{p(\cdot)}$ and the norm $\|\cdot\|_{p(\cdot)}$ are lower semicontinuous[19] with respect to (sequential) weak convergence and almost everywhere convergence.
- Hölder's inequality holds.

[19]**Theorem:** Let ϱ be a semimodular on X. Then ϱ is lower semicontinuous on X_ϱ, i.e.

$$\varrho(x) \leq \liminf_{k\to\infty} \varrho(x_k),$$

for all $x_k, x \in X_\varrho$ with $x_k \to x$ (in norm) for $k \to \infty$.

- $L^{p(\cdot)}$ is a Banach function space.
- $(L^{p(\cdot)})' \approxeq L^{p'(\cdot)}$ and the norm conjugate formula holds.

For Any Measurable Bounded Exponent p

- $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ are separable spaces.
- The Δ_2-condition holds, i.e. modular convergence and norm convergence are the same.
- Bounded functions are dense in $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$.
- C_0^{∞} is dense in $L^{p(\cdot)}$.

For Any Measurable Exponent p with $1 < p_1 \leq p_2 < \infty$

- $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ are reflexive.
- $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ are uniformly convex.

1.2.4 Warnings!

In this subsection, we list some results, properties, and techniques from constant exponent spaces which essentially never hold in the variable exponent setting even when the exponent is very regular, e.g, $p \in \mathcal{P}^{log}$[20] or $p \in C^{\infty}\left(\overline{\Omega}\right)$ with $1 < p_1 \leq p_2 < \infty$.

- The space $L^{p(\cdot)}$ is not rearrangement invariant.

- The translation operator

$$T_h : L^{p(\cdot)} \to L^{p(\cdot)}, T_h f(x) := f(x+h),$$

is not bounded.

- Young's convolution inequality

$$\|f * g\|_{p(.)} \leq c \, \|f_1\| \, \|g\|_{p(.)} \, ,$$

does not hold.

[20] $\mathcal{P}^{log}\left(\Omega\right) := \left\{p \in \mathcal{P}\left(\Omega\right) : 1/p \text{ is globally log-Hölder continuous.}\right\}$, such that $\mathcal{P}\left(\Omega\right)$: Set of variable exponents.

- The formula

$$\int_{\Omega} |f(x)|^p \, dx = p \int_0^{\infty} t^{p-1} |\{x \in \Omega : |f(x)| > t\}| \, dt,$$

has no variable exponent analogue.

- Maximal, Poincaré, Sobolev, etc., inequalities do not hold in a modular form. For instance, A. Lerner showed that

$$\int_{\mathbb{R}^n} |Mf|^{p(x)} \, dx \leq c \int_{\mathbb{R}^n} |f|^{p(x)} \, dx$$

if and only if $p \in [1, \infty)$ is constant.

1.2.5 Similarity

In general, variable-exponent and classical Lebesgue spaces are similar in many aspects. For the following assertions, see [40]:

- The Hölder inequality holds.

- They are reflexive if and only if $1 < p_1 \leq p_2 < \infty$.

- Continuous functions are dense if $p_2 < \infty$.

- If Ω has a finite measure and p, q are variable exponents so that $p(x) \leq q(x)$ almost everywhere in Ω, then the embedding $L^{q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ holds.

- The spaces $W_0^{1,p(.)}(\Omega)$ and $W^{-1,p'(.)}(\Omega)$ are defined by the same way as the usual Sobolev spaces where $p'(.)$ is the function such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

1.3 Notions of Blow-Up

We are interested sometimes by the behavior of solutions of a specific problem for an evolution PDE, particularly, if this PDE describe a concrete phenomenon, for example, propagation of pollutant in the air, if we indicate the concentration of this pollutant in the point x at the time t by $u(t, x)$, so it is reasonable that one has $\underset{t \to \infty}{lim} u(t, x) = 0$ since there will be no pollutant in the great distance.

From this point of view we begin, and have the following definition

Definition 1.13. *Let $\Omega \subset \mathbb{R}^N$ and $u = u(t,x)$ be a solution of a given evolution PDE on the set $\Omega := [0,T] \times A$. We say that u blows up in finite time T if such that*

$$\lim_{t \to T^-} |u(t,x)| = +\infty.$$

In this case one has

$$\sup_{x \in \Omega} |u(t,x)| = +\infty,$$

and T is called the time of Blow-up.

1.3.1 Referential Examples

Case of ODE

The simplest example to show the blow-up[21] phenomena in the case of ordinary differential equations (ODE) is the following (non-linear) Cauchy problem

$$x'(t) = x^2(t),\ t > 0,\ x(0) = x_0.$$

One can show immediately that if $x_0 > 0$ for some $T > 0$ then, the previous Cauchy problem admits the unique solution $x(t) = \dfrac{1}{T-t}$ in the interval $]0,\, T[$. This solution is a smooth function on $]0,\, T[$ and satisfies in particular at $\underset{t \to T^-}{lim} x(t) = +\infty$. This means that, according to the previous definition, the solution blows up in finite time. One can think to generalize this remark as the main phenomenon of ODEs and PDEs.

Case of PDE

The Blow-up's phenomena appear especially when the unknown function in the considered problem depends not only on time but also on the spacial variable, especially in the reaction-diffusion problems, propagation evolution problems, the famous example is the following Cauchy problem of Fujita's equation

$$\begin{cases} u_t = \Delta u + u^p \\ u(0,x) = u_0(x), x \in \mathbb{R}^N. \end{cases}$$

[21] If $T_{max} < \infty$, we say that the solution of our problems blows up and that T_{max} is the blow-up time. If $T_{max} = \infty$, we say that the solution is global.

Where the unknown function $u = u(t,x)$ is real-valued, $t > 0$, $p > 1$, and Δ is the classical Laplace[22] operator.

This equation is studied by Fujita in **1966**, particularly, he showed that if $1 < p < 1 + 2/N$ then all solutions in a given class blow up in finite time.

[22] Pierre-Simon Laplace, born March 23, 1749, in Beaumont-en-Auge (Calvados), died March 5 1827 in Paris, was a French mathematician, astronomer, and physicist particularly famous for his work in five volumes Céleste Mechanics.

Chapter 2

Existence and Blow-Up of a New Class of Nonlinear Damped Wave Equation

Key Words and Phrases: Wave equation, existence and uniqueness, Faedo-Galerkin, blow-up.

Our purpose in this chapter is to demonstrate the well-posedness and the finite-time blow-up of solutions of the following nonlinear wave equation with variable exponents:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + a u_t |u_t|^{m(\cdot)-2} = b u |u|^{p(\cdot)-2}, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega, a, b > 0$ are constants and the exponents $m(\cdot)$ and $p(\cdot)$ are given measurable functions defined on Ω.

This chapter is divided into three sections: Some necessary assumptions needed in this chapter are presented **in Section 2.1**. **In Section 2.2**, we demonstrate the well-posedness of the problem by using the famous Faedo Galerkin method. Then, by using the well-known contraction mapping theorem, we can show the local existence of (2.1). **In Section2.3,** we list some technical lemmas and we state with the proof our main result of blow up.

2.1 Basic Assumptions

We present in this section the most important basic materials that we need in the proof of our results and achieve the well-posedness of the problem. We utilize the Sobolev space $H_0^1(\Omega)$ and the standard Lebesgue space $L^2(\Omega)$ with their usual scalar products and norms. First, we assume the following hypotheses:

(H1) The exponents m and p are measurable functions such that either $m, p \in C(\bar{\Omega})$ or they satisfy the following log-Hölder continuity condition:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{for a.e } x, y \in \Omega, \text{with } |x-y| < \delta. \tag{2.2}$$

$A > 0, 0 < \delta < 1$[1].

(H2) We suppose for the nonlinearity in the damping that

$$2 \leq m_1 \leq m(x) \leq m_2 \leq \frac{2n}{n-2},\ n \geq 3. \tag{2.3}$$
$$2 \leq m_1 \leq m(x) \leq m_2 < +\infty,\ n < 3.$$

(H3) We suppose for the nonlinearity in the source term that

$$2 \leq p_1 \leq p(x) \leq p_2 \leq 2\frac{n-1}{n-2},\ n \geq 3. \tag{2.4}$$
$$2 \leq p_1 \leq p(x) \leq p_2 < +\infty,\ n < 3.$$

(H4) We furthermore suppose that

$$2 \leq m_1 \leq m(x) \leq m_2 < p_1 \leq p(x) \leq p_2 \leq \frac{2n}{n-2}, \tag{2.5}$$

this condition is necessary for the result of blow-up.

The energy associated to the problem (2.1) is presented as follows

$$E(t) := \frac{1}{2}\int_\Omega \left[u_t^2 + |\nabla u|^2 + |\nabla u_t|^2\right] dx - b\int_\Omega \frac{|u|^{p(x)}}{p(x)} dx,\ t \geq 0, \tag{2.6}$$

direct derivative of (2.6) and using problem (2.1), gives us

$$E'(t) = -a\int_\Omega |u_t(x,t)|^{m(x)} dx. \tag{2.7}$$

[1]almost everywhere, that is to say everywhere except possibly on a set of zero measure.

2.2 The Well-Posedness of the Problem

Our aim in this chapter is to study the local existence and uniqueness (or better local well-posedness) of the weak solution of the problem (2.1). We consider for this goal the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + a u_t |u_t|^{m(\cdot)-2} = f(x,t), & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{2.8}$$

where $a > 0$ is a constant, $f \in L^2(\Omega \times (0,T)), (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the exponent $m(\cdot)$ is a given measurable function satisfying **(H1)-(H2)** and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, we will prove the local existence of problem (2.8) by using the Faedo-Galerkin method. Then, by using the well-known contraction mapping theorem, we can appear the local existence of (2.1). In our proof, we followed closely the techniques due to Georgiev and Todorova [31], with appropriate modifications imposed by the nature of our problem.

Theorem 2.1. *Let $m \in C(\bar{\Omega})$. Under condition **(H2)**, problem $(2,8)$ has a unique local solution*

$$\begin{aligned} u &\in L^\infty((0,T), H_0^1(\Omega)), \\ u_t &\in L^\infty((0,T), H_0^1(\Omega)) \cap L^{m(\cdot)}(\Omega \times (0,T)), \\ u_{tt} &\in L^2((0,T), H^{-1}(\Omega)). \end{aligned}$$

2.2.1 Proof of Theorem 2. 1

Existence

Proof. Here, we prove the local existence by using Faedo-Galerkin's method, which consists to construct approximations of the solutions, then we get prior estimates necessary to guarantee the convergence of approximations. This method has proven to be an effective tool in the study of nonclassical problems, such problems have been studied by several authors for different types of parabolic, hyperbolic, and mixed type equations. We divide our proof into three steps:

- In the first step, we introduce an approach problem in a bounded dimension space V_n which has a unique solution v_n.

- In the second step, we derive the various a priori estimates.

- In the third step, we will pass to the limit of the approximations by using the compactness of some embedding in the Sobolev spaces.

Let $\{v_j\}_{j=1}^{\infty}$ be an orthonormal basis of $H_0^1(\Omega)$, with

$$\begin{aligned} -\Delta v_j &= \lambda_j v_j, \text{ in } \Omega, \\ v_j &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and represent for every $n \geq 1$, the finite-dimensional subspace $V_k = span\{v_1, ..., v_k\}$. By normalization, we get $\|v_j\|_2 = 1$, denote by λ_j the related eigenvalues, where v_j are solutions of the previous initial boundary value problem[2].

We look for functions

$$u^k(x,t) = \sum_{j=1}^{k} a_j(t) v_j,$$

which satisfy the following approximate problems

$$\begin{aligned} &\int_\Omega u_{tt}^k(x,t)\, v_j(x)\, dx + \int_\Omega \nabla u^k(x,t)\, \nabla v_j(x)\, dx \\ &+ \int_\Omega \nabla u_{tt}^k(x,t)\, \nabla v_j(x)\, dx + a \int_\Omega \left|u_t^k(x,t)\right|^{m(x)-2} u_t^k(x,t)\, v_j(x)\, dx \\ = &\int_\Omega f(x,t)\, v_j(x)\, dx, \quad u^k(x,0) = u_0^k, \quad u_t^k(x,0) = u_1^k, \quad \forall j = 1, 2, \cdots, k, \end{aligned} \tag{2.9}$$

where $u_0^k = \sum_{i=1}^{k} (u_0, v_i) v_i$, $u_1^k = \sum_{i=1}^{k} (u_1, v_i) v_i$ are two sequences in $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively, such that

$$u_0^k \to u_0 \text{ in } H_0^1(\Omega) \text{ and } u_1^k \to u_1 \text{ in } L^2(\Omega).$$

[2] Dirichlet's spectral problem

$$\begin{aligned} -\Delta e_j &= \lambda_j e_j, \text{ in } \Omega, \quad j = 1, \cdots, m, \\ e_j &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

admits a sequence of non-zero solutions e_j, corresponding to a sequence of eigenvalues $\lambda_j > 0$. The functions e_j will be used as special bases in the Faedo-Galerkin method.

This generates the system of k ordinary differential equations

$$\begin{cases} a_j''(t) + \lambda_j a_j(t) + \lambda_j a_j''(t) = g_j(t) + G_j(a_1'(t), ..., a_k'(t)) \\ a_j(0) = (u_0, v_j), \quad a_j'(0) = (u_1, v_j), \qquad \forall j = 1, 2, ..., k, \end{cases} \tag{2.10}$$

where

$$g_j(t) = \int_\Omega f(x,t) v_j(x) dx,$$

and

$$G_j(a_1'(t), ..., a_k'(t)) = -a \int_\Omega \left| \sum_{i=1}^{k} a_i'(t) v_i(x) \right|^{m(x)-2} a_i'(t) v_i(x) v_j(x) dx.$$

Because

$$\int_\Omega u_{tt}^k(x,t) - \int_\Omega \Delta u^k(x,t) - \int_\Omega \Delta u_{tt}^k(x,t) + a \int_\Omega \left| u_t^k(x,t) \right|^{m(x)-2} u_t^k(x,t) = \int_\Omega f(x,t),$$

then

$$\begin{aligned} &\int_\Omega u_{tt}^k(x,t) v_j(x) - \int_\Omega \Delta u^k(x,t) v_j(x) - \int_\Omega \Delta u_{tt}^k(x,t) v_j(x) \\ &+ a \int_\Omega \left| u_t^k(x,t) \right|^{m(x)-2} u_t^k(x,t) v_j(x) = \int_\Omega f(x,t) v_j(x), \end{aligned}$$

hence

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{k} a_j''(t) v_j(x) v_j(x) dx - \int_{\partial\Omega} \nabla u^k(x,t) v_j(x) dx \\ &+ \int_\Omega \nabla u^k(x,t) \nabla v_j(x) dx - \int_{\partial\Omega} \nabla u_{tt}^k(x,t) v_j(x) dx \\ &+ \int_\Omega \nabla u_{tt}^k(x,t) \nabla v_j(x) dx + a \int_\Omega \left| u_t^k(x,t) \right|^{m(x)-2} u_t^k(x,t) v_j(x) dx \\ = &\int_\Omega f(x,t) v_j(x) dx. \end{aligned}$$

The term $\int_{\partial\Omega} \nabla u^k(x,t) v_j(x)$ and $\int_{\partial\Omega} \nabla u_{tt}^k(x,t) v_j(x)$ equal zero because $v_j(x) = 0$ on $\partial\Omega$, so we get

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{k} a_j''(t) v_j(x) v_j(x) + \int_\Omega \nabla u^k(x,t) \nabla v_j(x) \\ &+ \int_\Omega \nabla u_{tt}^k(x,t) \nabla v_j(x) + a \int_\Omega \left| u_t^k(x,t) \right|^{m(x)-2} u_t^k(x,t) v_j(x) \\ = &\int_\Omega f(x,t) v_j(x), \end{aligned}$$

then

$$\begin{aligned} & \int_{\Omega}\sum_{i=1}^{k} a_j^{''}(t)\, v_j(x)\, v_j(x) - \int_{\Omega}\Delta v_j(x)\, u^k(x,t) \\ & + \int_{\partial\Omega}\nabla v_j(x)\, u^k(x,t) - \int_{\Omega}\Delta v_j(x)\, u_{tt}^k(x,t) \\ & + \int_{\partial\Omega}\nabla v_j(x)\, u_{tt}^k(x,t) + a\int_{\Omega}\left|u_t^k(x,t)\right|^{m(x)-2} u_t^k(x,t)\, v_j(x) \\ = & \int_{\Omega} f(x,t)\, v_j(x), \end{aligned}$$

thus

$$\begin{aligned} & \int_{\Omega}\sum_{i=1}^{k} a_j^{''}(t)\, v_j(x)\, v_j(x) + \int_{\Omega}\lambda_j v_j(x)\sum_{i=1}^{k} a_j(t)\, v_j(x) \\ & + \int_{\partial\Omega}\nabla v_j(x)\, u^k(x,t) + \int_{\Omega}\lambda_j v_j(x)\sum_{i=1}^{k} a_j^{''}(t)\, v_j(x) \\ & + \int_{\partial\Omega}\nabla v_j(x)\, u_{tt}^k(x,t) + a\int_{\Omega}\left|u_t^k(x,t)\right|^{m(x)-2} u_t^k(x,t)\, v_j(x) \\ = & \int_{\Omega} f(x,t)\, v_j(x), \end{aligned}$$

so

$$\begin{aligned} & a_j^{''}(t) + \lambda_j a_j(t) + \lambda_j a_j^{''}(t) + \int_{\partial\Omega}\nabla v_j(x)\, u^k(x,t) \\ & + \int_{\partial\Omega}\nabla v_j(x)\, u_{tt}^k(x,t) + a\int_{\Omega}\left|u_t^k(x,t)\right|^{m(x)-2} u_t^k(x,t)\, v_j(x) \\ = & \int_{\Omega} f(x,t)\, v_j(x). \end{aligned}$$

Hence

$$a_j^{''}(t) + \lambda_j a_j(t) + \lambda_j a_j^{''}(t) = g_j(t) + G_j(a_1'(t), \cdots, a_k'(t)),$$

where

$$g_j(t) = \int_{\Omega} f(x,t)\, v_j(x),$$

and

$$\begin{aligned} G_j(a_1'(t), \cdots, a_k'(t)) = & -a\int_{\Omega}\left|\sum_{i=1}^{k} a_i'(t)\, v_i(x)\right|^{m(x)-2}\sum_{i=1}^{k} a_i'(t)\, v_i(x)\, v_j(x)\, dx \\ & -\int_{\partial\Omega}\nabla v_j(x)\sum_{i=1}^{k} a_i(t)\, v_i(x) - \int_{\partial\Omega}\nabla v_j(x)\sum_{i=1}^{k} a_i^{''}(t)\, v_i(x). \end{aligned}$$

Now, if we have $v_j = 0$ on $\partial\Omega$ so $\nabla v_j = 0$ also on $\partial\Omega$, and we obtain that

$$G_j(a_1'(t), \cdots, a_k'(t)) = -a \int_\Omega \left| \sum_{i=1}^{k} a_i'(t)\, v_i(x) \right|^{m(x)-2} \sum_{i=1}^{k} a_i'(t)\, v_i(x)\, v_j(x)\, dx.$$

This system can be solved by standard ODE theory. Thence, we get functions

$$a_j : [0, t_k) \to \mathbb{R}, \quad 0 < t_k < T.$$

Next, we have to appear that $t_k = T, \forall k \geq 1$.

Multiplying (2.9) by $a_j'(t)$ and sum over j to obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left[\int_\Omega \left(\begin{array}{c} |u_t^k(x,t)|^2 dx + |\nabla u^k(x,t)|^2 \\ +|\nabla u_t^k(x,t)|^2 \end{array} \right) dx \right] \\ & +a\int_\Omega |u_t^k(x,t)|^{m(x)} dx \\ = & \int_\Omega f(x,t) u_t^k(x,t) dx. \end{aligned}$$

Integrating over $(0, t)$ to get

$$\begin{aligned} & \frac{1}{2}\int_\Omega \left(|u_t^k(x,t)|^2 dx + |\nabla u^k(x,t)|^2 + |\nabla u_t^k(x,t)|^2\right) dx + a\int_0^t \int_\Omega |u_t^k(x,s)|^{m(x)} dx ds \\ = & \frac{1}{2}\int_\Omega \left(|u_1^k|^2 + |\nabla u_0^k|^2 + |\nabla u_1^k|^2\right) dx + \int_0^t \int_\Omega f(x,s) u_t^k(x,s) dx ds \qquad (2.11) \\ \leq & \frac{1}{2}\int_\Omega \left(u_1^2 + |\nabla u_0|^2 + |\nabla u_1|^2\right) dx + \varepsilon \int_0^t \int_\Omega |u_t^k|^2 dx ds + c_\varepsilon \int_0^t \int_\Omega f^2 dx ds \\ \leq & C_\varepsilon + \varepsilon \sup_{(0,t_k)} \int_\Omega |u_t^k(x,t)|^2 dx, \quad \forall t \in [0, t_k). \end{aligned}$$

Where

$$C_\varepsilon = \frac{1}{2}\int_\Omega \left(u_1^2 + |\nabla u_0|^2 + |\nabla u_1|^2\right) dx + c_\varepsilon \int_0^t \int_\Omega f^2 dx ds.$$

Then, we obtain

$$\begin{aligned} & \frac{1}{2}\sup_{(0,t_k)} \int_\Omega |u_t^k(x,t)|^2 dx + \frac{1}{2}\sup_{(0,t_k)} \int_\Omega |\nabla u^k(x,t)|^2 dx \qquad (2.12) \\ & +\frac{1}{2}\sup_{(0,t_k)} \int_\Omega |\nabla u_t^k(x,t)|^2 dx + a\int_0^{t_k} \int_\Omega |u_t^k(x,s)|^{m(x)} dx ds \\ \leq & C_\varepsilon + \varepsilon \sup_{(0,t_k)} \int_\Omega |u_t^k(x,t)|^2 dx, \quad \forall t \in [0, t_k). \end{aligned}$$

Picking $\varepsilon = \frac{1}{4}$, we arrive at

$$\begin{aligned} & \sup_{(0,t_k)} \int_\Omega |u_t^k(x,t)|^2 dx + \sup_{(0,t_k)} \int_\Omega |\nabla u^k(x,t)|^2 dx \\ & + \sup_{(0,t_k)} \int_\Omega |\nabla u_t^k(x,t)|^2 dx + a \int_0^{t_k} \int_\Omega |u_t^k(x,s)|^{m(x)} dx ds \\ \leq \; & C. \end{aligned}$$

Therefore, the solution can be expanded to $[0,T)$ and, in addition, we get

$$\begin{aligned} & (u^k) \text{ is a bounded sequence in } L^\infty((0,T), H_0^1(\Omega)) \\ & (u_t^k) \text{ is a bounded sequence in } L^\infty((0,T), H_0^1(\Omega)) \cap L^{m(\cdot)}(\Omega \times (0,T)). \end{aligned}$$

Thus, we can extract a subsequence (u^ℓ) such that

$$\begin{aligned} u^\ell & \rightarrow u \text{ weakly} * \text{in } L^\infty((0,T), H_0^1(\Omega)) \\ u_t^\ell & \rightarrow u_t \text{ weakly} * \text{in } L^\infty((0,T), H_0^1(\Omega)) \text{ and weakly in } L^{m(\cdot)}(\Omega \times (0,T)). \end{aligned}$$

We can conclude by Lion's Lemma [48] that $u \in C([0,T], H_0^1(\Omega))$ so that $u(x,0)$ has a meaning[3]. Since (u_t^ℓ) is bounded in $L^{m(\cdot)}(\Omega \times (0,T))$ then $|u_t^\ell|^{m(x)-2}u_t^\ell$ is bounded in $L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega \times (0,T))$; thence, up to a subsequence,

$$|u_t^\ell|^{m(x)-2}u_t^\ell \rightarrow \psi \text{ weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega \times (0,T)).$$

We have to show that $\psi = |u_t|^{m(x)-2}u_t$. We utilize u^ℓ instead of u^k in (2.9) and integrate over $(0,t)$ to obtain

$$\begin{aligned} & \int_\Omega u_t^\ell v_j - \int_\Omega u_1^\ell v_j + \int_0^t \int_\Omega \nabla u^\ell . \nabla v_j + \int_0^t \int_\Omega \nabla u_{tt}^\ell . \nabla v_j + a \int_0^t \int_\Omega |u_t^\ell|^{m(x)-2} u_t^\ell v_j \\ = \; & \int_0^t \int_\Omega f v_j dx, \quad \forall j < \ell. \end{aligned}$$

As ℓ goes to $+\infty$, we facilely check that

$$\begin{aligned} & \int_\Omega u_t v_j - \int_\Omega u_1 v_j + \int_0^t \int_\Omega \nabla u . \nabla v_j + \int_0^t \int_\Omega \nabla u_{tt} . \nabla v_j + a \int_0^t \int_\Omega |u_t|^{m(x)-2} u_t v_j \\ = \; & \int_0^t \int_\Omega f v_j dx, \quad \forall j \geq 1. \end{aligned}$$

[3] In the case $p = \infty$ the symbol * is posed to show that the definition of weak convergence in $L^\infty(\Omega)$ is not entirely the same as in the spaces $L^p(\Omega), 1 \leq p < \infty$. Indeed, the dual of $L^\infty(\Omega)$ is strictly larger than $L^1(\Omega)$.

Therefore,

$$\begin{aligned}
&\int_\Omega u_t v - \int_\Omega u_1 v + \int_0^t \int_\Omega \nabla u.\nabla v + \int_0^t \int_\Omega \nabla u_{tt}.\nabla v + a \int_0^t \int_\Omega |u_t|^{m(x)-2} u_t v \\
= \; & \int_0^t \int_\Omega f v dx, \quad \forall v \in H_0^1(\Omega).
\end{aligned}$$

All terms define absolute continuous functions; so we obtain, for a.e $t \in [0, T]$,

$$\frac{d}{dt}\int_\Omega u_t v + \int_\Omega (\nabla u.\nabla v + \nabla u_{tt}.\nabla v + a\psi v) = \int_\Omega f v, \forall v \in H_0^1(\Omega). \tag{2.13}$$

This implies that

$$u_{tt} - \Delta u - \Delta u_{tt} + \psi = f, \text{ in } D'(\Omega \times (0, T)). \tag{2.14}$$

For simplicity, let $A(v) = |v|^{m(x)-2} v$ and define

$$X^\ell = \int_0^T \int_\Omega (A(u_t^\ell) - A(v))(u_t^\ell - v) dt \geq 0, \ \forall v \in L^{m(\cdot)}((0, T), H_0^1(\Omega)).$$

Employing (2.11) and exchaging u^k by u^ℓ to obtain

$$\begin{aligned}
X^\ell \; = \; & \int_0^T \int_\Omega f u_t^\ell + \frac{1}{2}\int_\Omega \left(\left|u_1^\ell\right|^2 + \left|\nabla u_0^\ell\right|^2 + \left|\nabla u_1^\ell\right|^2\right) \\
& -\frac{1}{2}\int_\Omega \left|u_t^\ell(x, T)\right|^2 - \frac{1}{2}\int_\Omega \left|\nabla u^\ell(x, T)\right|^2 \\
& -\frac{1}{2}\int_\Omega \left|\nabla u_t^\ell(x, T)\right|^2 - \int_0^T \int_\Omega A(u_t^\ell) v \\
& - \int_0^T \int_\Omega A(v)(u_t^\ell - v).
\end{aligned} \tag{2.15}$$

Taking $\ell \to \infty$, we get

$$\begin{aligned}
0 \; \leq \; & \limsup X^\ell \leq \int_0^T \int_\Omega f u_t + \frac{1}{2}\int_\Omega \left(u_1^2 + |\nabla u_0|^2 + |\nabla u_1|^2\right) \\
& -\frac{1}{2}\int_\Omega |u_t(x, T)|^2 - \frac{1}{2}\int_\Omega |\nabla u(x, T)|^2 \\
& -\frac{1}{2}\int_\Omega |\nabla u_t(x, T)|^2 - \int_0^T \int_\Omega \psi v \\
& - \int_0^T \int_\Omega A(v)(u_t - v).
\end{aligned} \tag{2.16}$$

Remplacing v by u_t in (2.13) and integrating over $(0,T)$ to obtain

$$\begin{aligned}\int_0^T\int_\Omega fu_t &= \frac{1}{2}\int_\Omega |u_t(x,T)|^2 \\ &-\frac{1}{2}\int_\Omega u_1^2+\frac{1}{2}\int_\Omega |\nabla u(x,T)|^2 \\ &-\frac{1}{2}\int_\Omega |\nabla u_0|^2+\frac{1}{2}\int_\Omega |\nabla u_t(x,T)|^2 \\ &-\frac{1}{2}\int_\Omega |\nabla u_1|^2+\int_0^T\int_\Omega \psi u_t.\end{aligned} \tag{2.17}$$

Addition of (2.16) and (2.17) yields

$$0\le \limsup_\ell X^\ell \le \int_0^T\int_\Omega \psi u_t-\int_0^T\int_\Omega \psi v-\int_0^T\int_\Omega A(v)(u_t-v).$$

That is,

$$\int_0^T\int_\Omega (\psi-A(v))(u_t-v)dt\ge 0,\ \forall v\in L^{m(\cdot)}\left((0,T),H_0^1(\Omega)\right).$$

Thence,

$$\int_0^T\int_\Omega (\psi-A(v))(u_t-v)dt\ge 0,\ \forall v\in L^{m(\cdot)}(\Omega\times(0,T)),$$

by density of $H_0^1(\Omega)$ in $L^{m(\cdot)}(\Omega)$ $(Lemma 1.5)$.

Now, let $v=\lambda w+u_t$, $w\in L^{m(\cdot)}(\Omega\times\ (0,T))$. Thus, we obtain

$$-\lambda\int_0^T\int_\Omega(\psi-A(\lambda w+u_t))w\ge 0,\ \forall\lambda\ne 0,\forall w\in L^{m(\cdot)}(\Omega\times\ (0,T)).$$

For $\lambda>0$, we get

$$\int_0^T\int_\Omega(\psi-A(\lambda w+u_t))w\le 0,\ \forall w\in L^{m(\cdot)}(\Omega\times\ (0,T)).$$

As $\lambda\to 0$ and using the continuity of A with respect to λ, we have

$$\int_0^T\int_\Omega(\psi-A(u_t))w\le 0,\ \forall w\in L^{m(\cdot)}(\Omega\times\ (0,T)).$$

Likewise, for $\lambda<0$, we get

$$\int_0^T\int_\Omega(\psi-A(u_t))w\ge 0,\ \forall w\in L^{m(\cdot)}(\Omega\times\ (0,T)).$$

This means that $\psi=A(u_t)$. So (2.13) becomes

$$\int_\Omega\left(u_{tt}v+\nabla u.\nabla v+\nabla u_{tt}.\nabla v+a|u_t|^{m(x)-2}u_tv\right)=\int_\Omega fv,\ \forall v\in L^{m(\cdot)}((0,T)\times\ H_0^1(\Omega)),$$

which gives

$$u_{tt} - \Delta u - \Delta u_{tt} + a|u_t|^{m(x)-2}u_t = f, \quad \text{in } D'(\Omega \times \ (0,T)).$$

To deal with the initial conditions, we note that

$$\begin{aligned} u^l &\rightarrow u \quad \text{weakly} * \text{in} \quad L^\infty\left((0,T), H_0^1(\Omega)\right) \\ u_t^l &\rightarrow u_t \quad \text{weakly} * \text{in} \quad L^\infty\left((0,T), H_0^1(\Omega)\right). \end{aligned} \tag{2.18}$$

And so, employing Lions' Lemma [48] gives us

$$u^l \rightarrow u \text{ in } C([0,T], H_0^1(\Omega)). \tag{2.19}$$

Therefore, $u^l(x,0)$ makes sense and $u^l(x,0) \rightarrow u(x,0)$ in $H_0^1(\Omega)$.

Also we have that

$$u^l(x,0) = u_0^l(x) \rightarrow u_0(x) \text{ in } H_0^1(\Omega).$$

So

$$u(x,0) = u_0(x). \tag{2.20}$$

As in [49], let $\phi \in C_0^\infty([0,T])$ and substituting (u^k) by (u^l), we get from (2.9) and for any $j \leq l$ that

$$\begin{aligned} -\int_0^T \int_\Omega u_t^l(x,t)v_j(x)\phi'(t)dxdt &= -\int_0^T \int_\Omega \nabla u^l(x,t)\nabla v_j(x)\phi(t)dxdt \\ &-\int_0^T \int_\Omega \nabla u_{tt}^l(x,t)\nabla v_j(x)\phi(t)dxdt \\ &-a\int_0^T \int_\Omega \left|u_t^l(x,t)\right|^{m(x)-2} u_t^l(x,t)v_j(x)\phi(t)dxdt \\ &+\int_0^T \int_\Omega f(x,t)\, v_j(x)\phi(t)dxdt. \end{aligned} \tag{2.21}$$

As $l \rightarrow \infty$, so, we have

$$-\int_0^T \int_\Omega u_t(x,t)v_j(x)\phi'(t)dxdt$$

$$= -\int_0^T \int_\Omega \nabla u(x,t)\nabla v_j(x)\phi(t)dxdt$$

$$-\int_0^T \int_\Omega \nabla u_{tt}(x,t)\nabla v_j(x)\phi(t)dxdt$$

$$-a\int_0^T \int_\Omega |u_t(x,t)|^{m(x)-2}u_t(x,t)v_j(x)\phi(t)dxdt$$

$$+\int_0^T \int_\Omega f\left(x,t\right)v_j(x)\phi(t)dxdt, \tag{2.22}$$

for all $j \geq 1$. This implies

$$-\int_0^T \int_\Omega u_t(x,t)v(x)\phi'(t)dxdt = \int_0^T \int_\Omega \begin{bmatrix} \Delta u + \Delta u_{tt} \\ -a|u_t(x,t)|^{m(x)-2}u_t(x,t) \\ +f\left(x,t\right) \end{bmatrix} v(x)\phi(t)dxdt, \tag{2.23}$$

for all $v \in H_0^1(\Omega)$. This means $u_{tt} \in L^{\frac{m(\cdot)}{m(\cdot)-1}}([0,T), H^{-1}(\Omega))$ and u solves the equation

$$u_{tt} - \Delta u - \Delta u_{tt} + a|u_t|^{m(\cdot)-2}u_t = f. \tag{2.24}$$

Consequently, $u_t \in L^\infty([0,T), H_0^1(\Omega))$, $u_{tt} \in L^{\frac{m(\cdot)}{m(\cdot)-1}}([0,T), H^{-1}(\Omega))$. Thus,

$$u_t \in C([0,T), H^{-1}(\Omega)). \tag{2.25}$$

So, $u_t^l(x,0)$ makes sense $(see[49, p.116])$. And from it we conclude that

$$u_t^l(x,0) \to u_t(x,0) \text{ in } H^{-1}(\Omega).$$

But

$$u_t^l(x,0) = u_1^l(x) \to u^1(x) \text{ in } H_0^1(\Omega).$$

Thence

$$u_t(x,0) = u_1(x). \tag{2.26}$$

□

Uniqueness

Proof. Assume that (2.8) has two solutions u and v. Then, $w = u - v$ satisfies

$$\begin{cases} w_{tt} - \Delta w - \Delta w_{tt} + a u_t |u_t|^{m(\cdot)-2} - a v_t |v_t|^{m(\cdot)-2} = 0, & \text{in } \Omega \times (0,T) \\ w(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ w(x,0) = w_t(x,0) = 0, & \text{in } \Omega. \end{cases}$$

Multiply by w_t and integrate over Ω, to get

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega w_t^2 + \int_\Omega |\nabla w|^2 + \int_\Omega |\nabla w_t|^2\right] + a\int_\Omega \left(u_t|u_t|^{m(x)-2} - v_t|v_t|^{m(x)-2}\right)(u_t - v_t)\,dx = 0.$$

Integrate over $(0,t)$, to obtain

$$\int_\Omega \left(w_t^2 + |\nabla w|^2 + |\nabla w_t|^2\right) + 2a\int_0^t \int_\Omega \left(u_t|u_t|^{m(x)-2} - v_t|v_t|^{m(x)-2}\right)(u_t - v_t)\,dx = 0.$$

Using the inequality

$$(|a|^{m(x)-2}a - |b|^{m(x)-2}b).(a-b) \geq 0, \text{ for all } a,b \in \mathbb{R}^n \text{ and a.e.} x \in \Omega,$$

we find

$$\int_\Omega \left(w_t^2 + |\nabla w|^2 + |\nabla w_t|^2\right) = 0,$$

which conduces that $w = C = 0$, as $w = 0$ on $\partial\Omega$. Therefor, the uniqueness.

This ends the proof of $Theorem 2.1$. □

We need now the following lemma to present the result of well-posedness of our problem

Lemma 2.1. *For almost everywhere $x \in \Omega$ and $p(\cdot)$ satisfying*

$$2 < p_1 \leq p(x) \leq p_2 < +\infty,$$

the function $g(s) = b|s|^{p(x)-2}s$ is differentiable and $|g'(s)| = |b|\,|p(x)-1|\,|s|^{p(x)-2}$.

Theorem 2.2. *Suppose that $m, p \in \mathcal{C}\left(\bar{\Omega}\right)$ and*

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega).$$

*Under the assumptions **(H2),(H3)**, then problem (2.1) admits a unique local solution*

$$\begin{aligned} u &\in L^\infty((0,T), H_0^1(\Omega)), \\ u_t &\in L^\infty((0,T), H_0^1(\Omega)) \cap L^{m(\cdot)}(\Omega \times (0,T)), \\ u_{tt} &\in L^2((0,T), H^{-1}(\Omega)). \end{aligned} \tag{2.27}$$

2.2.2 Proof of Theorem 2. 2

Existence

Proof. Let $v \in L^{\infty}((0,T), H_0^1(\Omega))$. Then

$$\begin{aligned}\|g(v)\|_2^2 &= |b|^2 \int_{\Omega} |v|^{2(p(x)-1)} dx \\ &\leq |b|^2 \left[\int_{\Omega} |v|^{2(p_2-1)} dx + \int_{\Omega} |v|^{2(p_1-1)} dx \right] \\ &< +\infty,\end{aligned}$$

since

$$2(p_1 - 1) \leq 2(p_2 - 1) \leq \frac{2n}{n-2}.$$

Therefore, in this case,

$$g(v) \in L^{\infty}((0,T), L^2(\Omega)) \subset L^2(\Omega \times (0,T)).$$

So, for each $v \in L^{\infty}((0,T), H_0^1(\Omega))$ there exists a unique

$$\begin{aligned} u &\in L^{\infty}((0,T), H_0^1(\Omega)), \\ u_t &\in L^{\infty}((0,T), H_0^1(\Omega)) \cap L^{m(\cdot)}(\Omega \times (0,T)), \end{aligned}$$

satisfying the nonlinear problem

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + a u_t |u_t|^{m(\cdot)-2} = g(v), & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega. \end{cases} \tag{2.28}$$

We define a map $G : X_T \to X_T$ by $G(v) = u$, where

$$X_T = \left\{ w \in L^{\infty}((0,T), H_0^1(\Omega)) \ / \ w_t \in L^{\infty}((0,T), H_0^1(\Omega)) \right\}.$$

X_T is Banach space with respect to the norm

$$\|w\|_{X_T} = \|w\|_{L^{\infty}((0,T), H_0^1(\Omega))} + \|w_t\|_{L^{\infty}((0,T), H_0^1(\Omega))}.$$

Multiplying the first equation in (2.28) by u_t and integrating over $\Omega \times (0,t)$, to obtain

$$\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |\nabla u_t|^2 + a\int_0^t \int_\Omega |u_t|^{m(x)} = \frac{1}{2}\int_\Omega u_1^2 + \frac{1}{2}\int_\Omega |\nabla u_0|^2 + \frac{1}{2}\int_\Omega |\nabla u_1|^2$$
$$+b\int_0^t \int_\Omega |v|^{p(x)-2} v u_t. \tag{2.29}$$

Young's inequality gives us

$$\begin{aligned}\int_\Omega |v|^{p(x)-2} v u_t &\leq \frac{\varepsilon}{4}\int_\Omega u_t^2 dx + \frac{4}{\varepsilon}\int_\Omega |v|^{2p(x)-2} dx \\ &\leq \frac{\varepsilon}{4}\int_\Omega u_t^2 dx + \frac{4}{\varepsilon}\left[\int_\Omega |v|^{2p_2-2} + \int_\Omega |v|^{2p_1-2}\right] \\ &\leq \frac{\varepsilon}{4}\int_\Omega u_t^2 dx + \frac{c_e}{\varepsilon}\left[\|\nabla v\|_2^{2p_2-2} + \|\nabla v\|_2^{2p_1-2}\right].\end{aligned}$$

Thence (2.29) becomes

$$\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |\nabla u_t|^2 \leq \lambda_0 + \frac{|b|\varepsilon T}{4}\sup_{(0,T)}\int_\Omega u_t^2 + \frac{|b| c_e}{\varepsilon}\left[\int_0^T \|\nabla v\|_2^{2p_2-2} + \|\nabla v\|_2^{2p_1-2}\right];$$

hence we have

$$\frac{1}{2}\sup_{(0,T)}\int_\Omega u_t^2 + \frac{1}{2}\sup_{(0,T)}\int_\Omega |\nabla u|^2 + \frac{1}{2}\sup_{(0,T)}\int_\Omega |\nabla u_t|^2 \leq 2\lambda_0 + \frac{|b|\varepsilon T}{2}\sup_{(0,T)}\int_\Omega u_t^2 + T c_\varepsilon\left[\|v\|_{X_T}^{2p_2-2} + \|v\|_{X_T}^{2p_1-2}\right],$$

with

$$\lambda_0 := \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\|\nabla u_1\|_2^2,$$

and c_e is the embedding constant.

Choosing ε such that $\frac{|b|\varepsilon T}{2} = \frac{1}{4}$, we get

$$\|u\|_{X_T}^2 \leq \lambda + T\beta\left[\|v\|_{X_T}^{2p_2-2} + \|v\|_{X_T}^{2p_1-2}\right].$$

Assume that $\|v\|_{X_T} \leq M$, for some M large. Then

$$\|u\|_{X_T}^2 \leq \lambda + T\beta M^{2p_2-2} \leq M^2,$$

if

$$M^2 \geq \lambda \ \text{ and } \ T \leq T_0 < \frac{M^2 - \lambda}{\beta M^{2p_2-2}}.$$

We deduce that $G : B \to B$, where

$$B = \left\{ w \in L^{\infty}((0,T), H_0^1(\Omega)), w_t \in L^{\infty}((0,T), H_0^1(\Omega)) \text{ such that } \|w\|_{X_{T_0}} \le M \right\}.$$

Then, we clarify that, for T_0 (even smaller), G is a contraction. For this goal, let $u_1 = G(v_1)$ and $u_2 = G(v_2)$ and set $u = u_1 - u_2$ then u satisfies

$$\begin{cases} u_{tt} - \Delta u + a\left[u_{1t}|u_{1t}|^{m(\cdot)-2} - u_{2t}|u_{2t}|^{m(\cdot)-2}\right] - \Delta u_{tt} \\ \quad = b\left[|v_1|^{p(x)-2}v_1 - v_2|^{p(x)-2}v_2\right], \text{ in } \Omega \times (0,T\ ) \\ u(x,t) = 0, \qquad \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad \text{in } \Omega. \end{cases} \tag{2.30}$$

We multiply by u_t and integrate over $\Omega \times (0,t)$ to get

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |\nabla u_t|^2 \\ &+a\int_0^t\int_\Omega \left[|u_{1t}|^{m(x)-2}u_{1t} - |u_{2t}|^{m(x)-2}u_{2t}\right](u_{1t} - u_{2t}) \\ = \ & b\int_0^t\int_\Omega (g(v_1) - g(v_2))\, u_t dxds. \end{aligned} \tag{2.31}$$

And then, we have

$$\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |\nabla u_t|^2 \le b\int_0^t\int_\Omega (g(v_1) - g(v_2))\, u_t dxds. \tag{2.32}$$

We calculate now the term

$$I = \int_\Omega |g(v_1) - g(v_2)|\,|u_t| = \int_\Omega |g'(\xi)|\,|v|\,|u_t|,$$

where $v = v_1 - v_2$ and

$$\xi = \alpha v_1 + (1-\alpha)v_2, 0 \le \alpha \le 1.$$

Young's inequality implies

$$\begin{aligned} I &\le \frac{\delta}{2}\int_\Omega u_t^2 + \frac{2}{\delta}\int_\Omega |g'(\xi)|^2 |v|^2 \\ &\le \frac{\delta}{2}\int_\Omega u_t^2 + \frac{2a^2(p_2-1)^2}{\delta}\int_\Omega |\alpha v_1 + (1-\alpha)v_2|^{2(p(x)-2)}|v|^2 \\ &\le \frac{\delta}{2}\int_\Omega u_t^2 + c_\delta\left(\int_\Omega |v|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\left[\left(\int_\Omega |\alpha v_1 + (1-\alpha)v_2|^{n(p_2-2)}\right)^{\frac{2}{n}}\right. \\ &\quad \left. + \left(\int_\Omega |\alpha v_1 + (1-\alpha)v_2|^{n(p_1-2)}\right)^{\frac{2}{n}}\right]. \end{aligned}$$

Exploit (2.4) to obtain

$$\begin{aligned} I &\leq \frac{\delta}{2}\int_\Omega u_t^2 + c_\delta c_e \left\|\nabla v\right\|_2^2 \left[\left\|\nabla v_1\right\|_2^{2(p_2-2)} + \left\|\nabla v_1\right\|_2^{2(p_1-2)} + \left\|\nabla v_2\right\|_2^{2(p_2-2)} + \left\|\nabla v_2\right\|_2^{2(p_1-2)}\right] \\ &\leq \frac{\delta}{2}\int_\Omega u_t^2 + 4c_\delta c_e M^{2(p_2-2)}\left\|\nabla v\right\|_2^2. \end{aligned}$$

Thus, (2.32) takes the form

$$\frac{1}{2}\left\|u\right\|_{X_T}^2 \leq \frac{\delta}{2}T_0 b\left\|u\right\|_{X_T}^2 + C_\delta M^{2(p_2-2)}T_0 b\left\|v\right\|_{X_T}^2.$$

Choosing δ small enough, we arrive at

$$\left\|u\right\|_{X_T}^2 \leq 4C_\delta M^{2(p_2-2)}T_0 b\left\|v\right\|_{X_T}^2 = \gamma T_0\left\|v\right\|_{X_T}^2.$$

By taking T_0 small enough, we get

$$\left\|u\right\|_{X_T}^2 \leq d\left\|v\right\|_{X_T}^2, \quad \text{for } \ 0<d<1.$$

Therefore G is a contraction. The Banach[4] fixed theorem implies the existence of a unique $u \in B$ satisfying $G(u) = u$. So, u is a local solution of (2.1). □

Uniqueness

Proof. Assume that we have two solutions u and v. So $w = u - v$ satisfies

$$\begin{cases} w_{tt} - \Delta w - \Delta w_{tt} + a u_t|u_t|^{m(\cdot)-2} - a v_t|v_t|^{m(\cdot)-2} & \\ \quad = bu|u|^{p(\cdot)-2} - bv|v|^{p(\cdot)-2}, \ \text{in } \Omega\times(0,T) & \\ w(x,t) = 0, & \text{on } \partial\Omega\times(0,T) \\ w(x,0) = w_t(x,0) = 0, & \text{in } \Omega. \end{cases}$$

We multiply the previous equation by w_t and integrate over $\Omega\times(0,t)$ to obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega w_t^2 + \frac{1}{2}\int_\Omega |\nabla w|^2 + \frac{1}{2}\int_\Omega |\nabla w_t|^2 \\ &+a\int_0^t\left(\int_\Omega u_t|u_t|^{m(x)-2} - v_t|v_t|^{m(x)-2}\right)(u_t - v_t) \\ =\ & b\int_0^t\left(\int_\Omega u|u|^{p(x)-2} - v|v|^{p(x)-2}\right)w_t dx, \end{aligned} \tag{2.33}$$

[4]Stefan Banach (1892 - 1945) was a Polish mathematician.

this implies

$$\frac{1}{2}\int_{\Omega} w_t^2 + \frac{1}{2}\int_{\Omega} |\nabla w|^2 + \frac{1}{2}\int_{\Omega} |\nabla w_t|^2 \leq b\int_0^t \int_{\Omega} \left(u|u|^{p(x)-2} - v|v|^{p(x)-2}\right) w_t dx.$$

As in above, we repeat the same estimates to arrive at

$$\int_{\Omega} w_t^2 + |\nabla w|^2 + |\nabla w_t|^2 \leq C\int_0^t \int_{\Omega} \left(w_t^2(x,s) + |\nabla w(x,s)|^2 + |\nabla w_t(x,s)|^2\right) dxds.$$

Gronwell's inequality yields

$$\int_{\Omega} \left(w_t^2 + |\nabla w|^2 + |\nabla w_t|^2\right) = 0.$$

Consequently, $w \equiv 0$. So the uniqueness is evident.

The proof of $Theorem$ 2.2 is finished. □

2.3 The Main Blow-Up Result

The focus of this chapter is to study the blow-up phenomenon of our problem. We first list several lemmas that we need to prove our result.

2.3.1 Technical Lemmas

Lemma 2.2. *Suppose the conditions of* $Corollary$1.1 *hold. Then there exists a positive* $C > 1$, *depending on* Ω *only, such that*

$$\varrho^{\frac{s}{p_1}}(u) \leq C(\|\nabla u\|_2^2 + \varrho(u)), \tag{2.34}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p_1$.

Proof. If $\varrho(u) > 1$, then $\varrho^{\frac{s}{p_1}}(u) \leq \varrho(u) \leq C(\|\nabla u\|_2^2 + \varrho(u))$, where $C > 1$. If $\varrho(u) \leq 1$ so, by $Lemma$1.3 (i), $\|u\|_{p(\cdot)} \leq 1$. Then, $Corollary$1.1 and $Lemma$1.4 imply

$$\begin{aligned} \varrho^{\frac{s}{p_1}}(u) &\leq \varrho^{\frac{2}{p_1}}(u) \leq \left[\max\left\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right\}\right]^{\frac{2}{p_1}} \\ &= \|u\|_{p(\cdot)}^2 \leq C\|\nabla u\|_2^2. \end{aligned}$$

The proof of $Lemma$2.2 is finished. □

As a special case, we have

Corollary 2.1. *Assume that the assumptions of Lemma2.2 hold. Then we have*

$$\|u\|_{p_1}^{s} \leq C(\|\nabla u\|_2^2 + \|u\|_{p_1}^{p_1}), \tag{2.35}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p_1$.

We set

$$H(t) := -E(t),$$

throughout these steps, we use C to denote a generic positive constant depending on Ω only.

As a result of (2.6) and (2.34), we get:

Corollary 2.2. *Assume that the assumptions of Lemma2.2 hold. Then we have*

$$\varrho^{\frac{s}{p_1}}(u) \leq C(|H(t)| + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \varrho(u)). \tag{2.36}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p_1$.

As a special case, we obtain:

Corollary 2.3. *Assume that the assumptions of Lemma2.2 hold. Then we have*

$$\|u\|_{p_1}^{s} \leq C(|H(t)| + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_{p_1}^{p_1}), \tag{2.37}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p_1$.

Lemma 2.3. *Assume that the assumptions of Lemma2.2 hold and let* u *be the solution of* (2.1). *Then,*

$$\varrho(u) \geq C\,\|u\|_{p_1}^{p_1}. \tag{2.38}$$

Proof. We have

$$\varrho(u) = \int_\Omega |u|^{p(x)}\,dx = \int_{\Omega_+} |u|^{p(x)}\,dx + \int_{\Omega_-} |u|^{p(x)}\,dx,$$

where

$$\Omega_+ = \{x \in \Omega \ / \ |u(x,t)| \geq 1\} \text{ and } \Omega_- = \{x \in \Omega \ / \ |u(x,t)| < 1\},$$

thence, we get

$$\begin{aligned}\varrho(u) &\geq \int_{\Omega_+} |u|^{p_1} + \int_{\Omega_-} |u|^{p_2} \\ &\geq \int_{\Omega_+} |u|^{p_1} + c_1 \left(\int_{\Omega_-} |u|^{p_1} \right)^{\frac{p_2}{p_1}}.\end{aligned}$$

This implies that

$$c_2 \left(\varrho(u)\right)^{\frac{p_1}{p_2}} \geq \int_{\Omega_-} |u|^{p_1} \text{ and } \varrho(u) \geq \int_{\Omega_+} |u|^{p_1},$$

and, so,

$$c_2 \left(\varrho(u)\right)^{\frac{p_1}{p_2}} + \varrho(u) \geq \|u\|_{p_1}^{p_1}. \tag{2.39}$$

Since

$$0 < H(0) \leq H(t) \leq \frac{b}{p_1} \varrho(u),$$

then (2.39) leads to

$$\varrho(u) \left[1 + c_2 \left(\frac{p_1}{b} H(0) \right)^{\frac{p_1}{p_2} - 1} \right] \geq \|u\|_{p_1}^{p_1}.$$

Thus, (2.38) follows. □

Lemma 2.4. *Let u be the solution of* (2.1) *and suppose that* (2.5) *holds. Then,*

$$\int_\Omega |u|^{m(x)}\, dx \leq C \left(\left(\varrho(u)\right)^{\frac{m_1}{p_1}} + \left(\varrho(u)\right)^{\frac{m_2}{p_1}} \right). \tag{2.40}$$

Proof.

$$\begin{aligned}\int_\Omega |u|^{m(x)}\, dx &\leq \int_{\Omega_-} |u|^{m_1}\, dx + \int_{\Omega_+} |u|^{m_2}\, dx \\ &\leq C \left[\left(\int_{\Omega_-} |u|^{p_1}\, dx \right)^{\frac{m_1}{p_1}} + \left(\int_{\Omega_+} |u|^{p_1}\, dx \right)^{\frac{m_2}{p_1}} \right] \\ &\leq C \left(\|u\|_{p_1}^{m_1} + \|u\|_{p_1}^{m_2} \right) \\ &\leq C \left(\left(\varrho(u)\right)^{\frac{m_1}{p_1}} + \left(\varrho(u)\right)^{\frac{m_2}{p_1}} \right),\end{aligned}$$

by *Lemma*2.3. □

2.3.2 The Main Result

In this subsection, we are in the process to state and proving our blow-up result, for this goal, we give the following theorem

Theorem 2.3. *Let the conditions of* $Theorem 2.2$ *be fulfilled. Assume further that* $(H4)$ *holds and*

$$E(0) < 0. \tag{2.41}$$

Then the solution of problem (2.1) *belonging to the class* (2.27) *blows up in finite time.*

2.3.3 Proof of the Main Result

Proof. Multiplying (2.1) by u_t and integrating over Ω to obtain

$$E'(t) = -a \int_{\Omega} |u_t(x,t)|^{m(x)}\, dx, \tag{2.42}$$

for almost every t in $[0,T)$ since $E(t)$ is absolutely continuous (*see* [31]); thence $H'(t) \geq 0$ and

$$0 < H(0) \leq H(t) \leq \frac{b}{p_1}\varrho(u), \tag{2.43}$$

for every t in $[0,T)$, by virtue of (2.41). We then define

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u u_t(x,t)dx, \tag{2.44}$$

for ε small to be selected later and

$$0 < \alpha \leq \min\left\{\frac{p_1 - 2}{2p_1}, \frac{p_1 - m_2}{p_1(m_2 - 1)}\right\}. \tag{2.45}$$

We derive (2.44) and use Eq. (2.1) to get

$$L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_{\Omega} u_t^2(x,t)dx + \varepsilon \int_{\Omega} u u_{tt}(x,t)dx,$$

$$
\begin{aligned}
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega u_t^2(x,t)dx \\
&\quad + \varepsilon \int_\Omega u \left(\Delta u + \Delta u_{tt} - a u_t \left| u_t \right|^{m(x)-2} + b u \left| u \right|^{p(x)-2} \right), \\
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega u_t^2(x,t)dx \\
&\quad + \varepsilon \int_\Omega \left(u\Delta u + u\Delta u_{tt} - a u u_t \left| u_t \right|^{m(x)-2} + b \left| u \right|^{p(x)} \right), \\
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega \left[u_t^2 - \left| \nabla u \right|^2 + \left| \nabla u_t \right|^2 \right] \\
&\quad - \varepsilon \int_\Omega \frac{d}{dt} \left\{ \nabla u_t \nabla u \right\} - a\varepsilon \int_\Omega u u_t \left| u_t \right|^{m(x)-2} + \varepsilon b \int_\Omega \left| u \right|^{p(x)},
\end{aligned}
$$

$$
\begin{aligned}
L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \left\{ \nabla u_t \nabla u \right\} \right) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega \left[u_t^2 - \left| \nabla u \right|^2 + \left| \nabla u_t \right|^2 \right] \\
&\quad - a\varepsilon \int_\Omega u u_t \left| u_t \right|^{m(x)-2} + \varepsilon b \int_\Omega \left| u \right|^{p(x)},
\end{aligned}
$$

$$
\begin{aligned}
L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \left\{ \nabla u_t \nabla u \right\} \right) &= (1-\alpha)H^{-\alpha}(t)H'(t) \\
&\quad + \varepsilon \int_\Omega \left[u_t^2 - \left| \nabla u \right|^2 + \left| \nabla u_t \right|^2 \right] \\
&\quad + \varepsilon b \int_\Omega \left| u \right|^{p(x)} - a\varepsilon \int_\Omega u u_t \left| u_t \right|^{m(x)-2}.
\end{aligned} \tag{2.46}
$$

Then exploit Young's inequality

$$
XY \leq \frac{\delta^r}{r} X^r + \frac{\delta^{-q}}{q} Y^q, \ X, Y \geq 0, \quad \text{for all} \ \ \delta > 0, \ \frac{1}{r} + \frac{1}{q} = 1,
$$

with $r = m$ and $q = m/(m-1)$ to estimate the last term in (2.46) as follows

$$
\int_\Omega \left| u_t \right|^{m(x)-1} \left| u \right| dx \leq \frac{1}{m_1} \int_\Omega \delta^{m(x)} \left| u \right|^{m(x)} + \frac{m_2 - 1}{m_2} \int_\Omega \delta^{-m(x)/m(x)-1} \left| u_t \right|^{m(x)},
$$

which yields, by substitution in (2.46)

$$
\begin{aligned}
L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \left\{ \nabla u_t \nabla u \right\} \right) &\geq \left[(1-\alpha)H^{-\alpha}(t) - \varepsilon \left(\frac{m_2 - 1}{m_2} \right) \delta^{-m(x)/m(x)-1} \right] H'(t) \\
&\quad + \varepsilon \int_\Omega \left[u_t^2 - \left| \nabla u \right|^2 + \left| \nabla u_t \right|^2 \right] + \varepsilon \left[\begin{array}{c} p_1 H(t) + \frac{p_1}{2} \int_\Omega u_t^2 \\ + \frac{p_1}{2} \int_\Omega \left| \nabla u \right|^2 + \frac{p_1}{2} \int_\Omega \left| \nabla u_t \right|^2 \end{array} \right] \\
&\quad - a\varepsilon \frac{1}{m_1} \int_\Omega \delta^{m(x)} \left| u \right|^{m(x)}.
\end{aligned} \tag{2.47}
$$

Of course (2.47) remains valid even if δ is time dependant since the integral is taken over the x variable.

Thus by taking δ so that $\delta^{-m(x)/m(x)-1} = kH^{-\alpha}(t)$, for large k to be given later, and substituting in (2.47) we arrive at

$$\begin{aligned} L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\}\right) \geq & \left[(1-\alpha) - \frac{m_2-1}{m_2}\varepsilon k\right] H^{-\alpha}(t)H'(t) \\ & + \varepsilon \int_\Omega u_t^2 - \varepsilon \int_\Omega |\nabla u|^2 + \varepsilon \int_\Omega |\nabla u_t|^2 + \varepsilon p_1 H(t) \\ & + \frac{\varepsilon p_1}{2}\int_\Omega u_t^2 + \frac{\varepsilon p_1}{2}\int_\Omega |\nabla u|^2 \\ & + \frac{\varepsilon p_1}{2}\int_\Omega |\nabla u_t|^2 - a\varepsilon \frac{1}{m_1}\int_\Omega \delta^{m(x)} |u|^{m(x)}, \end{aligned}$$

then

$$\begin{aligned} L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\}\right) \geq & \left[(1-\alpha) - \frac{m_2-1}{m_2}\varepsilon k\right] H^{-\alpha}(t)H'(t) \\ & + \varepsilon\left(\frac{p_1}{2}+1\right)\int_\Omega u_t^2 + \varepsilon\left(\frac{p_1}{2}-1\right)\int_\Omega |\nabla u|^2 \\ & + \varepsilon\left(\frac{p_1}{2}+1\right)\int_\Omega |\nabla u_t|^2 \\ & + \varepsilon\left[p_1 H(t) - a\frac{k^{1-m_1}}{m_1} H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}\,dx\right]. \end{aligned} \tag{2.48}$$

By exploiting (2.43) and the inequality (2.40 (*lemma* 2.4)), we obtain

$$H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}\,dx \leq \left(\frac{b}{p_1}\right)^{\alpha(m_2-1)} C\left[||u||_{p_1}^{m_1+\alpha p_1(m_2-1)} + ||u||_{p_1}^{m_2+\alpha p_1(m_2-1)}\right],$$

hence (2.48) yields

$$\begin{aligned} L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\}\right) \geq & \left[(1-\alpha) - \frac{m_2-1}{m_2}\varepsilon k\right] H^{-\alpha}(t)H'(t) \\ & + \varepsilon\left(\frac{p_1}{2}+1\right)\int_\Omega u_t^2 + \varepsilon\left(\frac{p_1}{2}-1\right)\int_\Omega |\nabla u|^2 \\ & + \varepsilon\left(\frac{p_1}{2}+1\right)\int_\Omega |\nabla u_t|^2 \\ & + \varepsilon\left[p_1 H(t) - a\frac{k^{1-m_1}}{m_1}\left(\frac{b}{p_1}\right)^{\alpha(m_2-1)} \times \right. \\ & \left. \left(||u||_{p_1}^{m_1+\alpha p_1(m_2-1)} + ||u||_{p_1}^{m_2+\alpha p_1(m_2-1)}\right)\right]. \end{aligned} \tag{2.49}$$

We use $Lemma 2.2$ and (2.45), for $s = m_2 + \alpha p_1(m_2 - 1) \leq p_1$ and $s = m_1 + \alpha p_1(m_2 - 1) \leq p_1$, to deduce from (2.49)

$$\begin{aligned} L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\}\right) \geq & \left[(1-\alpha) - \frac{m_2 - 1}{m_2}\varepsilon k\right] H^{-\alpha}(t) H'(t) \\ & +\varepsilon\left(\frac{p_1}{2}+1\right)\int_\Omega u_t^2 + \varepsilon\left(\frac{p_1}{2}-1\right)\int_\Omega |\nabla u|^2 \\ & +\varepsilon\left(\frac{p_1}{2}+1\right)\int_\Omega |\nabla u_t|^2 + \varepsilon\left[p_1 H(t)\right. \\ & \left. -k^{1-m_1} C_1 \left(H(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_{p_1}^{p_1}\right)\right], \end{aligned} \tag{2.50}$$

where $C_1 = 2a\left(\frac{b}{p_1}\right)^{\alpha(m_2-1)} C/m_1$. By noting that

$$H(t) = \frac{b}{p_1}\|u\|_{p_1}^{p_1} - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\|\nabla u_t\|_2^2,$$

and writing $p_1 = (p_1+2)/2 + (p_1-2)/2$, (2.50) yields

$$\begin{aligned} M'(t) \geq & \left[(1-\alpha) - \frac{m_2-1}{m_2}\varepsilon k\right] H^{-\alpha}(t) H'(t) + \varepsilon\left(\frac{p_1}{2}+1\right)\|u_t\|_2^2 \\ & +\varepsilon\left(\frac{p_1}{2}-1\right)\|\nabla u\|_2^2 + \varepsilon\left(\frac{p_1}{2}+1\right)\|\nabla u_t\|_2^2 \\ & +\varepsilon p_1 H(t) - \varepsilon k^{1-m_1} C_1 H(t) - \varepsilon C_1 k^{1-m_1}\|u_t\|_2^2 \\ & -\varepsilon C_1 k^{1-m_1}\|\nabla u_t\|_2^2 - \varepsilon C_1 k^{1-m_1}\|u\|_{p_1}^{p_1}, \end{aligned}$$

$$\begin{aligned} M'(t) \geq & \left[(1-\alpha) - \frac{m_2-1}{m_2}\varepsilon k\right] H^{-\alpha}(t) H'(t) + \varepsilon\left(\left(\frac{p_1}{2}+1\right) - C_1 k^{1-m_1}\right)\|u_t\|_2^2 \\ & +\varepsilon\left(\frac{p_1}{2}-1\right)\|\nabla u\|_2^2 + \varepsilon\left(\frac{p_1}{2}+1-C_1 k^{1-m_1}\right)\|\nabla u_t\|_2^2 \\ & +\left(\varepsilon p_1 - \varepsilon k^{1-m_1} C_1\right) H(t) - \varepsilon C_1 k^{1-m_1}\|u\|_{p_1}^{p_1}, \\ M'(t) \geq & \left[(1-\alpha) - \frac{m_2-1}{m_2}\varepsilon k\right] H^{-\alpha}(t) H'(t) + \varepsilon\left(\left(\frac{p_1}{2}+1\right) - C_1 k^{1-m_1}\right)\|u_t\|_2^2 \\ & +\varepsilon\left(\frac{p_1}{2}-1\right)\|\nabla u\|_2^2 + \varepsilon\left(\frac{p_1}{2}+1-C_1 k^{1-m_1}\right)\|\nabla u_t\|_2^2 \\ & +\left(\varepsilon\frac{p_1+2}{2} - \varepsilon k^{1-m_1} C_1\right) H(t) - \varepsilon C_1 k^{1-m_1}\|u\|_{p_1}^{p_1} + \varepsilon\frac{p_1-2}{2} H(t), \\ M'(t) \geq & \left[(1-\alpha) - \frac{m_2-1}{m_2}\varepsilon k\right] H^{-\alpha}(t) H'(t) + \varepsilon\left(\left(\frac{p_1}{2}+1\right) - C_1 k^{1-m_1}\right)\|u_t\|_2^2 \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\left(\frac{p_1}{2}-1\right)\|\nabla u\|_2^2+\varepsilon\left(\frac{p_1}{2}+1-C_1k^{1-m_1}\right)\|\nabla u_t\|_2^2 \\
&+\left(\varepsilon\frac{p_1+2}{2}-\varepsilon k^{1-m_1}C_1\right)H(t)-\varepsilon C_1k^{1-m_1}\|u\|_{p_1}^{p_1} \\
&+\varepsilon\frac{p_1-2}{2}\left(\frac{b}{p_1}\|u\|_{p_1}^{p_1}-\frac{1}{2}\|u_t\|_2^2-\frac{1}{2}\|\nabla u\|_2^2-\frac{1}{2}\|\nabla u_t\|_2^2\right), \\
\geq\ &\left[(1-\alpha)-\frac{m_2-1}{m_2}\varepsilon k\right]H^{-\alpha}(t)H'(t)+\varepsilon\left(\left(\frac{p_1}{2}+1\right)-C_1k^{1-m_1}\right)\|u_t\|_2^2 \\
&+\varepsilon\left(\frac{p_1}{2}-1\right)\|\nabla u\|_2^2+\varepsilon\left(\frac{p_1}{2}+1-C_1k^{1-m_1}\right)\|\nabla u_t\|_2^2 \\
&+\left(\varepsilon\frac{p_1+2}{2}-\varepsilon k^{1-m_1}C_1\right)H(t)-\varepsilon C_1k^{1-m_1}\|u\|_{p_1}^{p_1} \\
&+\varepsilon\frac{p_1-2}{2}\frac{b}{p_1}\|u\|_{p_1}^{p_1}-\varepsilon\frac{p_1-2}{4}\|u_t\|_2^2 \\
&-\varepsilon\frac{p_1-2}{4}\|\nabla u\|_2^2-\varepsilon\frac{p_1-2}{4}\|\nabla u_t\|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
M'(t)\ \geq\ &\left[(1-\alpha)-\frac{m_2-1}{m_2}\varepsilon k\right]H^{-\alpha}(t)H'(t)+\varepsilon\left(\left(\frac{p_1+6}{4}\right)-C_1k^{1-m_1}\right)\|u_t\|_2^2 \\
&+\varepsilon\left(\frac{p_1-2}{4}\right)\|\nabla u\|_2^2+\left(\varepsilon\frac{p_1+2}{2}-\varepsilon k^{1-m_1}C_1\right)H(t) \\
&+\left(\varepsilon\frac{p_1-2}{2}\frac{b}{p_1}-\varepsilon C_1k^{1-m_1}\right)\|u\|_{p_1}^{p_1}+\varepsilon\left(\frac{p_1+6}{4}-C_1k^{1-m_1}\right)\|\nabla u_t\|_2^2, \qquad (2.51)
\end{aligned}
$$

where

$$L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t\nabla u\}\right)=M'(t).$$

At this point, we choose k large enough so that the coefficients of $H(t)$, $\|u_t\|_2^2$, $\|\nabla u_t\|_2^2$ and $\|u\|_{p_1}^{p_1}$ in (2.51) are strictly positive, hence we get

$$
\begin{aligned}
L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t\nabla u\}\right)\ \geq\ &\left[(1-\alpha)-\frac{m_2-1}{m_2}\varepsilon k\right]H^{-\alpha}(t)H'(t) \qquad (2.52) \\
&+\varepsilon\gamma\left[H(t)+\|u_t\|_2^2+\|\nabla u_t\|_2^2+\|u\|_{p_1}^{p_1}\right],
\end{aligned}
$$

where $\gamma>0$ is the minimum of these coefficients. Once k is fixed (hence γ), we pick ε small enough so that $(1-\alpha)-\varepsilon k(m_2-1)/m_2\geq 0$ and

$$L(0)=H^{1-\alpha}(0)+\varepsilon\int_\Omega u_0u_1(x)\,dx>0.$$

Therefore (2.52) takes the form

$$L'(t)+\frac{d}{dt}\left(\varepsilon\int_\Omega\{\nabla u_t\nabla u\}\right)\geq\varepsilon\gamma\left[H(t)+\|u_t\|_2^2+\|\nabla u_t\|_2^2+\|u\|_{p_1}^{p_1}\right]. \qquad (2.53)$$

Thus we get

$$L(t) \geq L(0) > 0, \text{ for all } \ t \geq 0.$$

Next we would like to show that

$$L'(t) + \frac{d}{dt}\left(\varepsilon \int_{\Omega} \{\nabla u_t \nabla u\}\right) \geq \Gamma L^{1/(1-\alpha)}(t), \text{ for all } \ t \geq 0, \tag{2.54}$$

where Γ is a positive constant depending on $\varepsilon\gamma$ and C (the constant of $Corollary 2.1$).

Once (2.54) is determined, we obtain in a standard way the finite time blow up of $L(t)$, hence of u.

To prove (2.54), we first estime

$$\begin{aligned} \left|\int_{\Omega} uu_t\,(x,t)\,dx\right| &\leq \|u\|_2 \|u_t\|_2 \\ &\leq C\left(\|u\|_{p_1} \|u_t\|_2\right), \end{aligned}$$

which implies

$$\left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)} \leq C \|u\|_{p_1}^{1/(1-\alpha)} \|u_t\|_2^{1/(1-\alpha)}.$$

Again Young's inequality gives

$$\left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_{p_1}^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\right], \tag{2.55}$$

for $1/\mu + 1/\theta = 1$. Let $\theta = 2/(1-\alpha)$, to obtain $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p_1$ by (2.45).

Therefore (2.55) becomes

$$\left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_{p_1}^{s} + \|u_t\|_2^{2}\right],$$

where $s = 2/(1-2\alpha) \leq p_1$. By using $Corollary 2.3$, we get

$$\left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)} \leq C\left[H(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_{p_1}^{p_1}\right], \text{ for all } t \geq 0. \tag{2.56}$$

Finally by noting that

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon \int_{\Omega} uu_t\,(x,t)\,dx\right)^{1/(1-\alpha)} \\ &\leq 2^{1/(1-\alpha)}\left(H(t) + \left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)}\right), \end{aligned}$$

and combining it with (2.53) and (2.56), the inequality (2.54) is established. A simple integration of (2.54) over $(0.t)$ then yields

$$\begin{aligned}
\int_0^t \frac{dL(t)}{dt} &\geq \int_0^t \Gamma L^{1/(1-\alpha)}(t) - \int_0^t \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\}\right) \\
\int_0^t \frac{dL(t)}{L^{1/(1-\alpha)}(t)} &\geq \int_0^t \Gamma dt + \frac{\varepsilon}{L^{1/(1-\alpha)}(t)} \int_\Omega \Delta u_t u dx \\
\int_0^t L^{-1/(1-\alpha)}(t) dL(t) &\geq \int_0^t \Gamma dt + \frac{\varepsilon}{L^{1/(1-\alpha)}(t)} \int_\Omega \Delta u_t u dx \\
L^{\alpha/(1-\alpha)}(t) &\geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Gamma t\alpha/(1-\alpha)} + \frac{\varepsilon}{L^{1/(1-\alpha)}(t)} \int_\Omega \Delta u_t u dx \\
L^{\alpha/(1-\alpha)}(t) &\geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Gamma t\alpha/(1-\alpha)}.
\end{aligned} \tag{2.57}$$

Thence (2.57) shows that $L(t)$ blows up in finite time

$$T^* \leq \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}}, \tag{2.58}$$

where Γ and α are positive constant with $\alpha < 1$ and L is given by (2.44) above. This ends the proof. □

Remark 2.1. *The estimate* (2.58) *shows that the larger* $L(0)$ *is the quicker the blow-up takes place.*

Chapter 3

Global Existence and Finite Time Blow-Up in a Class of Non-Linear Viscoelastic Wave Equation

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Key Words and Phrases: Global existence, blow-up, source term, wave equation, viscosity.

In this chapter, we are in the process of studying the following non-linear viscoelastic wave equation:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \int\limits_0^t h(t-s)\Delta u(s)ds + cu_t\left|u_t\right|^{m-2} = du\left|u\right|^{p-2}, x \in \Omega, t > 0 \\ u(x,t) = 0, \qquad x \in \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \qquad x \in \Omega, \end{cases} \tag{3.1}$$

here Ω be an open bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geq 1)$, with a Lipschitz-countinuous boundary $\partial\Omega$, $p > 2, m \geq 1$, and c, d are strictly positive constants. Our chapter is divided as follows:

- In the first section, we present some assumptions needed in our chapter.

- In the second section, we show that solutions with arbitrary data continue to exist globally if $m \geq p$.

- In the third section, we prove a finite time blow-up for solutions with negative initial energy if $m < p$.

We study in this work the interaction between the damping and source terms in the presence of the viscoelastic and dispersion terms when $c = d = 1$. Our first intent is to itemize an appropriate domain for the parameters m, p, where the damping term dominates over the source and the global solution exists for any initial data. Secondly, we define another domain, where the blowup of the solution occurs for a finite time because the influence of the source is stronger.

3.1 Basic Assumptions

We provide in this section some information needed to demonstrate our results. During this work, C is used to indicate generic positive constant depending only on Ω. First, we mention the theory of local existence, for this purpose, we need to:

(G1) Suppose $m \geq 1, p > 2$, and

$$\max\{m, p\} \leq \frac{2(n-1)}{n-2}, n \geq 3, \tag{3.2}$$

this condition is necessary to determine the result of local existence (see[19] , [31]). The nonlinearity is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$ under this condition.

(G2) Assume that h is a C^1 function satisfying

$$1 - \int_0^\infty h(s)ds = l > 0, \tag{3.3}$$

we need this condition to assure the well-posedness and hyperbolicity of (3.1).

We define the energy functional associated to the problem (3.1) as follows

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t h(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(h \circ \nabla u)(t) - \frac{d}{p}\|u\|_p^p, \tag{3.4}$$

where

$$(h \circ v)(t) = \int_0^t h(t-s)\|v(t) - v(s)\|_2^2 ds,$$

and h satisfying the following assumptions

$$h(s) \geq 0, \quad h'(s) \leq 0, \quad \int_0^\infty h(s)ds < \frac{(p/2)-1}{(p/2)-1+(1/2p)}. \tag{3.5}$$

Remark 3.1. *By closely following the Theorem3.3 proof steps, with a small modification in the proof, we can see easily that the result of blow-up remains valid even for $m = 1$ (damping caused only by viscosity)*

Remark 3.2. *Without condition (3.5), we can determine a similar result provided that $\int_0^\infty h(s)ds < 1$ and E_0 is sufficiently negative.*

Remark 3.3. *There is a strong relation between the damping (caused by the viscosity) and the nonlinearity in the source (condition* (3.5) *shows that). More clarification the closer the value of* $\int_0^\infty h(s)ds$ *to* 1*, the larger* p *should be to ensure the blow-up.*

Theorem 3.1. *Assume that* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and suppose that the assumptions* $(G1)$ *and* $(G2)$ *hold. Then for some* $T_m > 0$ *the problem* (3.1) *admits a unique local solution*

$$u \in C([0, T_m), H_0^1(\Omega)), u_t \in C([0, T_m), H_0^1(\Omega)) \cap L^{m+1}(\Omega \times [0, T_m)). \tag{3.6}$$

Proof. Can be established by combination of the argument in [19] and [31] . □

3.2 Global Existence Result

We clarify in this section that the solution (3.6) is global if the exponent $m \geq p$

Theorem 3.2. *Let* $E_0 < 0, 2 \leq p \leq m$ *and let the condition*

$$m \leq \frac{2(n-1)}{n-2}, \quad n \geq 3, \tag{3.7}$$

hold. Then problem (3.1) *admits a unique global solution*

$$u \in C\left([0, \infty), H_0^1(\Omega)\right), \quad u_t \in C\left([0, \infty), H_0^1(\Omega)\right) \cap L^{m+1}\left((\Omega) \times (0, \infty)\right), \tag{3.8}$$

for any

$$u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega).$$

Proof. As in [31], we defined the following functional [1]

$$\begin{aligned} K(t) &= -H(t) + \frac{2d}{p}\|u\|_p^p \\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t h(s)\,ds\right)\|\nabla u\|_2^2 \\ &\quad + \frac{1}{2}(h \circ \nabla u)(t) + \frac{d}{p}\|u\|_p^p. \end{aligned}$$

[1] $K(t)$ denote the modified energy.

After differentiating $K(t)$ and exploiting (3.17), we obtain

$$K'(t) = -c\int_\Omega |u_t|^m \, dx + \frac{1}{2}\left(h' \circ \nabla u\right)(t) - \frac{1}{2}h\left(t\right)\left\|\nabla u\left(t\right)\right\|^2 + 2d\int_\Omega |u|^{p-2}\, uu_t dx.$$

We apply now the Young inequality in the form

$$XY \leq \delta X^\alpha + C_\delta Y^\beta,$$

where $X, Y, \alpha, \beta, \delta, C_\delta$ are positive constants such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. So we get

$$\left|\int_\Omega |u|^{p-2}\, uu_t dx\right| \leq \delta \left\|u_t\right\|_p^p + C_\delta \left\|u\right\|_p^p,$$

thus

$$\begin{aligned} K'(t) &\leq -c\left\|u_t\right\|_m^m + \frac{1}{2}\left(h' \circ \nabla u\right)(t) - \frac{1}{2}h\left(t\right)\left\|\nabla u\left(t\right)\right\|^2 + \delta \left\|u_t\right\|_p^p + C_\delta \left\|u\right\|_p^p \\ &\leq -c\left\|u_t\right\|_m^m + \delta \left\|u_t\right\|_p^p + C_\delta \left\|u\right\|_p^p, \end{aligned}$$

where C_δ is a constant depends on δ $(\delta > 0)$.

Having in mind that $m \geq p$, so we find

$$K'(t) \leq -c\left\|u_t\right\|_m^m + C\delta \left\|u_t\right\|_m^p + C_\delta \left\|u\right\|_p^p,$$

for $C = C(\Omega, p, m)$ is the embedding constant. Currently, we identify the following cases:

1) If $\left\|u_t\right\|_m^m > 1$, then we pick δ so small that

$$-c\left\|u_t\right\|_m^m + C\delta \left\|u_t\right\|_m^p \leq 0.$$

Subsequently

$$K'(t) \leq C_\delta \left\|u\right\|_p^p.$$

2) Otherwise $\left\|u_t\right\|_m^m \leq 1$, we get $K'(t) \leq C\delta + C_\delta \left\|u\right\|_p^p$.

So we have in either case

$$\begin{aligned} K'(t) &\leq c_1 + C_\delta \left\|u\right\|_p^p \\ &\leq c_1 + C_\delta K(t). \end{aligned} \tag{3.9}$$

We integrate (3.9) over $(0,t)$ to get

$$K(t) \leq \left(K(0) + \frac{c_1}{C_\delta}\right) e^{C_\delta t}.$$

From the last estimate and the continuation principle, we terminate our proof. □

3.3 Finite-Time Blow-Up

In order to carry the proof of our result, we need the following:

Lemma 3.1. *Assume the condition **(G1)** hold. Then there exists a positive constant $C > 1$ which depends only on Ω, such that*

$$\|u\|_p^s \leq C(\|\nabla u\|_2^2 + \|u\|_p^p), \tag{3.10}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

We let

$$H(t) := -E(t).$$

Corollary 3.1. *Suppose that the conditions (3.4) and (3.10) are satisfying, then*

$$\|u\|_p^s \leq C(-H(t) - \|u_t\|_2^2 - \|\nabla u_t\|_2^2 - (h \circ \nabla u)(t) + \|u\|_p^p), \text{ for all } \quad t \in [0,T), \tag{3.11}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

Theorem 3.3. *Let $m > 1, p > max\{2, m\}$ satisfying **(G1)**. Let (3.5) be fulfilled and assume that*

$$E_0 = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\|\nabla u_1\|_2^2 - \frac{d}{p}\|u_0\|_p^p < 0. \tag{3.12}$$

Then there exist a finite time T^* such that

$$T^* \leq \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}}, \tag{3.13}$$

where Γ, α $(\alpha < 1)$ are positive constant and L is given by (3.19) below.

Remark 3.4. *Our proof uses the same basic steps in* [52] , *with some modifications that relate to the nature of the problem that is being studied.*

Proof. To prove the $Theorem$3.3, we multiply (3.1) by $-u_t$ and integrate over Ω to get

$$\begin{aligned} & \frac{d}{dt}\left\{-\frac{1}{2}\int_\Omega |u_t|^2\,dx - \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_\Omega |\nabla u_t|^2\,dx \right. \\ & \left. +\frac{d}{p}\int_\Omega |u|^p\,dx\right\} + \int_0^t h(t-\tau)\int_\Omega \nabla u_t(t).\nabla u(\tau)\,dxd\tau \\ = \; & c\int_\Omega |u_t|^m\,dx, \end{aligned} \tag{3.14}$$

for any regular solution. We can extended this result to weak solutions through density argument. But

$$\begin{aligned} \int_0^t h(t-\tau)\int_\Omega \nabla u_t(t).\nabla u(\tau)\,dxd\tau &= \int_0^t h(t-\tau)\int_\Omega \nabla u_t(t).\left[\nabla u(\tau)-\nabla u(t)\right]dxd\tau \\ &+ \int_0^t h(t-\tau)\int_\Omega \nabla u_t(t).\nabla u(t)\,dxd\tau, \\ &= -\frac{1}{2}\int_0^t h(t-\tau)\frac{d}{dt}\int_\Omega |\nabla u(\tau)-\nabla u(t)|^2\,dxd\tau \\ &+ \int_0^t h(\tau)\left(\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla u(t)|^2\,dx\right)d\tau, \end{aligned}$$

$$\begin{aligned} = \; & -\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(t-\tau)\int_\Omega |\nabla u(\tau)-\nabla u(t)|^2\,dxd\tau\right] \\ & +\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(\tau)\int_\Omega |\nabla u(t)|^2\,dxd\tau\right] \\ & +\frac{1}{2}\int_0^t h'(t-\tau)\int_\Omega |\nabla u(\tau)-\nabla u(t)|^2\,dxd\tau \\ & -\frac{1}{2}h(t)\int_\Omega |\nabla u(t)|^2\,dxd\tau. \end{aligned} \tag{3.15}$$

Substitution of (3.15) in (3.14) gives us

$$
\begin{aligned}
&\frac{d}{dt}\left\{-\frac{1}{2}\int_{\Omega}|u_t|^2\,dx-\frac{1}{2}\int_{\Omega}|\nabla u|^2\,dx-\frac{1}{2}\int_{\Omega}|\nabla u_t|^2\,dx+\frac{d}{p}\int_{\Omega}|u|^p\,dx\right\} \\
&-\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(t-\tau)\int_{\Omega}|\nabla u(\tau)-\nabla u(t)|^2\,dxd\tau\right]+\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(\tau)\|\nabla u(t)\|^2\,d\tau\right] \quad (3.16) \\
=\;& c\int_{\Omega}|u_t|^m\,dx-\frac{1}{2}\int_0^t h'(t-\tau)\int_{\Omega}|\nabla u(\tau)-\nabla u(t)|^2\,dxd\tau+\frac{1}{2}h(t)\|\nabla u(t)\|^2 .
\end{aligned}
$$

After exploiting the definition of $H(t)$, the estimate (3.16) takes the form

$$
H'(t)=c\int_{\Omega}|u_t|^m\,dx-\frac{1}{2}(h'\circ\nabla u)(t)+\frac{1}{2}h(t)\|\nabla u(t)\|^2\geq 0. \quad (3.17)
$$

Hence

$$
0<H(0)\leq H(t)\leq\frac{d}{p}\|u\|_p^p, \quad (3.18)
$$

for every t in $[0,T)$, by virtue of (3.4), (3.17). We next define

$$
L(t):=H^{1-\alpha}(t)+\varepsilon\int_{\Omega}uu_t(x,t)dx, \quad (3.19)
$$

where ε (small) to be selected later and

$$
0<\alpha\leq\min\left\{\frac{p-2}{2p},\frac{p-m}{p(m-1)}\right\}. \quad (3.20)
$$

By differentiating (3.19) and using Eq. (3.1), we arrive at

$$
\begin{aligned}
L'(t) \;=\;& (1-\alpha)H^{-\alpha}(t)\left\{c\|u_t\|_m^m-\frac{1}{2}(h'\circ\nabla u)(t)+\frac{1}{2}h(t)\|\nabla u\|_2^2\right\} \\
&+\varepsilon\int_{\Omega}\left[u_t^2-|\nabla u|^2+|\nabla u_t|^2\right](x,t)\,dx \\
&+\varepsilon\int_0^t h(t-\tau)\int_{\Omega}\nabla u(t).\nabla u(\tau)\,dxd\tau \\
&+\varepsilon d\int_{\Omega}|u(x,t)|^p\,dx-\varepsilon c\int_{\Omega}u(x,t)\,u_t\,|u_t|^{m-2}\,dx \\
&-\varepsilon\int_{\Omega}\frac{d}{dt}\{\nabla u_t\nabla u\},
\end{aligned}
$$

$$\begin{aligned}
&\geq c(1-\alpha)H^{-\alpha}(t)\left\|u_t\right\|_m^m+\varepsilon \int_{\Omega}\left[u_t^2-|\nabla u|^2+|\nabla u_t|^2\right](x,t)\,dx \qquad (3.21)\\
&+\varepsilon d\int_{\Omega}|u(x,t)|^p\,dx-\varepsilon c\int_{\Omega}u(x,t)\,u_t\,|u_t|^{m-2}\,dx\\
&+\varepsilon\int_0^t h(t-\tau)\int_{\Omega}\nabla u(t)\,.\left[\nabla u(\tau)-\nabla u(t)\right]dxd\tau\\
&+\varepsilon\int_0^t h(t-\tau)\left\|\nabla u(t)\right\|_2^2 d\tau-\varepsilon\int_{\Omega}\frac{d}{dt}\left\{\nabla u_t\nabla u\right\}.
\end{aligned}$$

After using Schwarz inequality, (3.21) becomes

$$\begin{aligned}
L'(t)\;&\geq c(1-\alpha)H^{-\alpha}(t)\left\|u_t\right\|_m^m+\varepsilon \int_{\Omega}\left[u_t^2-|\nabla u|^2+|\nabla u_t|^2\right](x,t)\,dx\\
&+\varepsilon d\int_{\Omega}|u(x,t)|^p\,dx-\varepsilon c\int_{\Omega}u(x,t)\,u_t\,|u_t|^{m-2}\,dx \qquad (3.22)\\
&+\varepsilon\int_0^t h(t-\tau)\int_{\Omega}\left\|\nabla u(t)\right\|_2\left\|\nabla u(\tau)-\nabla u(t)\right\|_2 d\tau\\
&+\varepsilon\int_0^t h(t-\tau)\left\|\nabla u(t)\right\|_2^2 d\tau-\varepsilon\int_{\Omega}\frac{d}{dt}\left\{\nabla u_t\nabla u\right\}.
\end{aligned}$$

We next exploit (3.4) to replace the third term and apply Young's inequality for the fifth term in the right-hand side of (3.22). Therefore, we get

$$\begin{aligned}
&L'(t)\geq c(1-\alpha)H^{-\alpha}(t)\left\|u_t\right\|_m^m+\varepsilon\int_{\Omega}u_t^2(x,t)\,dx\\
&+\varepsilon\int_{\Omega}|\nabla u_t(x,t)|^2\,dx-\left(1-\int_0^t h(s)ds\right)\left\|\nabla u(t)\right\|_2^2\\
&+\varepsilon\Big(pH(t)+\frac{p}{2}(h\circ\nabla u)(t)+\frac{p}{2}\left\|u_t\right\|_2^2+\frac{p}{2}\left\|\nabla u_t\right\|_2^2\\
&+\frac{p}{2}\left(1-\int_0^t h(s)ds\right)\left\|\nabla u(t)\right\|_2^2\Big)\\
&-c\varepsilon\int_{\Omega}u(x,t)\,u_t\,|u_t|^{m-2}\,dx-\varepsilon\eta\,(h\circ\nabla u)(t) \qquad (3.23)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{\varepsilon}{4\eta}\int_0^t h(s)ds\,\|\nabla u(t)\|_2^2-\varepsilon\int_\Omega \frac{d}{dt}\{\nabla u_t \nabla u\},\\
\geq\; & c(1-\alpha)H^{-\alpha}(t)\,\|u_t\|_m^m+\varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx\\
& +\varepsilon\left(\frac{p}{2}+1\right)\int_\Omega |\nabla u_t(x,t)|^2\,dx+\varepsilon p H(t)\\
& +\varepsilon\left(\frac{p}{2}-\eta\right)(h\circ\nabla u)(t)-c\varepsilon\int_\Omega u(x,t)\,u_t\,|u_t|^{m-2}\,dx\\
& +\varepsilon\left(\left(\frac{p}{2}-1\right)-\left(\frac{p}{2}-1+\frac{1}{4\eta}\right)\int_0^t h(s)ds\right)\|\nabla u(t)\|_2^2\\
& -\varepsilon\int_\Omega \frac{d}{dt}\{\nabla u_t \nabla u\},
\end{aligned}
$$

for some $0<\eta<p/2$.

We recall (3.5), then (3.23) becomes

$$
\begin{aligned}
L'(t)+\varepsilon\int_\Omega \frac{d}{dt}\{\nabla u_t\nabla u\} \;\geq\; & c(1-\alpha)H^{-\alpha}(t)\,\|u_t\|_m^m+\varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx\\
& +\varepsilon\left(\frac{p}{2}+1\right)\int_\Omega |\nabla u_t(x,t)|^2\,dx+\varepsilon pH(t)\\
& +\varepsilon b_1 (h\circ\nabla u)(t)+\varepsilon b_2\,\|\nabla u(t)\|_2^2\\
& -c\varepsilon\int_\Omega u(x,t)\,u_t\,|u_t|^{m-2}\,dx,
\end{aligned}
\tag{3.24}
$$

where

$$
b_1=\frac{p}{2}-\eta>0,\, b_2=\left(\frac{p}{2}-1\right)-\left(\frac{p}{2}-1+\frac{1}{4\eta}\right)\int_0^t h(s)ds>0.
$$

Again, we apply Young's inequality on the last term in (3.24), for all $\delta>0$, $\dfrac{1}{r}+\dfrac{1}{s}=1$

$$
YZ\leq \frac{\delta^r}{r}Y^r+\frac{\delta^{-s}}{s}Z^s,\; Y,Z\geq 0,
$$

and $r=m, s=m/(m-1)$, to get

$$
\int_\Omega |u_t|^{m-1}|u|\,dx\leq \frac{1}{m}\int_\Omega \delta^m|u|^m+\frac{m-1}{m}\int_\Omega \delta^{-m/m-1}|u_t|^m,
$$

so (3.24) becomes

$$\begin{aligned} L'(t)+\frac{d}{dt}\left(\varepsilon\int_{\Omega}\{\nabla u_t \nabla u\}\right) &\geq c\left[(1-\alpha)H^{-\alpha}(t)-\varepsilon\left(\frac{m-1}{m}\right)\delta^{-m/m-1}\right]\|u_t\|_m^m \\ &+\varepsilon\left(\frac{p}{2}+1\right)\int_{\Omega}u_t^2(x,t)\,dx \\ &+\varepsilon\left(\frac{p}{2}+1\right)\int_{\Omega}|\nabla u_t(x,t)|^2\,dx \\ &+\varepsilon b_1(h\circ\nabla u)(t)+\varepsilon b_2\|\nabla u(t)\|_2^2 \\ &+\varepsilon pH(t)-c\varepsilon\frac{\delta^m}{m}\|u\|_m^m, \end{aligned} \tag{3.25}$$

for all $\delta > 0$.

The estimate (3.25) still valid, even if δ is time dependant since the integral is taken over the x variable. Thus by picking δ so that $\delta^{-m/m-1} = kH^{-\alpha}(t)$, for large k to be given later, and replacing in (3.25) we reach to

$$\begin{aligned} L'(t)+\frac{d}{dt}\left(\varepsilon\int_{\Omega}\{\nabla u_t \nabla u\}\right) &\geq c\left[(1-\alpha)-\varepsilon\left(\frac{m-1}{m}\right)k\right]H^{-\alpha}(t)\|u_t\|_m^m \\ &+\varepsilon\left(\frac{p}{2}+1\right)\int_{\Omega}u_t^2(x,t)\,dx \\ &+\varepsilon\left(\frac{p}{2}+1\right)\int_{\Omega}|\nabla u_t(x,t)|^2\,dx \\ &+\varepsilon b_1(h\circ\nabla u)(t)+\varepsilon b_2\|\nabla u(t)\|_2^2 \\ &+\varepsilon\left[pH(t)-\frac{k^{1-m}}{m}cH^{\alpha(m-1)}(t)\|u\|_m^m\right]. \end{aligned} \tag{3.26}$$

By using (3.18) and the inequality $\|u\|_m^m \leq C\|u\|_p^m$, we have

$$H^{\alpha(m-1)}(t)\int_{\Omega}|u|^m\,dx \leq \left(\frac{d}{p}\right)^{\alpha(m-1)}C||u||_p^{m+\alpha p(m-1)},$$

then (3.26) becomes

$$\begin{aligned} L'(t)+\frac{d}{dt}\left(\varepsilon\int_{\Omega}\{\nabla u_t \nabla u\}\right) &\geq c\left[(1-\alpha)-\varepsilon\left(\frac{m-1}{m}\right)k\right]H^{-\alpha}(t)\|u_t\|_m^m \\ &+\varepsilon\left(\frac{p}{2}+1\right)\int_{\Omega}u_t^2(x,t)\,dx \\ &+\varepsilon\left(\frac{p}{2}+1\right)\int_{\Omega}|\nabla u_t(x,t)|^2\,dx \\ &+\varepsilon b_1(h\circ\nabla u)(t)+\varepsilon b_2\|\nabla u(t)\|_2^2 \\ &+\varepsilon\left[pH(t)-\frac{k^{1-m}}{m}c\left(\frac{d}{p}\right)^{\alpha(m-1)}C||u||_p^{m+\alpha p(m-1)}\right]. \end{aligned} \tag{3.27}$$

We exploit $Corollary$3.1 and condition (3.20) with $s = m + \alpha p(m-1) \leq p$, to conclude

$$\begin{aligned} L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\}\right) &\geq c\left[(1-\alpha) - \varepsilon\left(\frac{m-1}{m}\right)k\right] H^{-\alpha}(t) \|u_t\|_m^m \\ &+\varepsilon\left(\frac{p}{2}+1\right)\int_\Omega u_t^2(x,t)\,dx \\ &+\varepsilon\left(\frac{p}{2}+1\right)\int_\Omega |\nabla u_t(x,t)|^2\,dx \\ &+\varepsilon b_1 (h \circ \nabla u)(t) + \varepsilon b_2 \|\nabla u(t)\|_2^2 \\ &+\varepsilon\left[pH(t) - C_1 k^{1-m}\left\{-H(t) - \|u_t\|_2^2\right.\right. \\ &\left.\left.- \|\nabla u_t\|_2^2 - (h \circ \nabla u)(t) + \|u\|_p^p\right\}\right], \\ &\geq c\left[(1-\alpha) - \varepsilon\left(\frac{m-1}{m}\right)k\right] H^{-\alpha}(t) \|u_t\|_m^m \\ &+\varepsilon\left(\frac{p}{2}+1+C_1 k^{1-m}\right)\|u_t\|_2^2 \\ &+\varepsilon\left(\frac{p}{2}+1+C_1 k^{1-m}\right)\|\nabla u_t\|_2^2 \\ &+\varepsilon\left(b_1 + C_1 k^{1-m}\right)(h \circ \nabla u)(t) \\ &+\varepsilon b_2 \|\nabla u(t)\|_2^2 + \varepsilon\left(p + C_1 k^{1-m}\right) H(t) \\ &-\varepsilon C_1 k^{1-m} \|u\|_p^p, \end{aligned} \tag{3.28}$$

where $C_1 = c\left(\frac{d}{p}\right)^{\alpha(m-1)} C/m$.

Noting that

$$H(t) \geq \frac{d}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\|\nabla u_t\|_2^2 - \frac{1}{2}(g \circ \nabla u)(t),$$

and puting $p = 2b_3 + (p - 2b_3)$, where $b_3 = min\{b_1, b_2\}$, (3.28) yields

$$\begin{aligned} M'(t) &\geq c\left[(1-\alpha) - \varepsilon\left(\frac{m-1}{m}\right)k\right] H^{-\alpha}(t) \|u_t\|_m^m \\ &+\varepsilon\left(\frac{p}{2}+1+C_1 k^{1-m} - b_3\right)\|u_t\|_2^2 \\ &+\varepsilon\left(\frac{p}{2}+1+C_1 k^{1-m} - b_3\right)\|\nabla u_t\|_2^2 \\ &+\varepsilon\left(b_1 + C_1 k^{1-m} - b_3\right)(h \circ \nabla u)(t) \\ &+\varepsilon(b_2 - b_3)\|\nabla u(t)\|_2^2 + \varepsilon(p - 2b_3 \\ &+C_1 k^{1-m}) H(t) + \varepsilon\left(\frac{2db_3}{p} - C_1 k^{1-m}\right)\|u\|_p^p, \end{aligned} \tag{3.29}$$

where

$$M'(t) = L'(t) + \frac{d}{dt}\left(\varepsilon \int_{\Omega} \{\nabla u_t \nabla u\}\right).$$

For this goal, we pick k large enough so that the coefficients of $H(t), \|u_t\|_2^2, \|\nabla u_t\|_2^2, \|u\|_p^p$ and $(h \circ \nabla u)(t)$ in (3.29) are strictly positive, therefore we obtain

$$\begin{aligned} M'(t) \geq\ & c\left[(1-\alpha) - \varepsilon\left(\frac{m-1}{m}\right)k\right] H^{-\alpha}(t)\|u_t\|_m^m \\ & + \varepsilon\gamma\left[H(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_p^p + (h \circ \nabla u)(t)\right], \end{aligned} \tag{3.30}$$

where $\gamma > 0$ is the minimum of these coefficients. Once k is fixed (thus γ), we choose ε small enough so that

$$(1-\alpha) - \varepsilon k(m-1)/m \geq 0,$$

and

$$L(0) = H^{1-\alpha}(0) + \varepsilon \int_{\Omega} u_0 u_1(x)\, dx > 0.$$

Subsequently (3.30) becomes

$$L'(t) + \frac{d}{dt}\left(\varepsilon \int_{\Omega} \{\nabla u_t \nabla u\}\right) \geq \varepsilon\gamma\left[H(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_p^p + (h \circ \nabla u)(t)\right]. \tag{3.31}$$

Therefore

$$L(t) \geq L(0) > 0, \text{ for all } t \geq 0.$$

To achieve our result, we first estimate

$$\begin{aligned} \left|\int_{\Omega} u u_t(x,t)\, dx\right| &\leq \|u\|_2 \|u_t\|_2 \\ &\leq C\left(\|u\|_p \|u_t\|_2\right), \end{aligned}$$

thence

$$\left|\int_{\Omega} u u_t(x,t)\, dx\right|^{1/(1-\alpha)} \leq C\|u\|_p^{1/(1-\alpha)} \|u_t\|_2^{1/(1-\alpha)}.$$

Again Young's inequality leads to

$$\left|\int_{\Omega} u u_t(x,t)\, dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\right], \tag{3.32}$$

where $1/\mu + 1/\theta = 1$.

We put $\theta = 2(1-\alpha)$, then $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p$ by (3.20). Thus (3.32) turn into

$$\left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_p^s + \|u_t\|_2^2\right],$$

for $s = 2/(1-2\alpha) \leq p$.

We utilize $Corollary$3.1 to get

$$\left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)} \leq C\left[H\,(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_p^p + (h \circ \nabla u)\,(t)\right], \text{for all } t \geq 0. \tag{3.33}$$

By noting that

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon \int_{\Omega} uu_t\,(x,t)\,dx\right)^{1/(1-\alpha)} \qquad (3.34) \\ &\leq 2^{1/(1-\alpha)}\left(H\,(t) + \left|\int_{\Omega} uu_t\,(x,t)\,dx\right|^{1/(1-\alpha)}\right) \\ &\leq C\left[H\,(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_p^p + (h \circ \nabla u)\,(t)\right], \end{aligned}$$

for all $t \geq 0$, and collecting with (3.31) and (3.34), we find

$$L'(t) \geq \Gamma L^{1/(1-\alpha)}(t) \quad \text{for all } t \geq 0, \tag{3.35}$$

where C (the constant of $Lemma$3.1) and Γ is a positive constant depending on $\varepsilon\gamma$ only.

Finally we integrate (3.35) over $(0,t)$ to arrive at

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Gamma t\alpha/\left(1-\alpha\right)}. \tag{3.36}$$

Thence (3.36) shows that $L(t)$ blows up in finite time given by (3.13) above.

The proof is completed. □

Chapter 4

Blow-Up Results for a Quasilinear Wave Equation with Variable Exponents Non-Linearities

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Key Words and Phrases: Blowing up, negative initial energy, variable exponents, positive initial energy.

The following new category of a quasilinear wave equation with variable exponents nonlinearities is studied in this chapter

$$\begin{cases} u_{tt} - \operatorname{div}\left(\left|\nabla u^{s(.)-2}\right| \nabla u\right) - \Delta u_{tt} + \eta u_t \left|u_t\right|^{q(.)-2} = \mu u \left|u\right|^{p(.)-2}, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega. \end{cases} \tag{4.1}$$

We care to find sufficient conditions on $s(.), q(.), p(.)$ and the initial data for which the blowup happens, here $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, be a bounded domain with a smooth boundary $\partial\Omega.\eta, \mu >$

0 are constants and the exponents $q(\cdot), p(\cdot)$, and $s(\cdot)$ are given measurable functions on Ω.

Our chapter is divided into four sections: In the first section, we present some advanced assumptions needed in this chapter. The second section deals with some technical lemmas and the statement without demonstration of the well-posedness of our problem, the third one deals with the result of blow-up for solutions with negative initial energy, and in the fourth one, we present and demonstrate the theorem of blow-up for certain solutions with positive initial energy.

4.1 Basic Assumptions

Some hypotheses required in the proof of our result will be given in this section[1]. Firstly, we suppose the following assumptions:

$(B1)$

$$2 \leq \max\{q_2, s_2\} < p_1 \leq p(x) \leq p_2 \leq s^*(x), \tag{4.2}$$

with

$$\begin{aligned} p_1 &: = ess\inf_{x\in\Omega} p(x), \qquad p_2 := ess\sup_{x\in\Omega} p(x), \\ s_1 &: = ess\inf_{x\in\Omega} s(x), \qquad s_2 := ess\sup_{x\in\Omega} s(x), \\ q_1 &: = ess\inf_{x\in\Omega} q(x), \qquad q_2 := essp\sup_{x\in\Omega} q(x), \end{aligned}$$

and

$$s^*(x) = \begin{cases} \frac{ns(x)}{ess\sup_{x\in\Omega}(n-s(x))} & \text{if } s_2 < n \\ +\infty & \text{if } s_2 \geq n \end{cases},$$

and

$$ess\inf_{x\in\Omega}(s^*(x) - p(x)) > 0.$$

$(B2)$ Also, we suppose that the exponents $q(\cdot), p(\cdot)$, and $s(\cdot)$ are measurable functions such that either satisfy the log-Hölder continuity condition:

$$|m(x) - m(y)| \leq -\frac{A}{\log|x-y|} \text{ for a.e. } x, y \in \Omega, \text{ with } |x-y| < \delta, \tag{4.3}$$

$A > 0, 0 < \delta < 1$, or $q(\cdot), p(\cdot)$, and $s(\cdot) \in C(\bar{\Omega})$.

In (4.3), if $x = y$ the inequality is undefined because log 0 is undefined. The inequality is defined for x not equal to y. But the condition that δ is completely greater than zero always makes x not equal to y because $|x-y| < \delta$. The term $\Delta_{s(\cdot)}u = \text{div}\left(\left|\nabla u^{s(\cdot)-2}\right| \nabla u\right)$ is called $s(.)$ $-$Laplacian.

The energy function associated to the problem (4.1) is the following

$$E(t) := \frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \frac{1}{s(x)} |\nabla u|^{s(x)} dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx - \mu \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx, \ t \geq 0. \tag{4.4}$$

[1] We use the Lebesgue space $L^2(\Omega)$ and the variable-exponent Sobolev space $W_0^{1,s(\cdot)}(\Omega)$ with their norms.

We derive the energy relation and use (4.1) to get

$$E'(t) = -\eta \int_{\Omega} |u_t(x,t)|^{q(x)}\, dx, \text{ for a.e. } t \in [0,T). \tag{4.5}$$

4.2 Statement and Well-Posedness of Problem

This section contains some essential lemma which will be useful to us later in the proof of our blow-up result, before that we introduce the statement without proof of the well-posedness of the problem (4.1)

Proposition 4.1. *Let* $(u_0, u_1) \in \left(W_0^{1,s(\cdot)}(\Omega) \times L^2(\Omega)\right)$ *and suppose that the exponents* p, q, s *satisfy* $(B1)$ *and* $(B2)$*. Then problem* (4.1) *admits a unique weak solution such that*

$$\begin{aligned} u &\in L^\infty((0,T), W_0^{1,s(\cdot)}(\Omega)), \\ u_t &\in L^\infty((0,T), H_0^1(\Omega)), \\ u_{tt} &\in L^\infty((0,T), W_0^{1,s'(\cdot)}(\Omega)), \end{aligned}$$

where $\dfrac{1}{s(\cdot)} + \dfrac{1}{s'(\cdot)} = 1.$

Remark 4.1. *As in the second chapter, we can achieve the proof of the previous proposition by using the Galerkin method. You can see also* [2] .

Lemma 4.1. *Suppose the conditions of* $Lemma 1.14$ *hold. Then, we have*

$$\varrho_{p(.)}^{\frac{r}{p_1}}(u) \le C(\|\nabla u\|_{s(.)}^{s_1} + \varrho_{p(.)}(u)), \; s_1 \le r \le p_1, \tag{4.6}$$

for any $u \in W_0^{1,s(\cdot)}(\Omega)$*, where* $C > 1$ *is a positive constant that depends on* Ω *only.*

Proof. If $\varrho_{p(.)}(u) > 1$, then $\varrho_{p(.)}^{\frac{r}{p_1}}(u) \le \varrho_{p(.)}(u) \le C\left(\|\nabla u\|_{s(.)}^{s_1} + \varrho_{p(.)}(u)\right)$.

If $\varrho_{p(.)}(u) \le 1$, then, by $Lemma 1.3$, $\|u\|_{p(.)} \le 1$. Then, $Lemma 1.14$ and $Lemma 1.4$ imply

$$\begin{aligned} \varrho_{p(.)}^{\frac{r}{p_1}}(u) &\le \varrho_{p(.)}^{\frac{s_1}{p_1}}(u) \le \max\left[\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\}\right]^{\frac{s_1}{p_1}} \\ &= \|u\|_{p(.)}^{s_1} \le C\,\|\nabla u\|_{s(.)}^{s_1}, \end{aligned}$$

where $C > 1$. Therefore (4.6) follows. ◻

Now, we will take the following special case

Corollary 4.1. *Let the assumptions of the previous Lemma hold. Then for any $u \in W_0^{1,s(\cdot)}$ we get*

$$\|u\|_{p_1}^{r} \leq C(\|\nabla u\|_{s_{(.)}}^{s_1} + \|u\|_{p_1}^{p_1}), \tag{4.7}$$

where $s_1 \leq r \leq p_1$ and C is a positive constant.

Now, we set

$$H(t) := -E(t),$$

and use, throughout this chapter, C to denote a generic positive constant depending on Ω only.

As a result of (4.4) and (4.6), we have

Corollary 4.2. *Let the assumptions of Lemma4.1 hold. Then we have*

$$\varrho_{p(.)}^{\frac{r}{p_1}}(u) \leq C(|H\,(t)| + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \varrho_{p(.)}(u)), \tag{4.8}$$

for any $u \in W_0^{1,s(\cdot)}$ and $s_1 \leq r \leq p_1$.

As a particular case, we have the following

Corollary 4.3. *Let the assumptions of Lemma4.1 hold. Then we have*

$$\|u\|_{p_1}^{r} \leq C(|H\,(t)| + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_{p_1}^{p_1}), \tag{4.9}$$

for any $u \in W_0^{1,s(\cdot)}$ and $s_1 \leq r \leq p_1$.

Lemma 4.2. *Assume that* (4.2) *and* (4.3) *hold and $E(0) < 0$. Then the solution of* (4.1) *satisfies, for some $c > 0$,*

$$\varrho_{p(.)}(u) \geq c\,\|u\|_{p_1}^{p_1}. \tag{4.10}$$

Proof. Similar in the proof of *Lemma*2.3. □

Lemma 4.3. *Let u be the solution of problem* (4.1) *and assume that* (4.2) *holds. Then,*

$$\int_\Omega |u|^{q(x)}\,dx \leq C\left(\left(\varrho_{p(.)}(u)\right)^{\frac{q_1}{p_1}} + \left(\varrho_{p(.)}(u)\right)^{\frac{q_2}{p_1}}\right). \tag{4.11}$$

Proof.

$$
\begin{aligned}
\int_{\Omega}|u|^{q(x)}\,dx &\leq \int_{\Omega_-}|u|^{q_1}\,dx+\int_{\Omega_+}|u|^{q_2}\,dx \\
&\leq C\left[\left(\int_{\Omega_-}|u|^{p_1}\,dx\right)^{\frac{q_1}{p_1}}+\left(\int_{\Omega_+}|u|^{p_1}\,dx\right)^{\frac{q_2}{p_1}}\right] \\
&\leq C\left(\|u\|_{p_1}^{q_1}+\|u\|_{p_1}^{q_2}\right) \\
&\leq C\left(\left(\varrho_{p(.)}(u)\right)^{\frac{q_1}{p_1}}+\left(\varrho_{p(.)}(u)\right)^{\frac{q_2}{p_1}}\right),
\end{aligned}
$$

by *Lemma*4.2. □

Lemma 4.4. *Let u be the solution of* (4.1) *with $E(0)<0$. Then, there exists a constant $c_1>0$ such that*

$$\|\nabla u(.,t_k)\|_{s(.)}\geq c_1,\quad \forall t\geq 0. \tag{4.12}$$

Proof. Assume, by contradiction, there exists a sequence t_j such that

$$\|\nabla u(.,t_j)\|_{s(.)}\to 0 \quad \text{as } j\to\infty.$$

Then, *Lemmas*1.4 and 1.14 gives us

$$\varrho_{p(.)}(u(.,t_j))\to 0 \quad \text{as } j\to\infty,$$

this yields

$$\lim_{j\to\infty}E(t_j)\geq 0, \tag{4.13}$$

that contrasts with the fact that $E(t)\leq E(0)<0,\forall t\geq 0$. □

4.3 Blowing-Up for Negative Initial Energy

The main purpose of this section is to introduce and demonstrate the first results of the blow-up.

Theorem 4.1. *Assume that the assumptions of Proposition*4.1 *hold and suppose that*

$$E(0)<0. \tag{4.14}$$

Then the solution of problem (4.1) *blows up in finite time.*

Proof. As usual, multiplying by u_t and integrating over Ω in (4.1), to get

$$E'(t) = -\eta \int_\Omega |u_t(x,t)|^{q(x)}\, dx \leq 0, \tag{4.15}$$

for almost every t in $[0,T)$ since $E(t)$ is absolutely continuous function (see Georgiev and Todorova [31]); hence $H'(t) \geq 0$ and

$$0 < H(0) \leq H(t) \leq \frac{\mu}{p_1} \varrho_{p(.)}(u), \tag{4.16}$$

for every t in $[0,T)$, by remembring the condition that $E(0) < 0$. We then introduce

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t)dx, \tag{4.17}$$

for ε small to be chosen later and

$$0 < \alpha \leq \min\left\{ \frac{p_1 - 2}{2p_1}, \frac{p_1 - q_2}{p_1(q_2 - 1)} \right\}. \tag{4.18}$$

By taking the derivative of (4.17) and using Eq. (4.1), we obtain

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega u_t^2(x,t)dx + \varepsilon \int_\Omega u u_{tt}(x,t)dx, \qquad (4.19)\\ L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega \left[u_t^2 - |\nabla u|^{s(x)} + |\nabla u_t|^2 \right] \\ &\quad + \varepsilon\mu \int_\Omega |u|^{p(x)} - \eta\varepsilon \int_\Omega u u_t |u_t|^{q(x)-2} - \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\} \right). \end{aligned}$$

Adding and subtracting the term $\varepsilon(1-\xi)p_1 H(t)$, for $0 < \xi < 1$, in the right side of (4.19), to get

$$\begin{aligned} L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\} \right) &\geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon(1-\xi)p_1 H(t) \\ &\quad + \varepsilon\mu\xi \int_\Omega |u|^{p(x)} + \varepsilon \left(\frac{(1-\xi)p_1}{2} + 1 \right) \|u_t\|_2^2 \qquad (4.20)\\ &\quad + \varepsilon \left(\frac{(1-\xi)p_1}{s_2} - 1 \right) \int_\Omega |\nabla u|^{s(x)} \\ &\quad + \varepsilon \left(\frac{(1-\xi)p_1}{2} + 1 \right) \int_\Omega |\nabla u_t|^2 \\ &\quad - \eta\varepsilon \int_\Omega u u_t |u_t|^{q(x)-2}\, dx. \end{aligned}$$

So, for ξ small enough, we obtain

$$\begin{aligned} L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\} \right) &\geq \varepsilon\beta \left[H(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \varrho_{s(.)}(\nabla u) + \varrho_{p(.)}(u) \right] \qquad (4.21)\\ &\quad + (1-\alpha)H^{-\alpha}(t)H'(t) - \eta\varepsilon \int_\Omega u u_t |u_t|^{q(x)-2}\, dx, \end{aligned}$$

where

$$\beta = \min \left\{ (1-\xi)\, p_1, \mu\xi, \frac{(1-\xi)\, p_1}{2} + 1, \frac{(1-\xi)\, p_1}{s_2} - 1 \right\} > 0.$$

By using Young's inequality, the last term in (4.21) yields

$$\int_\Omega |u_t|^{q(x)-1} |u|\, dx \leq \frac{1}{q_1} \int_\Omega \delta^{q(x)} |u|^{q(x)} + \frac{q_2 - 1}{q_2} \int_\Omega \delta^{-q(x)/q(x)-1} |u_t|^{q(x)}\, dx, \forall \delta > 0. \tag{4.22}$$

Thus, by picking δ such that

$$\delta^{-q(x)/q(x)-1} = kH^{-\alpha}(t),$$

for a large constant k to be given later, and replacing in (4.22) we reach to

$$\int_\Omega |u_t|^{q(x)-1} |u|\, dx \leq \frac{1}{q_1} \int_\Omega k^{1-q(x)} |u|^{q(x)} H^{\alpha(q(x)-1)}(t) + \frac{q_2 - 1}{q_2 \mu} kH^{-\alpha}(t)H'(t), \forall \delta > 0. \tag{4.23}$$

Combining (4.21) and (4.23) yields

$$\begin{aligned} L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\} \right) \geq\ & \varepsilon\beta \left[H(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \varrho_{s(.)}(\nabla u) + \varrho_{p(.)}(u) \right] \\ & + \left[(1-\alpha) - \varepsilon \frac{q_2 - 1}{q_2} k \right] H^{-\alpha}(t)H'(t) \\ & -\eta\varepsilon \frac{k^{1-q_1}}{q_1} C_1 H^{\alpha(q_2-1)}(t) \int_\Omega |u|^{q(x)}\, dx. \end{aligned} \tag{4.24}$$

Exploiting *Lemma*4.3 and (4.16) to get

$$H^{\alpha(q_2-1)}(t) \int_\Omega |u|^{q(x)}\, dx \leq C \left[(\varrho(u))^{\frac{q_1}{p_1} + \alpha(q_2-1)} + (\varrho(u))^{\frac{q_2}{p_1} + \alpha(q_2-1)} \right]. \tag{4.25}$$

Now, we employ *Lemma*4.1 and (4.18) for

$$r = q_2 + \alpha p_1(q_2 - 1) \leq p_1 \text{ and } r = q_1 + \alpha p_1(q_2 - 1) \leq p_1,$$

it is easy to see from (4.25) that

$$H^{\alpha(q_2-1)}(t) \int_\Omega |u|^{q(x)}\, dx \leq C \left(\|\nabla u\|_{s(.)}^{s_1} + \varrho_{p(.)}(u) \right), \tag{4.26}$$

then, using *Lemmas*4.4 to obtain

$$\|\nabla (u/c_1)\|_{s(.)} \geq 1. \tag{4.27}$$

*Lemma*1.4 and (4.27) leads to

$$\begin{aligned} \varrho_{s(.)}(\nabla (u/c_1)) &\geq \min \left\{ \|\nabla (u/c_1)\|_{s(.)}^{s_1}, \|\nabla (u/c_1)\|_{s(.)}^{s_2} \right\} \\ &= \|\nabla (u/c_1)\|_{s(.)}^{s_1}. \end{aligned} \tag{4.28}$$

Thus (4.28) becomes

$$\varrho_{s(.)}(\nabla u) \geq c_2 \|\nabla u\|_{s(.)}^{s_1}. \tag{4.29}$$

Collecting of (4.24), (4.26), and (4.29) reach to

$$\begin{aligned} L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\}\right) &\geq \left[(1-\alpha) - \frac{q_2 - 1}{q_2}\varepsilon k\right] H^{-\alpha}(t) H'(t) \\ &+\varepsilon\left(\beta - \eta\frac{k^{1-q_1}}{q_1}C\right)\left[H(t) + \|u_t\|_2^2\right. \\ &\left. + \|\nabla u_t\|_2^2 + \|\nabla u\|_{s(.)}^{s_1} + \varrho_{p(.)}(u)\right]. \end{aligned} \tag{4.30}$$

In this step, we choose k so large that the coefficient

$$\gamma = \beta - \eta\frac{k^{1-q_1}}{q_1}C > 0.$$

Once k is fixed (thus γ), we put sufficiently small ε so that

$$(1-\alpha) - \frac{q_2-1}{q_2}\varepsilon k \geq 0 \text{ and } L(0) = H^{1-\alpha}(0) + \varepsilon \int_\Omega u_0 u_1(x)\, dx > 0.$$

Subsequently (4.30) becomes

$$\begin{aligned} L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\}\right) &\geq \varepsilon\gamma\left[H(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla u\|_{s(.)}^{s_1} + \varrho_{p(.)}(u)\right] \\ &\geq \varepsilon\gamma\left[H(t) + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}\right], \end{aligned} \tag{4.31}$$

by virtue of (4.10). Therefore

$$L(t) \geq L(0) > 0, \text{ for all } t \geq 0.$$

Next, we are in the position to obtain an inequality of the form

$$G'(t) \geq \Gamma L^{1/(1-\alpha)}(t), \quad \text{for all } t \geq 0, \tag{4.32}$$

here Γ is a positive constant depends on $\varepsilon\gamma$, C (the constant of $Corollary$4.1) and

$$L'(t) + \frac{d}{dt}\left(\varepsilon \int_\Omega \{\nabla u_t \nabla u\}\right) = G'(t).$$

When we prove (4.32), we get in a standard way the finite-time blow-up of the functional $L(t)$.

To achieve (4.32), we estime the term

$$\begin{aligned}\left|\int_{\Omega} uu_t(x,t)\,dx\right| &\leq \|u\|_2\|u_t\|_2\\ &\leq C\left(\|u\|_{p_1}\|u_t\|_2\right),\end{aligned}$$

thence

$$\left|\int_{\Omega} uu_t(x,t)\,dx\right|^{1/(1-\alpha)} \leq C\|u\|_{p_1}^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}.$$

Young's inequality gives us the following estimate

$$\left|\int_{\Omega} uu_t(x,t)\,dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_{p_1}^{\omega/(1-\alpha)} + \|u_t\|_2^{\chi/(1-\alpha)}\right], \tag{4.33}$$

where $1/\omega + 1/\chi = 1$. Putting $\chi = 2(1-\alpha)$, we find $\omega/(1-\alpha) = 2/(1-2\alpha) \leq p_1$ by (4.18).

Thus (4.33) becomes

$$\left|\int_{\Omega} uu_t(x,t)\,dx\right|^{1/(1-\alpha)} \leq C\left[\|u\|_{p_1}^{r} + \|u_t\|_2^{2}\right],$$

with $r = 2/(1-2\alpha) \leq p_1$. We obtain after using $Corollary 4.3$

$$\left|\int_{\Omega} uu_t(x,t)\,dx\right|^{1/(1-\alpha)} \leq C\left[H(t) + \|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|u\|_{p_1}^{p_1}\right], \text{ for all } t \geq 0. \tag{4.34}$$

In the end, by noting that

$$\begin{aligned}L^{1/(1-\alpha)}(t) &= \left[H^{1-\alpha}(t) + \varepsilon\int_{\Omega} uu_t(x,t)\,dx\right]^{1/(1-\alpha)}\\ &\leq 2^{1/(1-\alpha)}\left[H(t) + \left|\int_{\Omega} uu_t(x,t)\,dx\right|^{1/(1-\alpha)}\right],\end{aligned}$$

and combining it with (4.31) and (4.34), the inequality (4.32) is achieved.

Integrate (4.32) over $(0,t)$ to obtain

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Gamma t\alpha/(1-\alpha)}. \tag{4.35}$$

So $L(t)$ blows up in finite time

$$T^* \leq \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}}, \tag{4.36}$$

where Γ and α are positive constant with $\alpha < 1$ and L is given by (4.17) above.

The proof is completed. □

Remark 4.2. *The estimate* (4.36) *shows that the larger* $L(0)$ *is, the quicker the blow-up takes place.*

4.4 Blowing-Up for Positive Initial Energy

Now, we are in the position to present and prove one of the main results of this section which is the blowup for certain solutions with positive energy. For this goal, let A be the best constant of the Sobolev embedding $W_0^{1,s(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ and let

$$\begin{aligned} A_1 &= \max\left\{1, A, \left(\frac{1}{\mu}\right)^{1/s_2}\right\}, \\ \alpha_1 &= \left(\left(\frac{1}{\mu A_1^{p_1}}\right)^{s_2/(p_1-s_2)}\right), \\ \alpha_0 &= \|\nabla u_0\|_{s(.)}^{s_2}, \\ E_1 &= \left(\frac{1}{s_2} - \frac{1}{p_1}\right)\alpha_1, \end{aligned}$$

$$H(t) = E_1 - E(t), \tag{4.37}$$

$$K(t) = H^{1-\lambda}(t) + \varepsilon \int_\Omega u u_t\,(x,t)\,dx, \tag{4.38}$$

for $0 < \lambda < 1, \varepsilon > 0$ are to be specified later.

We state here the following theorem which will be our main result.

Theorem 4.2. *Assume that the conditions of Proposition*4.1 *hold and suppose that*

$$E(0) < E_1, \alpha_1 < \alpha_0 \le A_1^{-s_2}. \tag{4.39}$$

Then the solution of (4.1) *blows up in a finite time.*

To demonstrate our theorem, we refer to the following *two lemmas*.

Lemma 4.5. *Let the assumptions in Theorem*4.2 *be fulfilled, then there exists a constant* $\alpha_2 > \alpha_1$ *such that*

$$\|\nabla u\,(.,t)\|_{s(.)}^{s_2} \ge \alpha_2, \qquad \forall t \ge 0. \tag{4.40}$$

Proof. Exploiting (4.4), we get

$$\begin{aligned} E(t) &\geq \frac{1}{s_2}\varrho_{s(.)}(\nabla u) - \frac{\mu}{p_1}\varrho_{p(.)}(u) \\ &\geq \frac{1}{s_2}\min\left\{\|\nabla u\|_{s(.)}^{s_1}, \|\nabla u\|_{s(.)}^{s_2}\right\} - \frac{\mu}{p_1}\max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\} \\ &\geq \frac{1}{s_2}\min\left\{\|\nabla u\|_{s(.)}^{s_1}, \|\nabla u\|_{s(.)}^{s_2}\right\} - \frac{\mu}{p_1}\max\left\{\left(A_1\|\nabla u\|_{s(.)}\right)^{p_1}, \left(A_1\|\nabla u\|_{s(.)}\right)^{p_2}\right\} \\ &= \frac{1}{s_2}\min\left\{\alpha^{\frac{s_1}{s_2}}, \alpha\right\} - \frac{\mu}{p_1}\max\left\{(A_1^{s_2}\alpha)^{\frac{p_1}{s_2}}, (A_1^{s_2}\alpha)^{\frac{p_2}{s_2}}\right\} \\ &: = h(\alpha), \forall \alpha \in [0,\infty), \end{aligned}$$

where $\alpha = \|\nabla u\|_{s(.)}^{s_2}$.

Let

$$g(\alpha) = \frac{1}{s_2}\alpha - \frac{\mu}{p_1}(A_1^{s_2}\alpha)^{\frac{p_1}{s_2}}.$$

By noting that $g(\alpha) = h(\alpha)$, for $0 < \alpha \leq A_1^{s_2}$. We can easily verify that the function $g(\alpha)$ is increasing for $0 < \alpha < \alpha_1$ and decreasing for $\alpha_1 < \alpha \leq +\infty$. Because $E(0) < E_1 = g(\alpha_1)$, there exists a positive constant $\alpha_2 \in (\alpha_1, \infty)$ such that $g(\alpha_2) = E(0)$. So we get $g(\alpha_0) = h(\alpha_0) \leq E(0) = g(\alpha_2)$. This means that $\alpha_0 \geq \alpha_2$.

To demonstrate (4.40), we suppose that $\|\nabla u(t_0)\|_{s(.)}^{s_2} < \alpha_2$, for some $t_0 > 0$. Then there exists $t_1 > 0$ such that $\alpha_1 < \|\nabla u(t_1)\|_{s(.)}^{s_2} < \alpha_2$. Exploiting the monotonicity of $g(\alpha)$ to find

$$E(t_1) \geq g\left(\|\nabla u(t_1)\|_{s(.)}^{s_2}\right) > g(\alpha_2) = E(0),$$

which contradicts $E(t) < E(0)$, for all $t \in (0,T)$. Consequently, (4.40) is determined. □

Lemma 4.6. *Let the assumptions in Theorem4.2 be fulfilled, so we have*

$$0 < H(0) \leq H(t) \leq \frac{\mu}{p_1}\varrho_{p(.)}(u).$$

Proof. Exploiting (4.4), (4.15), and (4.37) to get

$$\begin{aligned} 0 &< H(0) \leq H(t) \\ &\leq E_1 - \left[\frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \frac{1}{s(x)}|\nabla u|^{s(x)} dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx\right] \\ &+\mu\int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx, \end{aligned}$$

then from (4.40), we find

$$\begin{aligned}
E_1 - \left[\frac{1}{2}\int_{\Omega} u_t^2 dx + \int_{\Omega} \frac{1}{s(x)} |\nabla u|^{s(x)} dx + \frac{1}{2}\int_{\Omega} |\nabla u_t|^2 dx\right] &\leq E_1 - \int_{\Omega} \frac{1}{s_2} |\nabla u|^{s(x)} dx \\
&\leq E_1 - \frac{1}{s_2} \min\left\{\|\nabla u\|_{s(.)}^{s_1}, \|\nabla u\|_{s(.)}^{s_2}\right\} \\
&\leq E_1 - \frac{1}{s_2} \min\left\{\alpha_2^{\frac{s_1}{s_2}}, \alpha_2\right\} \\
&\leq E_1 - \frac{1}{s_2} \min\left\{\alpha_1^{\frac{s_1}{s_2}}, \alpha_1\right\} \\
&= E_1 - \frac{1}{s_2}\alpha_1 = -\frac{\alpha_1}{p_1} < 0, \forall t \geq 0.
\end{aligned}$$

Therefore,

$$0 < H(0) \leq H(t) \leq \frac{\mu}{p_1} \varrho_{p(.)}(u). \forall t \geq 0.$$

□

Proof of Theorem4.2. It is not hard to determine the proof precisely by repeating the same steps (4.17) to (4.34) of the proof of *Theorem*4.1. With the use of *Lemma*4.6.

Conclusion and Suggestions

Conclusion

We studied in this dissertation three classes of nonlinear hyperbolic problems with constant and variable exponents nonlinearities, and we obtained different results of existence and blow-up of these problems, of course under suitable assumptions on the exponents of nonlinearity and the initial data. Specially, we expanded the results of blow-up of some nonlinear wave equations studied by Messaoudi [51, 58, 60] , and exploit ideas by Georgiev and Todorova [31] in both cases of constant and variable exponents nonlinearities.

Perspectives and Some Open Problems

As a perspective, after the completion of this dissertation, our vision is devoted to illustrating the results of blow-up numerically.

As future work, we collect here some questions and open problems of other nonlinear hyperbolic equations with variable exponents that can be studied:

- A researcher can expand the result for the previous problem in unbounded domains, where the Poincaré's inequality and some of the results embedding are no longer valid.
- We also imposed another question related to the asymptotic behavior of solution for a system of a nonlinear damped wave equation with nonstandard nonlinearities.
- Expand the results of blow-up to some Fpde problems.
- Extend the blowup results to some Timoshenko equation with nonstandard nonlinearities.

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