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About Ensemble Controllability Of Parameter Dependent Systems

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بسم الله الرحمن الرحيم

﴿قالوا سبحانك لاعلم لنا إلا ما علمتنا إنك أنت العليم الحكيم﴾

صدق الله العظيم

" البقـــرة ٣٢ "

إهداء

الحمد الله الذي وفقني في تثمين هذه الخطوة في مسيرتي الدراسية بمذكرة أهديها إلى:

أوفى خلق الله وأحبهم إلى قلبي، إلى ذلك الحرف الامتناهي من الحب والحنان ،من ترفرف العين من وحشتها ، التي كانت تتمنى رؤيتي وأنا أحقق هذا النجاح، شاء الله أن يأتي هذا اليوم، رحمك الله وأسكنك فسيح جناته يا أمي .

درعي الذي به احتميت و تاج فخر لطالما حملته على رأسي لك، إلى من لم يبخل علي يوما ، أبي حفظه الله .

من قاسموني حلو الحياة ومرها و من كانو السند الدائم لي وكانو أسعد الناس بنجاحي إخوتي وأخواتي خصوصا الراحلة الباقية في قلبي أختي "غادة"رحمها الله .

إلى من أراه بسمتي وجمال الأيام هو "أخي خالد" .

إلى التي حين إلتقيتها أدركت أن الصدف تختبأ وراءها حياة، أختي ورفيقة دربي " براهمي بلقيس" أحسن من عرفني بهم القدر صديقاتي هيبي صفاء، دوايدي أميرة، بوزنادة صورية، قواسمية هديل، قدوري صفية.

وفي الأخير أهدي هذا العمل المتواضع لكل من نسيهم قلمي،وإلى كل من يفكر ويبحث للإرتقاء بالعلم .

"بلقاسمي لبنة"...

إهداء

الحمد الله الذي وفقني في إتمام هذا العمل المتواضع الذي أهديه بدوري إلى أعز الناس وأقربهم إلى قلبي إلى والدتي العزيزة والدي العزيز اللذان كانا عونا وسندا دائما لي، ولم يبخلا عليا بأي شيء، وكان لدعائهما المبارك أعظم الأثر في تسيير سفينة البحث حتى ترسو على هذه الصورة .

إلى الذين ظفرت بهم هدية من الأقدار إخوة فعرفو معنى الأخوة حماكم الله.

وإلى الشخص الذي سأخطو معه أسمى خطوات حياتي إلى رفيق القلب والعمر

لن أقول شكرا ،بل سأعيش الشكر معك دائما .

إلى رفاق الضحك والتعب والسهر صديقاتي هيبي صفاء ، دوايدي أميرة ، قواسمية هديل ، بوزنادة صورية ، قدوري صفية .

إلى أختي التي لم تلدها أمي وصديقة العمر قبل أن تكون رفيقتي في هذا العمل "بلقاسمي لبنة" .

إلى كل معلم علمني ولو حرفا ،إلى كل طالب علم يسعى لإكتساب المعرفة وتزويد رصيده المعرفي والعلمي .

" براهمي بلقيس"...

شكر وعرفان

قال رسول الله صلى الله عليه وسلم :

(من لم يشكر الناس لم يشكر الله ومن أهدى إليكم معروفا فكافئوه فإن لم تستطيعوا فادعوا له)

وعملا بهذا الحديث وإعترافا بالجميل نشكر الله عزَ وجل أولا الذي أعاننا وشَدَ من عزمنا لإكمال هذا العمل ،نشكر الله الذي وهبنا الصبر والشجاعة لنجعل هذا المشروع علما ينتفع به .

ونشكر أولياءنا خاصة وعائلتينا عامة على تضحياتهم المختلفة ،ودعمهم المعنوي والتشجيع طوال مشوار دراستنا .

ونتقدم بأجمل عبارات الشكر والإمتنان للأستاذ "**عبد الحق خفظ الله** " لقبوله الإشراف على هذا العمل والذي لم يدخر جهدا في تقديم النصيحة والتوجيه لنا ، كما لا يفوتنا أن نشكره لأنه كان دائما منتبها جدا وتتبع هذا البحث بكل إهتمام وكان متاحا طوال الوقت جعل الله ذلك في ميزان حسناته يوم الدين .

كما نشكر الأستاذان الفاضلان "**النوري بومعزة**" و "**فيصل مرغادي**" لقبولهما مناقشة هذا العمل .

ونتوجه بالشكر إلى هيئة التدريس والإدارة في كلية الرياضيات والإعلام الألي جامعة العربي التبسي .

ونشكر جميع الأهل والأصدقاء على دعمهم ومساندتهم ولو بكلمة طيبة .

دون أن ننسى شكر أنفسنا على إصرارنا لإكمال هذا العمل المتواضع رغم كل المعيقات والصعوبات التي واجهتنا ، تحت شعار

"كن عالما . . فإن لم تستطع فكن متعلما . . فإن لم تستطع فأحب العلماء . . فإن لم تستطع فلا تبغضهم "

Contents

Abstract

This memory provides an overview of recent developments in ensemble controllability. In parameter-dependent linear systems, we look at various ensemble control concepts that have been actively developed over the past decade As an example, we cite the work of Li [9] and Khaneja [10].

The goal of the control function is to steer the ensemble system to a desired parameter independent state, by implementing parameter-independent open loop controls, then necessary and sufficient conditions for ensemble control are established using methods from complex approximation theory .

We consider the problem of ensemble controllability for finite and infinite linear systems, and we provide an overview of the observational ensemble due to its strong connection with the memory theme.

Keywords: Ensemble controllability, parameter dependent systems, ensemble observability, uniform null ensemble controllability, averaged control.

ملخص

تقدم هذه المذكرة لمحة عامة عن التطورات الأخيرة حول موضوع قابلية التحكم في المجموعات في الأنظمة الخطية المعتمدة على وسائط، نحن نقدم مجموعة متنوعة من مفاهيم التحكم في المجموعة التي تم تطويرها بشكل فعال خلال العقد الماضي ،وكمثال على ذلك نذكر عمل Li [9] و Khaneja [10].

هدف وظيفة التحكم في المجموعة هو توجيه مجموعة الانظمة للحالة المرغوبة، من خلال تطبيق عناصر تحكم حلقة مفتوحة مستقلة عن الوسائط، ثم بعد ذلك يتم وضع الشروط الضرورية والكافية للتحكم في المجموعة باستخدام طرق من نظرية التقريب المعقدة .

نعتبر مشكلة قابلية التحكم في المجموعة للأنظمة الخطية المحدودة وغير المحدودة ونعطي لمحة عن مجموعة قابلية الملاحظة نظرًا لارتباطها القوي بموضوع المذكرة.

الكلمات المفتاحية : مجموعة التحكم، الأنظمة المعتمدة على المعلمات، مجموعة الملاحظة، إمكانية التحكم الموحد في المجموعة الفارغة، متوسط التحكم.

Notations & abbreviations

$\mathbb{R}$	Set of real numbers.
$\parallel .\parallel_H$	A norm in space H.
ODE	Ordinary differential equation.
PDE	Partial differential equation.
$\mathbb{C}$	Compound set.
L^p	Space of Lebesgue.
I	The identity operator.
$\lvert . \rvert$	A norm in $L^2\left(\Omega\right)$or absolute value.
$L^p\left(\Omega\right)$	$\left\{u : \Omega \longrightarrow \mathbb{R} \,\middle\vert\, \int_\Omega \lvert u\rvert^p \, dx < \infty \right\}, p \in \mathbb{R}, 1 \leq p < \infty.$
$\langle .,. \rangle_{E,E'}$	Duality Product.
Λ	The Jordan canonical form.
$\mathbf{R}H\infty$	The set of rational, proper and stable matrices with real coefficients.

Figures list

Introduction

The challenge of controlling a considerable, perhaps endless, number of objects is referred to as ensemble control. The heart of solid network theory involves regulation ensembles. Ensembles are driven by various applications for various physical and non-physical systems. The goal of this memory is to provide you with an overview of the new topic of ensemble control for linear systems, which is concerned with the control of families of linear systems.

Regulating ensembles is at the heart of network theory because it uses a single-input control function or a single-feedback controller to control a set of states or systems.

Controlling parameter-variable systems with parameter-independent open-loop or closed-loop controllers is another research topic. The so-called blending problem, which arose in the late 1970s, is one such example, we want to discover family-stabilizing parameters- independent feedback control principles. In physics, mathematics, and engineering, the goal is to control a family of systems or state variables. A major difficulty in ensemble control is the controllability of an ensemble using open-loop input signals independent of the system properties. This corresponds to the classic challenge of controlling many linear systems in parallel if the parameter set has a finite number of values. The key results on the ensemble controllability of linear parameter-dependent systems can be found in Helmke and Schönlein [6]; Li [9].

The underlying notions used to examine the ensembles of linear systems are the same as those used in quantum physics and other sciences to explore large-scale systems. It can also happen in regular life, such as when using an oven.

This memory is organized as follows: In the first chapter, we give a general statement about the ensemble controllability in finite dimensional systems, in which we dealt with two sections, namely, the ensemble controllability in continuous time and in discrete time. where we discussed in the first section the concepts of the ensemble controllability in one parameter and a set of parameters, similar to the concept of uniform null ensemble controllability, the difference between the ensemble control and the average control.

The second chapter contains a general statement about ensemble controllability in infinite dimensional systems, in which we discuss the problem of L^2 ensemble controllability without forgetting optimal control of a harmonic oscillator ensemble.

The third chapter deals with the concept and theories of ensemble observability. We finish the memory by the conclusion.

Chapter 1

Ensemble controllability for finite dimensional systems

1.1 Ensemble controllability of continuous time systems

1.1.1 In the case of a single parameter

In the linear instance, we consider the finite dimensional control system as follows:

$$\begin{cases} \frac{\partial}{\partial t} y(t;\theta) = A(\theta) y(t;\theta) + B(\theta) u(t) & t \in [0,T]\,, \\ y(0;\theta) = y_0(\theta), \end{cases} \tag{1.1}$$

Where θ is a parameter that generally changes in a compact interval $[a,b] \subset \mathbb{R}$, $A(\theta) \in \mathbb{R}^{n\times n}$, $B(\theta) \in \mathbb{R}^{n\times m}$ and $y_0(.)$ is continuous for all θ.

An important problem in ensemble control is that an ensemble can only be controlled using a common control input rather than applying individual input signals to individual systems, which is a practical boundary in normal ensemble control concerns.

A common control task is to guide the ensemble to a chosen terminal state at time $t = T$, which is frequently requested in order to range with a specific parameter. As a result, terminal states can be expressed as functions $\theta \to y(T;\theta)$.

The control input, on the other hand, is required to be independent of the parameter θ.

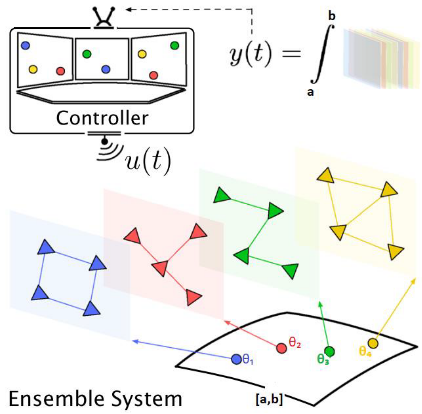

Figure 1 : A surface parameter indexes a continuum ensemble of systems.

A supervisor sends out a signal $u(t)$ as a common steering control input to each and every system in the ensemble. Meanwhile, receives the result of a measurement $y(t)$ information integration of the individual states systems

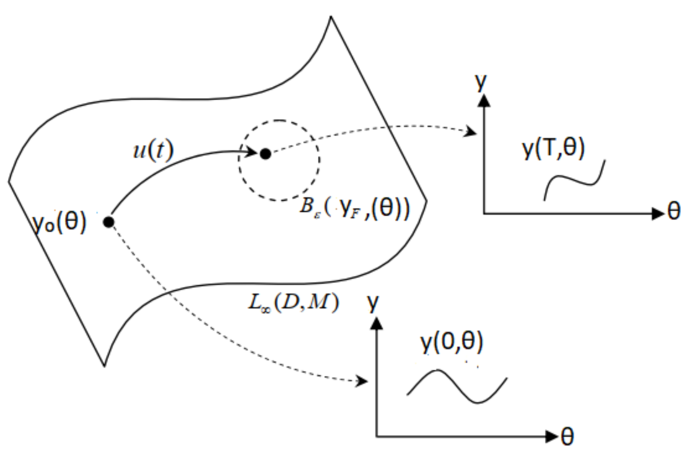

Figure 2 : The concept of ensemble controllability.

Two locations on the function space $L^\infty(D, M)$, $y_0(\theta)$ and $y(T, \theta)$, correspond to two functions on the $\theta - y$ domain. The system is ensemble controllable if there is a $u(t)$ that steers the system (1.1) from an initial position $y_0(\theta)$ to $y(T, \theta) \in B_\varepsilon(y_F(\theta))$ for some limited time T.

Let's outline the uniform ensemble controllability:

Definition 1.1 (Adu,D, 2017) ***(L^p-ensemble controllability)*** *For every y_0 and y_d in $L^p([a,b];\mathbb{R}^n)$, an ensemble $\sum_C([a,b],A,B)$ is L^p-ensemble controllable if and only if there exists a finite time $T>0$ and a control signal $u \in L^q([0,T];\mathbb{R}^m)$ that directs the trajectories of $\sum_C([a,b],A,B)$ for all $\theta \in [a,b]$ from $y(0;\theta)$ to $y_d(\theta)$, where $y(T;\theta)$ satisfies the relation*

$$\|y(T;\theta)-y_d\|_{L^p([a,b];\mathbb{R}^n)} = \left(\int_a^b \mid y(T;\theta)-y_d \mid^p d\theta\right)^{\frac{1}{p}} = 0. \tag{1.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Definition 1.2 (Adu,D, 2017) ***(Uniform ensemble controllability)****In $C([a,b],\mathbb{R}^n)$, an ensemble $\sum_C([a,b],A$ is uniform ensemble controllable if and only if, for any y_0 and y_d in $C([a,b],\mathbb{R}^n)$, and, there exists a finite time $T>0$ and a control signal $u \in L^1([0,T];\mathbb{R}^m)$ that steers the trajectories of $\sum_C([a,b],A,B)$ or all $\theta \in [a,b]$ from $y(0;\theta)$ to y_d , where $y(T;\theta)$ satisfies the relation*

$$\|y(T;\theta)-y_d\|_\infty = \sup_{\theta\in[a,b]} \| y(T;\theta)-y_d \| = 0. \tag{1.2}$$

Also, we introduce another notion of ensemble controllability.
To obtain a valid definition of ensemble controllability, we must first solve the $y(T;\theta)=y_d$ problem. Because y_d is a constant and $y(T;\theta)$ is a variable connected to θ, this assertion cannot be equal. To overcome this difficulty.

Remark 1.1 (Adu,D, 2017) *We find it easier to consider the system parameters in terms of $\mathbb{C}$. We analyze the ensemble of control systems $\sum_C([a,b],A,B)$ described in (1.1), where $(A;B) \in L^\infty(M;\mathbb{C}^{n\times n})\times L^2(M;\mathbb{C}^{n\times m})$. That is,$a_{ij} \in L^\infty(M;\mathbb{C})$ and $b_{ij} \in L^2(M;\mathbb{C})$, where a_{ij} and b_{ij} are the $ijth$ entries of A and B, respectively, and $i,\ j \in \{1,...,n\}$ is the number of entries.*
We assume in this chapter that $u \in L^2([0;T];C^m)$, We have the system $(1,1)$ solution of the form

$$y(T,\theta) = e^{A(\theta)T}y_0 + \int_0^T e^{A(T-t)}B(\theta)\,u(t)\,dt. \tag{1.4}$$

It should be noted that if $y(T,\theta)=y_d$, then we have that

$$\begin{aligned} F(\theta) &: \theta \in [a,b] \to \mathbb{R}^n, && (1.5)\\ &= \textstyle\int_0^T e^{A(\theta)T}B(\tau,\theta)u(\tau)d\tau. && (1.1)\end{aligned}$$

Let's start by giving some conditions for ensemble controllability:

Theorem 1.1 (Fuhrmann and Helmke ,2015) ***(Sufficient Condition for ensemble controllability)****. If these conditions are satisfied, let $\theta \in [a,b]$ be a compact interval and $(A(\theta),B(\theta))$ be a*

continuum family of uniformly controlled single-input systems (or, in general, L^q - controllable for $1 \leq q \leq \infty$):
(a) $(A(\theta), b(\theta))$ *is controllable for all* $\theta \in [a, b]$.
(b) *The spectra of* $A(\theta)$ *and* $A(\theta')$ *are disjoint i.e. For pairs of distinct parameters* $\theta, \theta' \in [a, b], \theta \neq \theta'$:

$$\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset.$$

(c) *For every* $\theta \in [a, b]$ *the eigenvalues of* $A(\theta)$ *have an algebraic multiplicity of one. Conditions* (a) *and* (b) *also important for uniform ensemble controllability.*

Proof. See (Fuhrmann and Helmke ,2015) page (612) ∎

1.1.2 In the case of a set of parameters

Our analysis of combinations of linear systems begins with linear systems that depend on many factors inside the model (1.6).

$$\begin{cases} \frac{\partial}{\partial t} y(t;\theta) = A(\theta) y(t;\theta) + B(\theta) u(t) \qquad t \in [0, T], \\ \qquad\qquad\qquad y(0;\theta) = y_0(\theta), \end{cases} \tag{1.6}$$

For simplicity, we assume that the system matrices $A(\theta) \in \mathbb{R}^{n\times n}$ and $B(\theta) \in \mathbb{R}^{n\times m}$ range continuously in a compact domain $\mathbf{P}$ of parameters θ in euclidean space $\mathbb{R}^d$.
The analysis of such linear system families can take different shapes. To begin with, finding parameter-dependent controllers that guide systems from a set of preliminary states to a set of preferred terminal states is a difficult task. The degree of consistency or smoothness within the parameters imposed at the controls is most likely a restriction here. It may be appropriate for remark controllers and input functions to be similar if the system matrices are polynomially dependent on a parameter, for example. It may be appropriate for the remark controllers and input functions to be similar if the system matrices are polynomially dependent on a parameter.

Definition 1.3 [*Fuhrmann and Helmke,* 2015] *Let* $1 \leq p \leq \infty$. *System* (1.6) *is uniformly ensemble controllable if there is a control* $u \in L^p([0, T], \mathbb{R}^m)$ *such that the caused nation trajectory satisfies for any continuous function* $y_d : \mathbf{P} \to \mathbb{R}^n$ *and each* $\varepsilon > 0$.

$$\sup_{\theta \in \mathbf{P}} \parallel y\,(T;\theta) - y_d \parallel < \varepsilon. \tag{1.7}$$

Rather than seeking for controls that meet the uniform ensemble controllability criterion (1.7)*, one can look for controls* $u(t)$ *that lower the* L^q*-norms for the ensemble.*

$$\left(\int_{\mathbf{P}} \mid y(T;\theta) - y_d \mid^p d\theta\right)^{\frac{1}{p}} < \varepsilon. \tag{1.8}$$

The system is then said to be L^q-ensemble controllable. The system is referred to as precisely ensemble controlled if the conditions in (1.7) *or* (1.8) *are met for* $\varepsilon = 0$. *Of course, the ability to select the input function independently of the parameter* θ *it isn't in any respect plain, and systems of this type do in truth exist.*

1.1.3 Some characterizations of ensemble controllability

Proposition 1.1 [Schönlein and Helmke, 2016]*Suppose that the ensemble in continuous time (or discrete time) is uniform ensemble controllable. Then*
(1) *For each* $\theta \in \mathbf{P}$, *the linear system* $(A(\theta), B(\theta))$ *is controllable.*
(2) *For every number* $s \geq m+1$ *of distinct parameters* $\theta_1, . . ., \theta_s \in \mathbf{P}$*, the spectra of* $A(\theta)$ *satisfy*

$$\sigma(A(\theta_1)) \cap . . . \cap \sigma(A(\theta_s)) = \emptyset.$$

The previous conditions are useful since they exclude non-ensemble controlled families. $A(\theta)$ *cannot have an* θ *independent eigenvalue, according to condition* (2). *We then show how to employ a polynomial approximation condition to characterize significant and necessary situations for uniform ensemble controllability.*

Remark 1.2 [Schönlein and Helmke, 2016] *Let* $\mathbf{P}$ *be compact. Assume that:*
(i) $A(\theta)$ *has simple spectra for all* θ.
(ii) *For all* $\theta \neq \theta'$ *the spectra of* $A(\theta)$ *and* $A(\theta')$ *are disjoint.*
Then, the related additives of

$$K = \bigcup_{\theta \in \mathbf{P}} \sigma\left(A(\theta)\right) \subset \mathbb{C}$$

They are linked. It's worth mentioning that the communication is phony in general. As a result, uniform ensemble controllability appears identical for continuous-time and discrete-time systems.

Theorem 1.2 [Schönlein and Helmke, 2016]*The union of the compact intervals is* $\mathbf{P} \subset \mathbb{R}$. *Assume that* $0 \in \mathbf{P}$. *If and only if* $rank\ A = n$ *and* $rank\ B = n$*, the family* $\sum = \{(\theta A, B) | \theta \in \mathbf{P}\}$ *is uniformly ensemble controllable.*
Proof. *We focus on the case of continuous time; the case of discrete time will pass as we consider the needs of the distinct situation. Assume that* $\sum = \{(\theta A, B) | \theta \in \mathbf{P}\}$ *is uniformly ensemble controllable. Given that* $0 \in \mathbf{P}$*, the essential condition* (1) *implies that* $rank\ B = n$. *We have in particular, Assume* $rank\ A < n$ *to demonstrate the second assertion. Then,* 0 *is an eigenvalue of* A*, and we have for awesome parameter values* $\{\theta_1, ..., \theta_{n+1}\} \in \mathbf{P}$*:*

$$0 \in \sigma\left(\theta_1 A\right) \cap . . . \cap \sigma\left(\theta_{n+1} A\right)$$

Contradicting the necessary condition (2).

On expect $rank\ A = n$ *and* $rank\ B = n$*, on the other hand. We can assume* $B = I_n$ *without lack of generality. The rank condition for* B *implies the controllability condition* (1)*. The Jordanian canonical form is denoted by* Λ*. It is sufficient to recall the ensemble.*

$$\frac{\partial}{\partial t} y(t,\theta) = \theta \Lambda y(t,\theta) + I u(t). \tag{1.9}$$

For, we will focus on the two-dimensional Jordan block; however, the higher-dimensional situation is supported by the induction argument. Let

$$\frac{\partial}{\partial t} z(t,\theta) = \begin{pmatrix} \theta\lambda & \theta \\ 0 & \theta\lambda \end{pmatrix} z(t,\theta) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u(t). \tag{1.10}$$

The solution to (1.10) *is given by*

$$\varphi(T,\theta,u) = \int_0^T \begin{pmatrix} e^{\theta\lambda(T-s)} u_1(s) + \theta\lambda(T-s) e^{\theta\lambda(T-s)} u_2(s) \\ e^{\theta\lambda(T-s)} u_2(s) \end{pmatrix} ds$$

Given $z^* = col(z_1^* z_2^*) \in C([a,b], \mathbb{R}^2)$ *and* $\varepsilon > 0$.

There may be an enter function $u_2 : [0,T] \to \mathbb{R}$ *so that* $|z_2^*(\theta) - \varphi_2(T,\theta,u_2)| < \varepsilon$ *for all* $\theta \in [a,b]$*. Let*

$$w^*(\theta) := z_1^*(\theta) - \int_0^T \theta(T-s) e^{\theta\lambda(T-s)} u_2(s) ds \in C(\mathbf{P}, \mathbb{R})$$

Following the equal reasoning there's an input $u_1 : [0,T] \to \mathbb{R}$ *so that*

$$|w^*(\theta) - \int_0^T e^{\theta\lambda(T-s)} u_1(s) ds| < \varepsilon$$

Consequently, we have

$$\sup_{\theta \in \mathbf{P}} \parallel z^*(\theta) - \varphi(T,\theta,u) \parallel < \varepsilon$$

And we are done. ■

1.1.4 Uniform Null Ensemble Controllability

We continue to introduce a definition of the controllability of a uniform empty set using the parameter theta in $[0,1]$.

Definition 1.4 [Adu,D, 2017] *Let* $\sum_C(\mathbf{P}, A, B)$ *be an ensemble of continuous-time single-enter systems. Then,* $\sum_C(\mathbf{P}, A, B)$ *is uniformly null ensemble controllable if and only if there exists an open* *set $V \subset \mathbb{R}^n$ containing the origin, a finite time $T > 0$, and a control signal $u \in L^1([0,T];[0,1])$ such that, for all $y(0,\theta) \neq 0 \in V$, u steer $y(0,\theta)$ to $y(T,\theta) \in V$, where*

$$\sup_{\theta \in \mathbf{P}} \|y(0,\theta)\| < \varepsilon.$$

The equal definition holds for the discrete-time single-input scenario.

Theorem 1.3 [*Adu,D*, 2017, *Theorem* 5.1.2] *An ensemble $\sum_C(\mathbf{P}, A, B)$ of continuous-time single-enter systems*
is uniformly null ensemble controllable if the subsequent condition holds:
1. *The eigenvalues of $A(\theta)$ has a nonzero imaginary part, for all $\theta \in \mathbf{P}$.*
2. *The pair $(A(\theta), B(\theta))$ is null controllable, for all $\theta \in \mathbf{P}$.*
3. *$\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset$, for any pair of wonderful parameter θ, $\theta' \in \mathbf{P}$.*
4. *The eigenvalues of $A(\theta)$ have algebraic multiplicity of one, for every θ.*

Proof. See [Adu,D, 2017, page 44] . ■

1.2 Ensemble controllability of discrete-time

In discrete-time, we consider a family of control systems of the form

$$y(t+1;\theta) = A(\theta)y(t;\theta) + B(\theta)u(t). \tag{1.11}$$

in which $A(\theta) \in \mathbb{R}^{n\times n}$, $B(\theta) \in \mathbb{R}^{n\times m}$ and $u(t) \in \mathbb{R}^m$ with $\theta \in [a,b] \subset \mathbb{R}$.
We will regularly discover an ensemble of control systems given by (1.9) with$\sum_D([a,b]; A;\ B)$,Given preliminary states $y(0;\theta)$, for all $\theta \in [a,b]$ and a finite time $T > 0$, the use of the variant of the constant formula, the overall answer for (1.11) is given by,

$$y(T;\theta) = A^T(\theta)y\left(0;\theta\right) + \sum_{\kappa=1}^{T-1} A^{\kappa}\left(\theta\right)B\left(\theta\right)u\left(T-1-\kappa\right)\ . \tag{1.12}$$

Following that, we present both the necessary and sufficient requirements for the uniform ensemble controllability of linear systems (1.6). These are the same criteria that apply to discrete-time systems. Let

$$\left(zI - A\left(\theta\right)\right)^{-1} B\left(\theta\right) = N_\theta\left(z\right) D_\theta\left(z\right)^{-1}$$

Be a right coprime factorization by a square polynomial matrix $N_\theta\left(z\right) \in \mathbb{R}^{n\times m}\left[z\right]$ and a non-singular polynomial matrix $D_\theta\left(z\right) \in \mathbb{R}^{m\times m}[z]$. We first notion the important conditions for uniform ensemble controllability.

Proposition 1.2 [*Fuhrmann and Helmke*, 2015] ***(Necessary Conditions)****. Let* $\mathbf{P}$ *be a subset of* $\mathbb{R}^d$ *such that the indoors factors of* $\mathbf{P}$ *are dense in* $\mathbf{P}$*. Assume that the family of linear systems* $(A(\theta), B(\theta))_{\theta\in\mathbf{P}}$ *is uniformly ensemble controllable. Then, these properties are satisfied:*
1. *For each* $\theta \in \mathbf{P}$ *the system* $(A(\theta), B(\theta))$ *is controllable.*
2. *For finitely many parameters* θ_1*,. . .,*$\theta_s \in \mathbf{P}$*, the* $m \times m$ *polynomial matrices* $D_{\theta 1}(z), ..., D_{\theta s}(z)$ *are mutually left coprime.*
3. *For* $m + 1$ *distinct parameters* $\theta_1, . . .,\theta_{m+1} \in \mathbf{P}$ *the spectra of* $A(\theta)$ *satisfy*

$$\sigma(A(\theta_1)) \cap \cdots \cap \sigma(A(\theta_{m+1})) = \emptyset.$$

4. *Assume* $m = 1$*. The dimension of* $\mathbf{P}$ *satisfies* $\dim \mathbf{P} \leq 2$*. If* $A(\theta)$ *has a simple real eigenvalue for some* $\theta \in \mathbf{P}$*, then dim* $\mathbf{P} \leq 1$*.*

1.2.1 Characterizations of ensemble controllability for discrete time systems

The uniform ensemble controllability requirement can be presented in a more comprehensible format. We will concentrate on the discrete-time situation with a single input for the sake of simplicity. In the continuous-time situation, corresponding characterizations are more complex and are not required for the subsequent analysis. Uniform ensemble controllability is defined by the following result.

Proposition 1.3 [*Helmke and Schönlein*, 2014] *A family* $\{(A(\theta), b(\theta)), \theta \in [a, b]$ *of discrete-time systems is uniformly ensemble controllable if and only if for all* $\varepsilon > 0$ *and all continuous functions* $y_d : [a, b] \to \mathbb{R}^n$ *there is a real scalar polynomial*
$[a, b] \in \mathbb{R}[z]$ *such that*

$$\underset{\theta\in[a,b]}{sup} \parallel p(A(\theta))b(\theta) - y_d \parallel < \varepsilon. \tag{1.13}$$

Proof. Recall that given inputs $u(0), ..., u(T - 1)$ the solution is given by

$$\begin{aligned} y(T, \theta) &= \sum_{k=0}^{T-1} A(\theta)^k b(\theta) u(T - 1 - k) \\ &= \left(\sum_{k=0}^{T-1} u(T - 1 - k) A(\theta)^k \right) b(\theta) \\ &= p(A(\theta))b(\theta). \end{aligned}$$

Where $p(z) = \sum_{k=0}^{T-1} u_{T-1-k} z^k$ is a parameter independent polynomial.
Suppose that $(A(\theta), b(\theta))$ is controllable for all $\theta \in [a, b]$. Then, by the controller canonical form, there exists a continuous family of invertible state-space transformations $S(\theta) = R(A(\theta), b(\theta))^{-1}$such that

$$\left(\tilde{A}(\theta), e_1\right) = \left(S(\theta)A(\theta)S(\theta)^{-1}, S(\theta)b(\theta)\right).$$

Is in (tall) control canonical form, where $R(A(\theta), b(\theta))$ denotes the $n \times n$ controllability matrix and $\tilde{A}(\theta)$ denotes the tall companion matrix of the characteristic polynomial $q_\theta(z) = det(zI - A(\theta))$ and e_1 is the first standard basis vector of $\mathbb{R}^n$. Given any continuous $y_d : [a, b] \to \mathbb{R}^n$ we consider the real polynomial u_θ in z defined by

$$u_\theta(z) := (1z...z^{n-1})R\left(A(\theta), b(\theta)\right)^{-1} y_d. \tag{1.14}$$

■

Proposition 1.4 [Helmke and Schönlein, 2014] *Assume that the discrete-time system $(A(\theta), b(\theta))$ is controllable for any*
$\theta \in [a, b]$. Then, the following are equivalent.
(1) *$(A(\theta), b(\theta))_\theta$ is uniformly ensemble controllable.*
(2) *For any continuous function $y_d \in C([a, b], \mathbb{R}^n)$ and any $\varepsilon > 0$ there exists a polynomial $p \in \mathbb{R}[z]$,*

$$\| \left(p - u_\theta\right)\left(A(\theta)\right) b(\theta) \| < \varepsilon.$$

For all $\theta \in [a, b]$.
(3) *For any continuous function $y_d \in C([a, b], \mathbb{R}^n)$ and any $\varepsilon > 0$ there exists a polynomial $p \in \mathbb{R}[z]$*

$$\| p\left(A(\theta)\right) - u_\theta\left(A(\theta)\right) \| < \varepsilon.$$

For all $\theta \in [a, b]$.
Assume, that for each $\theta \in [a, b]$, the eigenvalues of $A(\theta)$ are distinct. Let
$\{C := (z, \theta) \in \mathbb{C} \times [a, b], det\left(zI - A(\theta)\right) = 0\}$.
Each above conditions is equivalent to:
(4) *For any continuous function $y_d \in C([a, b], \mathbb{R}^n)$ and any $\varepsilon > 0$ there is a polynomial $p \in \mathbb{R}[z]$ with*
$|p(z) - u_\theta(z)| < \varepsilon, \forall (z, \theta) \in C$.

Proof. See [Helmke and Schönlein, 2014, page 72]. ■

1.3 Comparison between averaged controllability and ensemble controllability

The control function's goal is to guide the system to a state that satisfies a set of properties specified either at $T > 0$ or during a certain time interval. These dwellings can be split according to parameter values and may refer to a single system **(e.g.)** or solutions corresponding to the entire parameter range **(e.g. ensemble control, averaged control)**. Controls u is designed as a parameter invariant in the latter instance, suggesting that an equal control is to be applied to the system (1.1) regardless of the specific attention of the parameter θ, whereas controls u_θ range with θ in the first case.

The goal of the first concept **(averaged controllability)** is to persuade the system's expectancy to the target, whereas the goal of the second concept **(ensemble controllability)** is to persuade each system's attention to an arbitrarily small ball across the target. Of course, for the averaged controllability concept to work, the parameter attention must adhere to a few potential laws. Ensemble controllability is a more powerful concept, as evidenced by the fact that ensemble controllability implies controllability.
The difference between them is mathematically as follows:
Averaged controllability:

$$\int_0^1 y\,(T,\theta)\,d\theta = y_d.$$

Ensemble controllability:

$$\left(\int_a^b \mid y(T;\theta) - y_d \mid^p d\theta\right)^{\frac{1}{p}} = 0.$$

Chapter 2

Ensemble controllability for infinite dimensional systems

2.1 Statement of problem

Let $\mathcal{A} : Y \to Y$ and $\mathcal{B} : U \to Y$ be bound linear operators on Banach spaces Y and U, respectively. A linear system

$$y'(t) = \mathcal{A}y(t) + \mathcal{B}u(t) \tag{2.1}$$

If the controllable set of 0 is dense in Y, it is said to be approximately controllable.
The mathematical relationship between ensemble controllability and approximation controllability is simple to explain. Allow Y to indicate the Banach space of $\mathbb{R}^n$-valued continuous functions at the compact parameter space $\mathbf{P}$, equipped with a supremum norm, explicitly for uniform ensemble management. Similarly, choose $Y = L^q(\mathbf{P}, \mathbb{R}^n)$ for L^q-ensemble controllability A continuous family of linear systems $(A(\theta), B(\theta))$ on a Banach space Y with a finite-dimensional space of control values $U = \mathbb{R}^m$ defines a linear system of the type (2.1) in both cases. Here

$$\mathcal{A} : Y \to Y, (\mathcal{A}x)(\theta) := A(\theta)x(\theta) \tag{2.2}$$

Is the bounded linear multiplication operator, whereas the input operator is denoted by.

$$\mathcal{B} : \mathbb{R}^m \to Y, (\mathcal{B}u)(\theta) := B(\theta)u \tag{2.3}$$

Is described by an m-tuple of Banach-space elements, i.e. by the columns $B(.)$.

2.2 L^2-Ensemble Controllability

A few mathematical concepts must be remembered. Remember that the space $L^2\left([a,b];\mathbb{C}^k\right), a,b\in\mathbb{R}$. The inner product of , $k\in\mathbb{N}$ is defined by

$$\langle f,g\rangle=\int\limits_a^b f^{\dagger}(t)\,g(t)\,dt,$$

For all $f,g\in L^2\left([a,b];\mathbb{C}^k\right)$, where the conjugate transpose is denoted by $\dagger$. Let $H_1=L^2\left([0,T];\mathbb{C}^m\right)$ and $H_2=L^2\left(P;\mathbb{C}^m\right)$,We define an operator $L:H_1\rightarrow H_2$ by

$$(Lu)(\theta)=\int\limits_0^T e^{A(\theta)T}B(\tau,\theta)\,u(\tau)\,d\tau, \tag{2.4}$$

We can deduce from (1.5) and (2.4) that

$$(Lu)(\theta)=F(\theta), \tag{2.5}$$

For all $\theta\in P.$, Ensemble controllability is now equivalent to solving the operator equation with this new formulation (2.5). To put it another way, we want to find $u\in H_1$ that solves

$$Lu=F. \tag{2.6}$$

It is shown in $[Li, J.S.(2010).]$ that, the operator L described in (2.4) is proved to be bound and compact .
For completeness, we offer a demonstration of this fact in the appendix see ([Adu,D, 2017]Theorem $7.1.2$ and Proposition $7.1.3$). As a result, L is a compact bound linear operator. Under these circumstances, the fact that L has an adjoint operator L^* is well-known in [Kreyszig,E, 1991].
Which is a additionally a bound compact linear operator such that, for all $f\in H_2$ and $u\in H_1$, L^* fulfill the relationship

$$\langle f,Lu\rangle_{H_2}=\langle L^*f,u\rangle_{H_1}, \tag{2.7}$$

Inner products defined in the spaces H_1 and H_2 are $\langle .,.\rangle_{H_1}$ and $\langle .,.\rangle_{H_1}$, respectively. From (2.7) we can see that L^* is provided for every $f\in H_2$:

$$(L^*f)(t)=\int\limits_P B^{\dagger}(\tau,\theta)\,S^{\dagger}(0,\tau,\theta)\,f(\theta)\,d\theta. \tag{2.8}$$

The operator equation (2.6) now has no unique solution since compact operators are not invertible (see Proposition 7.1.5). We give a result and direct the reader to [Luenberger,D.G, 1997] for the evidence.

Theorem 2.1 [*John Wiley and Sons,* 1997, *Theorem* 6.10] *Let H_1 and H_2 indicate Hilbert spaces, and $L \in B(H_1, H_2)$ denote the range space of L, which is closed in H_2. The vector of least norm u satisfying $Lu = F$ is then provided by $u = L^* z$ for $F \in R(L)$, where z is any solution of $LL^* z = F$.*

Using (2.4) and (2.8), it is possible to demonstrate that the operator $LL^* : H_2 \to H_2$ has the form

$$(LL^* z)(\theta) = \int\limits_P \int\limits_0^T S(0, \tau, \theta) B(\tau, \theta) B^{\dagger}(\tau, \theta') S^{\dagger}(0, \tau, \theta') z(\theta') \, d\tau d\theta'. \tag{2.9}$$

We provide the following definition before starting and proving the main results.

Definition 2.1 [*Li,J.S.*(2010)] *Let H_1 and H_2 represent Hilbert spaces, and $L : H_1 \to H_2$ represent the compact operator. If (λ_j^2, ψ_j) is an eigen system of LL^* and (λ_j^2, ϕ_j) is an eigen system of LL^*, then, the two systems are connected by the equations $LL^* \psi_j = \lambda_j^2 \psi_j$, $\psi_j \in H_2$ and $LL_j^* \phi_j = \lambda_j^2 \phi_j$, $\phi_j \in H_1$, where $\lambda_j > 0$:*

$$L\phi_j = \lambda_j \psi_j \ \ and \ \ L^* \psi_j = \lambda_j \phi_j. \tag{2.10}$$

The triple $(\lambda_j, \ \phi_j, \psi_j)$ is referred to as a singular L system. Now we'll declare and prove the chapter's key conclusion.

Theorem 2.2 [*Li,J.S.*(2010)] *An ensemble $\sum_C (P, A, B)$ is L^2-ensemble controllable in $L^2(P; \mathbb{R}^n)$, If and only if, for any given initial and intended state x_0 and $x_d \in L^2(P; \mathbb{R}^n)$ and for $F(\theta) = S(0, T; \theta) x_d(\theta) - x_0(\theta)$, the condisions*

1. $\sum_{j=1}^{\infty} \frac{|\langle F, \psi_j \rangle|^2}{\lambda_j^2} < \infty$
2. *$F \in \overline{R(L)}$ hold, where $\overline{R(L)}$ signifies the range space closure for L.*

Additionally, the control law

$$u = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \langle \varphi, \psi_j \rangle \phi_j, \tag{2.11}$$

Satisfy

$$\langle u, u \rangle \leq \langle u_0, u_0 \rangle ,$$

For all $u_0 \in \mathrm{F}$ and $u \neq u_0$, where ,With conditions 1 and 2 of Theorem 3.1.3 met, $\mathrm{F} = \{u \in L_2[0, T]; \mathbb{R}^m\} \, |Lu =$ F

Furthermore, for a given $\epsilon > 0$,

$$u_r = \sum_{j=1}^{r} \frac{\langle F, \psi_j \rangle \phi_j}{\lambda_j},$$

In a way that

$$\| F - Lu_m \| < \epsilon, \tag{2.12}$$

For any $m \geq r$, where $r \in \mathbb{N}$ and depends on ϵ

$$u_m = \sum_{j=1}^{m} \frac{\langle F, \psi_j \rangle \phi_j}{\lambda_j}. \tag{2.13}$$

Proof. We begin by demonstrating the requirement. Assume that there is $u \in H_1$ that satisfies (2.3). Then,

$$\langle F, \psi_j \rangle = \langle Lu, \psi_j \rangle, \tag{2.14}$$

It denotes

$$\frac{1}{\lambda_j} \langle F, \psi_j \rangle = \langle u, \phi_j \rangle. \tag{2.15}$$

The sequences $\{\phi_j\}_{j\geq 1} \subset H_1$ and $\{\psi_j\}_{j\geq 1} \subset H_2$ are orthonormal because LL^* is a self-adjoint compact operator (see $[5$,page$.248])$. We have it using Bessel's inequality. we have that,

$$\sum_{j=1}^{\infty} \frac{| \langle F, \psi_j \rangle |^2}{\lambda_j^2} \leq \| u \|_2^2 < \infty.$$

The proof of the first statement is now complete. We also have that $\alpha \in H_2$ for every $\alpha \in N(L^*)$ such that

$$L^* \alpha = 0.$$

Following this logic,

$$\langle F, \alpha \rangle = \langle Lu, \alpha \rangle = \langle u, L^* \alpha \rangle = 0.$$

Hence,

$$F \in N(L^*)^{\perp} = \overline{R(L^*)}.$$

The second statement's proof is now complete. Assume, on the other hand, that the first and second requirements are met. Thus, let

$$\beta_j = \frac{\langle F, \psi_j \rangle}{\lambda_j}. \tag{2.16}$$

We can see from the first condition that

$$\sum_{j=1}^{\infty} | \beta_j |^2 < \infty. \tag{2.17}$$

According to (Proposition [1] 7.1.6), $u \in H_1$ exists.

$$u = \sum_{j=1}^{\infty} \beta_j \phi_j. \tag{2.18}$$

$\{\phi_j\}_{j\geq 1}$ and $\{\psi_j\}_{j\geq 1}$ have been proved to be orthonormal bases for $\overline{R(L^*)}$ and $\overline{R(L)}$, respectively, in [8], and since $u \in \overline{R(L^*)} \subset H_1$, we have that

$$u = \sum_{j=1}^{\infty} \langle u, \phi_j \rangle \phi_j. \tag{2.19}$$

Because $\{\phi_j\}_{j\geq 1}$ is an orthonormal basis, its coefficients are one-of-a-kind. Becauset of (2.18) and (2.19), we may deduce that

$$\langle u, \phi_j \rangle = \frac{\langle F, \psi_j \rangle}{\lambda_j}.$$

We assert that $u \in H_1$ in (2.19), for example, is not in $N(L)$. A contradiction argument is used to demonstrate this. If $u \in N(L)$ is true, then $Lu = 0$. Now, using L's linearity and continuity, we get

$$Lu = \sum_{j=1}^{\infty} \beta_j \left(L\phi_j\right) = \sum_{j=1}^{\infty} \langle F, \psi_j \rangle \psi_j = 0. \tag{2.20}$$

Given that $\{\psi_j\}_{j\geq 1}$ is an orthonormal basis, $\langle F, \psi_j \rangle = 0$ for $j \in \{1, 2, \ldots\}$. Concludes that $F = 0$, which is a contradiction. As a result, the assumption $u \in N(L)$ is incorrect.
We can show that the right-hand side of equation (2.20) is true since $F \in \overline{R(L)}$ and $\{\psi_j\}_{j\geq 1}$ are orthonormal basis in $\overline{R(L)}$.

$$\sum_{j=1}^{\infty} \langle F, \psi_j \rangle \psi_j = F.$$

As a result, the operator equation is solved by u in (2.19).(2.6). Let us also consider

$$u_N = \sum_{j=1}^{N} \frac{\langle F, \psi_j \rangle \phi_j}{\lambda_j}. \tag{2.21}$$

Where $N \in \mathbb{N}$. We get the following result by using the knowledge that $\{\phi_j\}_{j\geq 1}$ is an orthonormal sequence.

$$\| u - u_N \|_2^2 = \sum_{j=N+1}^{\infty} \frac{1}{\lambda_j^2} | \langle \varphi, \psi_j \rangle |^2 \to 0 \quad as \quad N \to \infty. \tag{2.22}$$

This suggests that

$$\| F - Lu_N \|_2^2 = \sum_{j=N+1}^{\infty} \lambda_j^2 | \langle u, \phi_j \rangle |^2 \to 0 \quad as \quad N \to \infty. \tag{2.23}$$

The proof is now complete. ■

2.3 Optimal control of an Ensemble of Harmonic Oscillators

To demonstrate the ensemble controller's construction, we use an example from [Li,J.S, 2010]. An ensemble of harmonic oscillators is subjected to a xed endpoint optimum control problem. Consider :

$$\frac{\partial}{\partial t}y(t;\theta) = A(\theta)y(t;\theta) + B(\theta)u(t), \tag{2.24}$$

Where $\theta \in \mathbf{P} \subset \mathbb{R}$,$y(t;\theta) = (y_1(t;\theta), y_2(t;\theta))^T \in \mathbb{R}^2$, $u(t) = (u_1(t), u_2(t))^T \in \mathbb{R}^2$ and $u_i \in L^2([0,T];\mathbb{R})$ for $i \in \{1,2\}$,

$$A(\theta) = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \quad and \quad B(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We want to find $u \in L^2([0;T];\mathbb{R}^2)$ that steers the trajectories of (2.24) from $y(0,\theta)$ to $y(T,\theta) \in \mathbb{R}^2$ in the sense of L^2-ensemble controllability, so that u minimizes the cost functional

$$\min_{u \in L^2([0;T];\mathbb{R}^2)} J(u) = \int_0^T \shortparallel u(t) \shortparallel^2 dt.$$

Let's take advantage of the fact that $\mathbb{R}^2$ is isomorphic to $\mathbb{C}$.

$$\begin{aligned} y(t,\theta) &= y_1(t,\theta) + iy_2(t,\theta), \\ u(t) &= u_1(t) + iu_2(t). \end{aligned}$$

As a result, (2.24) can be written as.

$$\frac{\partial y}{\partial t}(t,\theta) = i\theta y(t,\theta) + u(t).$$

We get the following from the fluctuation of constants formula:

$$y(t,\theta) = e^{i\theta}y(0,\theta) + \int_0^T e^{i\theta(t-s)}u(s)\,ds.$$

Consequently,

$$F(\theta) = \int_0^T e^{-i\theta T}u(s)\,ds, \tag{2.25}$$

Where,

$$F(\theta) = e^{-i\theta T}y(T,\theta) - y(0,\theta).$$

$H_1 = L^2([0,T];\mathbb{C})$ and $H_2 = L^2(\mathbf{P};\mathbb{C})$. We create the $L : H_1 \to H_2$ an operator by

$$(Lu)(\theta) = \int_0^T e^{-i\theta T} u(s)\, ds. \tag{2.26}$$

From (2.25) and (2.26) We get.

$$(Lu)(\theta) = F(\theta), \tag{3.27}$$

For both $\theta \in \mathbf{P}$ Because $u \in H_1$ and the kernel $k(t,\theta) = e^{-i\theta T}$ are both bounded, the operator L defined in (2.26) must be a bounded compact linear operator with an adjoint. It's worth noting that we have $f \in H_2$ for all of them.

$$\langle f, Lu\rangle_{H_2} = \int_0^T \int_{\theta^-}^{\theta^+} e^{-i\theta s} f(\theta)^{\dagger}\, d\theta u(s)\, ds. \tag{2.28}$$

As a result, the adjoint operator is satisfied.

$$(L^* f)(s) = \int_{\theta^-}^{\theta^+} e^{i\theta s} f(\theta)\, d\theta. \tag{2.29}$$

We know from Theorem 2.1 that

$$L^* z = u,$$

When z is true

$$LL^* z = F.$$

The operator $LL^* : H_2 \to H_2$ is obtained by substituting (2.29) into (2.26). is of the form

$$(LL^* z)(\theta_1) = \int_0^T \int_{\theta^-}^{\theta^+} e^{i(\theta' - \theta_1)s} z(\theta')\, d\theta' ds. \tag{2.30}$$

We obtain using Fubini's Theorem.

$$(LL^* z)(\theta_1) = \int_{\theta^-}^{\theta^+} \left(\int_0^T e^{i(\theta' - \theta_1)s} ds \right) z(\theta')\, d\theta'. \tag{2.31}$$

We have through straight calculation

$$(LL^* z)(\theta_1) = \int_{\theta^-}^{\theta^+} \left(\int_0^T \frac{e^{i(\theta' - \theta_1)T} - 1}{i(\theta' - \theta_1)} ds \right) z(\theta')\, d\theta'. \tag{2.32}$$

We have the following:

$$
\begin{aligned}
\frac{e^{i(\theta'-\theta_1)T}-1}{i(\theta'-\theta_1)} &= \frac{\cos\left((\theta'-\theta_1)T\right)-1+i\sin\left((\theta'-\theta_1)T\right)}{i(\theta'-\theta_1)}, \\
&= \frac{\left(\cos^2\left((\theta'-\theta_1)\frac{T}{2}\right)-1\right)-\sin^2\left((\theta'-\theta_1)\frac{T}{2}\right)+i\left(2\sin\left((\theta'-\theta_1)\frac{T}{2}\right)-1\right)\cos\left((\theta'-\theta_1)\frac{T}{2}\right)}{i(\theta'-\theta_1)}, \\
&= \frac{-2\sin^2\left((\theta'-\theta_1)\frac{T}{2}\right)+i\left(2\sin(\theta'-\theta_1)\frac{T}{2}\cos\left((\theta'-\theta_1)\frac{T}{2}\right)\right)}{i(\theta'-\theta_1)}, \\
&= \frac{2\pi\sin\left((\theta'-\theta_1)\frac{T}{2}\right)}{\pi(\theta'-\theta_1)}\left(\cos\left((\theta'-\theta_1)\frac{T}{2}\right)+i\sin\left(\sin(\theta'-\theta_1)\frac{T}{2}\right)\right), \\
&= 2\pi e^{i(\theta'-\theta_1)\frac{T}{2}}\left(\frac{\sin\left((\theta'-\theta_1)\frac{T}{2}\right)}{\pi(\theta'-\theta_1)}\right).
\end{aligned}
$$

Let $\omega' = \frac{\theta'}{\theta}$,$\omega = \frac{\theta_1}{\theta}$ and $\alpha = \frac{T\theta}{2}$ be true, then $\omega', \omega \in [-1, 1]$. Equation (2.32) can be rewritten using this observation as

$$
(LL^* z)(\omega) = \int_{-1}^{1} 2\pi e^{i(\omega'-\omega)\alpha}\left(\frac{\sin\left((\omega'-\omega)\alpha\right)}{\pi(\omega'-\omega)}\right) z(\omega')\, d\omega'. \tag{2.33}
$$

We look at the equation.

$$
\int_{-1}^{1}\left(\frac{\sin\left((\omega'-\omega)\alpha\right)}{\pi(\omega'-\omega)}\right)\beta_j(\omega',\alpha)\, d\omega' = v_j(\alpha)\,\beta_j(\omega,\alpha), \tag{2,34}
$$

where $\beta_j(\omega,\alpha)$ is the appropriate eigenvalue of a well-known prolate spheroidal wave function [Percival,D.B and Walden,A.T,1993],[Flammer,C,2014],[Slepian,D and Pollak,H.O,1961],[Landau,H.J.and Po and [Landau,H.J.and Pollak,H.O, 1962], and v_j is the jth eigenfunction. Similarly, \consider

$$
\left(LL^*\psi_j\right)(\omega,\alpha) = \int_{-1}^{1} 2\pi e^{i(\omega'-\omega)\alpha}\left(\frac{\sin\left((\omega'-\omega)\alpha\right)}{\pi(\omega'-\omega)}\right)\psi_j(\omega',\alpha)\, d\omega' = \rho_j(\alpha)\,\psi_j(\omega,\alpha). \tag{2.35}
$$

We have that rearranging (2.35)

$$
\int_{-1}^{1} 2\pi e^{i\omega'\alpha}\left(\frac{\sin\left((\omega'-\omega)\alpha\right)}{\pi(\omega'-\omega)}\right)\psi_j(\omega',\alpha)\, d\omega' = \frac{1}{2\pi}e^{i\omega\alpha}\rho_j(\alpha)\,\psi_j(\omega,\alpha). \tag{2.36}
$$

Let

$$
e^{i\omega'\alpha}\psi_j(\omega',\alpha) = \beta_j(\omega',\alpha). \tag{2.37}
$$

Then,

$$
v_j(\alpha)\,\beta_j(\omega,\alpha) = \frac{1}{2\pi}e^{i\omega\alpha}\rho_j(\alpha)\,\psi_j(\omega,\alpha). \tag{2.38}
$$

We get (2.38) when we evaluate it at ω'.

$$v_j(\alpha)\,\beta_j(\omega',\alpha) = \frac{1}{2\pi} e^{i\omega'\alpha} \rho_j(\alpha)\,\psi_j(\omega',\alpha)\,. \tag{2.39}$$

We can find the answer by comparing equations (2.37) and (2.39).

$$\rho_j = 2\pi v_{j.} \tag{2.40}$$

The eigenvectors and eigenvalues of the operator LL^* can thus be expressed in terms of v_j and $\beta_j$from (2.37) and (2.40), respectively. The fact that β_j's are orthogonal and complete on $L^2[-1,1]$ is well known (see, for example, [Percival,D.B and Walden,A.T.(1993)]). Now, let

$$z = \sum_{j=1}^{\infty} \frac{1}{\rho_j} \left\langle F, \tilde{\psi}_j \right\rangle \tilde{\psi}_j, \tag{2.41}$$

Where

$$\tilde{\psi}_j = e^{-i\omega\alpha} \frac{\beta_j}{\shortparallel \beta_j \shortparallel}. \tag{2.42}$$

Then there's that.

$$LL^* z = \sum_{j=1}^{\infty} \left\langle F, \tilde{\psi}_j \right\rangle \tilde{\psi}_j = F,$$

Theorem 2.1 is applied to LL^* with respect to the orthonormal basis $\left\{\tilde{\psi}_j\right\}_{j\geq 1}$ in $\overline{R(L)}$. We may clearly notice this.

$$u = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \left\langle F, \tilde{\psi}_j \right\rangle \tilde{\phi}_j, \tag{2.43}$$

Where

$$\lambda_j = \sqrt{\rho_j}. \tag{2.44}$$

The control signal can also be expressed solely in terms of $\tilde{\psi}_j$, with $\tilde{\phi}_j$ being produced using the same reasoning and the operator LL^*. It's also possible to write the control signal as

$$u(t) = \int_{-\theta}^{\theta} e^{i\tilde{\theta}t} \sum_{j=1}^{\infty} \frac{1}{\rho_j} \left\langle F\left(\tilde{\theta}\right), \tilde{\psi}_j\left(\tilde{\theta}\right) \right\rangle \tilde{\psi}_j\left(\tilde{\theta}\right) d\tilde{\theta}. \tag{2.45}$$

Let

$$z_N = \sum_{j=1}^{N} \frac{1}{\rho_j} \left\langle F, \tilde{\psi}_j \right\rangle \tilde{\psi}_j,$$

As a result, where $N \in \mathbb{N}$

$$LL^* z_N = \sum_{j=1}^{N} \rho_j \left\langle z, \tilde{\psi}_j \right\rangle \tilde{\psi}_j,$$

We have it.

$$\| F - LL^* z_N \|_2^2 = \sum_{j=N+1}^{\infty} | \left\langle F, \tilde{\psi}_j \right\rangle |^2 . \tag{2.46}$$

As $N \to \infty$, it goes to zero. As a result, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, we have $N > n$.

$$\| F - LL^* z_N \|_2 < \epsilon. \tag{2.47}$$

Now, since $LL^* z_N$ approximates F in this way, we can deduce that

$$\begin{aligned} \| \quad & LL^* z - LL^* z_N \|_2 = \| L \left(L^* z - LL^* z_N \right) \|_2, \\ \leq \quad & \| L \|_2 \| L^* z - LL^* z_N \|_2, \\ \leq \quad & \| L \|_2^2 \| z - z_N \|_2 < \epsilon, \end{aligned}$$

Since $z_N \to z$ equals $N \to \infty$, it follows that for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $N > n$.

$$\| L^* z - L^* z_N \| < \epsilon.$$

As a result, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, we have $N > n$.

$$\| u - u_N \|_2 < \epsilon,$$

Where $u_N = L^* z$ As a result, the sequence of control inputs provides the best approximation of the control signal u that achieves a minimum norm.

$$u_N = L^* z_N.$$

Proposition 2.1 *Controllability can be described as* (2.2) *and* (2.3) *for bounded linear operators* $\mathcal{A}$ *and* $\mathcal{B}$. *If and only if system* (2.1) *on the Banach space* X *is roughly controllable, the parameter-dependent system* (1.1) *is uniformly (or or* (L^q *– ensemble controllable*).

Proof. We focus on uniform ensemble controllability and the Banach space $Y = C(\mathbf{P}, \mathbb{R}^n)$; the proof for L^q-ensemble controllability is the same as that for uniform ensemble controllability. With $y(0) = 0$, let $t \mapsto y(t) \in Y$ signify a unique solution to (2.1). The specific solution to (1.1) is $t \mapsto y(t;\theta)$, with $y(0;\theta) = 0$ for $\theta \in \mathbf{P}$. The approximation controllability of (2.1) states that there exists $T > 0$ such that $\parallel y\,(T) - y_d \parallel = \sup_{\theta \in \mathbf{P}} \parallel y\,(T;\theta) - y_d\,(\theta) \parallel$ for the continuous function $y_d : \mathbf{P} \to \mathbb{R}^n$ and $\varepsilon > 0$. However, this is just a requirement for uniform ensemble controllability. Thus, if and only if the infinite-dimensional system (2.1) is roughly controllable, the parameter-dependent system (1.1) is uniformly ensemble controllable. Similarly, replacing the Banach space Y with the Hilbert space $H = L^2(\mathbf{P}, \mathbb{R}^n)$ concludes that the L^2-ensemble controllability of (1.1) equals the approximation controllability of the infinite-dimensional system (2.1).. ■

Chapter 3

Ensemble observability

If y_0 is an unknown vector, which could be a multivariate random variable, and $\mathbf{P}_0$ is the probability distribution.
Before we begin addressing this setup mathematically, we'd like to take a closer look at it in terms of population models, and so introduce some vocabulary. You'll think of the setup as an outline of a continuum of individual systems with the same dynamics and dimension outputs as (3.2), but exclusive preliminary states, whether you're using population models or ensembles. We name (3.2) the ensemble's structural system, and $\mathbf{P}_0$ the preliminary distribution, which accounts for the population is distinctive heterogeneity of preliminary states.
The fact that the preliminary nation is a random vector implies that the output $y(t)$ at any given time is a random vector as well. $\mathbf{P}_{y(t)}$ denotes its distribution, which we'll refer to as output distribution.

3.1 Ensemble Observability for finite dimensional systems

Definition 3.1 ($\mathbf{L}^p$ ***definition***) *[Fuhrmann,P.A and Helmke,U, 2015] Assume that the matrix $A(\theta) \in \mathbb{R}^{n\times n}, C(\theta) \in \mathbb{R}^{p\times n}$ range continuously in the compact parameter domain $\mathbf{P} \subset \mathbb{R}^d$.*
consider the following parameter dependent system :

$$\begin{aligned} \frac{\partial y\,(t,\theta)}{\partial t} &= A(\theta)y(t,\theta), \\ z(t) &= \int_{\mathbf{P}} C(\theta)y(t,\theta)d\theta, \\ y\,(0) &= y(0,\cdot) \text{ is unknown in } L^2(\mathbf{P},\mathbb{R}^n). \end{aligned} \tag{3.1}$$

If $T > 0$, $z(t) = 0$ on $[0,T]$ implies $y(0,\theta) = 0$ for all $\theta \in \mathbf{P}$, this is known as an L^2-ensemble observable.

This means that the L^2-initial state $y(0,\cdot)$ of (3.1) can be reconstructed from the average values, according to definition.

$$\int_{\mathbf{P}} C(\theta)y(t,\theta)d\theta, \quad 0 \leq t \leq T,$$

One of the outcomes is $C(\theta)y(t,\theta)$. As a result, ensemble observability is a very durable quality that is particularly valuable in biological parameter identification tasks, where the most effective and averaged form of output data is delivered on a regular basis.

3.2 Ensemble Observability for Infinite dimensional systems

The linear system is equivalent to the system (3.1).

$$\begin{aligned} y'(t) &= \mathcal{A}y(t), \quad y(0) \in L^2(\mathbf{P}, \mathbb{R}^n), \\ z(t) &= \mathcal{C}y(t) \end{aligned} \tag{3.2}$$

$X = L^2(\mathbf{P}, \mathbb{R}^n)$ in the Hilbert space Bounded linear operators $\mathcal{A} : X \to X, \mathcal{C} : X \to \mathbb{R}^p$ is defined by

$$(\mathcal{A}y)(\theta) = A(\theta)y(\theta), \quad (\mathcal{C}y)(\theta) = \int_{\mathbf{P}} C(\theta)y(\theta)d\theta,$$

Respectively. As a result, A stands for a multiplication factor, while C stands for an integration factor. The idea is similar to the preceding concept of a group note. Approximately observed (3.2), only observable if the dual system is present.

$$\frac{\partial}{\partial t}y(t,\theta) = A(\theta)^\top y(t,\theta) + C(\theta)^\top u(t), y(0,\theta) = 0, \tag{3.3}$$

As a result, when $p = 1$, Theorem (1.1) holds for (3.1). This establishes that every continuous one-parameter family $(A(\theta), C(\theta)), \theta \in [a,b]$, of single-output linear systems is L^2-ensemble observable if the following three conditions are met: 1.$(A(\theta), C(\theta))$ can be observed for all $[a,b]$.
2. $A(\cdot)$ spectra are pairwise disjoint, that is,

$$\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset, \forall \theta, \theta' \in [a,b], \theta = \theta'.$$

3. The eigenvalues of $A(\theta)$ have an algebraic multiplicity of one for each $\theta \in [a,b]$.

The ensemble observability problem is similar to the classical observability problem in that it seeks to reconstruct the beginning distribution $\mathbf{P}_0$ from the evolution of the output distribution. $\mathbf{P}_{y(t)}$. A linear system to which there is a unique solution. Using the time-evolution of the data, create an initial distribution. Ensemble observable refers to the output distribution.

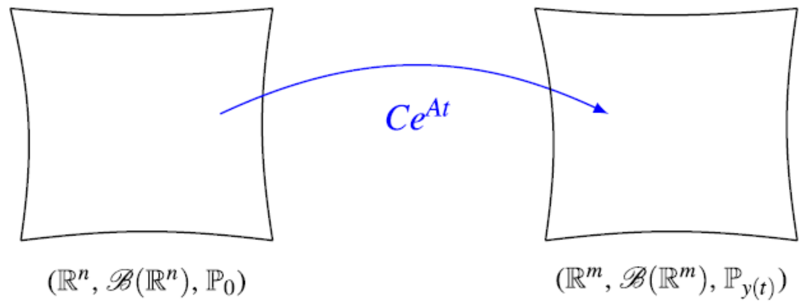

Figure3 : the connection between $\mathbf{P}_0$ and $\mathbf{P}_{y(t)}$.

The push for ward measure of $\mathbf{P}_0$ under the mapping Ce^{At} is the distribution $\mathbf{P}_{y(t)}$.

Definition 3.2 [*Zeng,S et al.* 2015] ***(Ensemble Observability of Linear Systems)****: It is considered an ensemble observable if a linear system is ensemble observable for a specified class of continuous probability distributions.*

$$(\forall t \geq 0 \mathbf{P}_{y(t)}|P_0' = \mathbf{P}_{y(t)}|P_0'') \Rightarrow P_0' = P_0''. \tag{3.4}$$

for all initial distributions P_0' and P_0'' in that class We limit ourselves to certain kinds of continuous probability distributions since achieving a general solution is unachievable. As the story progresses, this will become evident.

3.3 Observability of Structural System is Necessary

In this paragraph, we present a first theoretical discovery about the structural system's observability being required for the ensemble observability problem to accept a unique solution.

Theorem 3.1 [*Zeng,S et al.* 2015] ***(Necessary Condition)****: The observability of (A, C) is a needed condition for ensemble observability for the class of continuous starting distributions.*

Proof. Under the premise that (A, C) is unobservable, we prove that there are starting densities $P_0' \neq P_0''$ for which

$$\int_{(Ce^{At})^{-1}(B_y)} P_0' dx = \int_{(Ce^{At})^{-1}(B_y)} P_0'' dx. \tag{3.5}$$

$B_y \in \mathcal{B}(\mathbb{R}^n)$ for all $t \geq 0$ In other words, we create two initial densities, P_0' and P_0'', that are identical to the output distributions. We fixed an arbitrary probability density function P_0' to accomplish this. The intersection isn't visible because (A, C) isn't visible.

$$\bigcap_{t\geq 0} \ker Ce^{At} = \left\{x_0 \in \mathbb{R}^n : Ce^{At}x_0 \equiv 0\right\}. \tag{3.6}$$

It's not easy to figure out which subspace is unobservable. As a result, we can select a non-zero vector v from this unobservable subspace and define a second probability density function by

$$P_0''(x) := P_0'(x+v).$$

These two densities are clearly distinct. additionally, we have

$$\begin{aligned}\int_{(Ce^{At})^{-1}(B_y)} P_0''(x)dx &= \int_{(Ce^{At})^{-1}(B_y)} P_0'(x+v)dx. \\ &= \int_{v+(Ce^{At})^{-1}(B_y)} P_0'(x)dx.\end{aligned}$$

$B_y \in \mathcal{B}(\mathbb{R}^n)$ for every $t \geq 0$. Finally, we notice that

$$v + (Ce^{At})^{-1}(B_y) = (Ce^{At})^{-1}(B_y).$$

since $v \in \ker Ce^{At}$ for all $t \geq 0$ As a result, the claim is made. ■

3.4 Sufficient Conditions for Ensemble Observability

Theorem 3.2 [*Zeng,S et al.* 2015] *If $\theta \to \varphi_{X_0}(\theta v)$ is real analytic for every non-zero $v \in \mathbb{R}^n$, a linear system (A, C) is ensemble observable for the class of starting distributions .*

$$\bigcup_{t\geq 0} \left(\ker Ce^{At}\right)^{\perp} = \bigcup_{t\geq 0} \operatorname{Im}\left(Ce^{At}\right)^{\perp}. \tag{3.7}$$

There isn't a suitable algebraic subvariety of $\mathbb{R}^n$ that contains it. As a result, a sufficient requirement is that the directions given by $t \to Ce^{At}$ are rich in the sense that no appropriate algebraic variety contains the union (3.7). Remember that the zero set of a polynomial is an algebraic variety of $\mathbb{R}^n$, and that it is proper if it is not $\mathbb{R}^n$.

Proof. We show that knowing the characteristic function on (3.7) is sufficient to know the characteristic function everywhere under the analyticity requirement in φ_{X_0} and the assumption that the union (3.7) is not contained in a valid algebraic variety.
To begins, we assume two $\varphi_{X_0'}$ and $\varphi_{X_0''}$ such that their difference $h := \varphi_{X_0'} = \varphi_{X_0''}$ vanishes in the union (3.7), that is,

$$h(\xi) = 0 \text{ for all } \xi \in \bigcup_{t\geq 0} \operatorname{Im}\left(Ce^{At}\right)^{\intercal}.$$

We can write for any non-zero $\xi \in \mathbb{R}^n$ and any sufficiently small by analyticity λ.

$$h(\lambda\xi) = \sum_{p=0}^{\infty} \lambda^p a_p(\xi).$$

$a_p(\xi) = (i^p/p!)(\mathbf{E}(\langle \xi, X_0' \rangle^p) - \mathbf{E}(\langle \xi, X_0'' \rangle^p))$ is found there is a homogeneous polynomial of degree p, as shown in Section IV-A. By analyticity
, the condition $h(\lambda\xi) = 0$, for all λ in the neighborhood around the origin, is equal to the vanishing of the polynomials a_p on the union for any arbitrary $\xi \in \bigcup_{t \geq 0} \mathrm{Im}(Ce^{At})$ (3.7). As a result, the algebraic varieties described by a_p contain the union (3.7). All polynomials must be trivial, i.e., a $a_p \equiv 0$, under the premise that the union (3.7) is not contained in a suitable algebraic variety.
Because the mapping $\lambda \to h(\lambda\xi)$ is real analytic in the vicinity of any point on the real axis for any non-zero $\xi \in \mathbb{R}^n$, $\lambda \to h(\lambda\xi)$ is totally determined by its power series about the origin, which is zero. As a result, we conclude that $h \equiv 0$, i.e., $\_\varphi_{X_0'} = \varphi_{X_0''}$, and hence $X_0' = X_0''$. ■

Theorem 3.3 [*Zeng,S et al.* 2015] *The union* (3.7) *is not contained in a valid algebraic variety if* (A, C) *is observable and* $rank\ C = n - 1$.

Proof. The dimension of $(\ker Ce^{At})^{\perp}$ is also $n - 1$ with $rank\ C = n - 1$. The intersection (3.6) is trivial due to the observability of (A, C), and consequently $(\ker Ce^{At})^{\perp}$ with t0, forms an infinite family of pairwise unique hyperplanes.
More exactly, it is impossible for an observable system (A, C) to occur.

$$\forall t \geq 0 \exists i = 1, 2, \dots \ker Ce^{At} = span\{v_i\}.$$

for arbitrary nonzero countable vectors $v_i \in \mathbb{R}^n$ Since $\ker Ce^{At} = span\{v_i\}$ is identical to $Ce^{At}v_i = 0$ with $rank\ C = n - 1$, using the definition

$$T_i := \{t \geq 0 : Ce^{At}v_i = 0\}.$$

$\cup_{i=1,2,\dots} T_i = [0, \infty)$ would be required. However, due to the observability of T_i, this is impossible because the sets T_i consist of isolated points (A, C). Finally, a valid algebraic variety cannot include an infinite family of unique hyperplanes. Additionally, we'd like to call attention to the exceptional case of $n = 2$, in which the richness feature is met just by the observability of (A, C).
■

Corollary 3.1 [*Zeng,S et al.* 2015] *The union* (3.7) *is not contained in a valid algebraic variety for an observable two-dimensional system* (A, C). *One immediate concern is whether an observable system* (A, C) *currently generates "directions" rich enough that the union* (3.7) *is excluded from an algebraic subvariety. This question is answered in the negative in the following example.*

Example 3.1 [*Zeng,S et al.* 2015] *Remember from Theorem* (3.3) *and Corollary* (3.1) *that we should to explore systems with at least three degrees to identify an observable system yet for which the union* (3.7) *is contained in a valid algebraic variety. Take into account the system.*

$$x'(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} x(t).$$

$$y\,(t) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} x(t).$$

Because the diagonal entries are pairwise distinct and every entry in the output matrix is non-zero, this is easily seen to be observable in the traditional sense. Now, if we calculate $Ce^{At} = \begin{pmatrix} 1 & e^{-t} & e^{-2t} \end{pmatrix}$
The algebraic variety provided by the homogeneous polynomial equation can be seen.

$$x_1 x_3 = x_2^2.$$

contains the union (3.7)*, therefore, breaking Theorem* (3.2) *is a richness criterion.*

Conclusion

Finally, following this scientific station, which required us to stand at many stations on ensemble controllability of parameter-dependent systems, we may say that it is a novel topic about which we attempted to familiarize ourselves despite the lack of knowledge by touching on its most essential aspects and emerging with a set of results, the most important of which are as follows:
We develop a single parameter-independent control that allows all parameter-dependent system realizations to approach the aim. This is the ideal situation, often known as the ensemble control. The idea arises from studying the complex Rotation dynamics in nuclear magnetic resonance spectroscopy. We show that the controllability of the group is related to the individual values of the operator that system dynamics.
In conclusion, we say that the topic **(Ensemble controllabillity of parameters-dependent systems)** still needs a lot of studies to discover its secrets, and this study was only an easy attempt, and we hope that we have succeeded, even in a small part.

Appendices

Definition 1 *(Right coprime factorization)* Two transfer matrices $M(\theta) \in \mathbf{R}H^{m\times m}$ and $N(\theta) \in \mathbf{R}H_{\infty}^{m\times p}$ constitute a right coprime factorization of a rational transfer matrix $H(\theta)$ of dimensions $p \times m$ if and only if the following:

$1 - M(\theta)$ is square, and $det\ (M(\theta)) \neq 0$;

$2 - \forall\theta \in \mathbf{P}$,$H(\theta) = N(\theta)\, M(\theta)^{-1}$;

$3 - \exists\, \overline{U}(\theta) \in \mathbf{R}H_{\infty}^{m\times p}, \exists\, \overline{V}(\theta) \in \mathbf{R}H^{m\times m}$ such that:

$$\forall\theta \in \mathbf{P}, \overline{V}(\theta)\, M(\theta) + \overline{U}(\theta)\, N(\theta) = I_m.$$

Definition 2 *(Left coprime factorization)* Two transfer matrices $\overline{M}(\theta) \in \mathbf{R}H_{\infty}^{p\times p}$ and $\overline{N}(s) \in \mathbf{R}H_{\infty}^{p\times m}$ constitute left coprime factorization of a rational transfer matrix $H(\theta)$ of dimensions $p \times\ m$ if and only if the following:

$1 - \overline{M}(\theta)$ is square, and $det\ \left(\overline{M}(\theta)\right) \neq 0$;

$2 - \forall\theta \in \mathbf{P}$,$H(\theta) = N(\theta)\, \overline{M}(\theta)^{-1}$;

$3 - \exists\, U(\theta) \in \mathbf{R}H_{\infty}^{m\times p}, \exists\, V(\theta) \in \mathbf{R}H_{\infty}^{p\times p}$ such that:

$$\forall\theta \in \mathbf{P}, \overline{M}(\theta)\, V(\theta) + \overline{N}(\theta)\, U(\theta) = I_P.$$

Definition 3 Let $H_0 = L_2(M; \mathbb{R}^{n\times m})$ be a vector space of all matrix-valued functions f whose $ijth$ entries $f_{ij}(t,\theta)$,
$i = \{1, ..., n\}$,$j = \{1, ..., m\}$, are complex- valued measurable function defined on M. We define an inner product

$$\langle f, g\rangle : H_0 \times H_0 \to \mathbb{R} \text{ to be } \langle f, g\rangle = tr \int_{\kappa}\int_0^T f(t,\theta)\, g\dagger(t,\theta)\, dtd\theta,$$

for all $f, g \in H_0$ and its corresponding norm

$$\|f\|^2 = \int_{\kappa}\int_0^T \|f(t,\theta)\|^2 dtd\theta.$$

Then, H_0 is a Hilbert space.

Theorem 1 Let $\sum_C(M, A, B)$ be an ensemble of continuous-time varying linear systems and suppose
$(A, B) \in L_{\infty}(M; \mathbb{R}^{n\times n}) \times L_2(M; \mathbb{R}^{n\times m})$. Let $\Phi(t, 0, \theta)$ be the transition matrix induced by $\sum_C(M, A, B)$ such that, for all $\theta \in \kappa$, $\Phi(t, 0, \theta)$ satisfies

$$\frac{\partial\Phi}{\partial t}(t, 0, \theta) = A(t,\theta)\, \Phi(t, 0, \theta), \qquad \Phi(0, 0, \theta) = I.$$

Then, the operator $L : H_1 \to H_2$ defined by

$$(Lu)(\theta) = \int_0^T \Phi(0, \tau, \theta) B(\tau, \theta) u(\tau) d\tau.$$

is compact.

Theorem 2 (*Fubini's theorem*) Let $X \times Y$ be an interval in $\mathbb{R}^{m+n}$, which is the direct product of intervals $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$. If the function $f : X \times Y \to \mathbb{R}$ is integrable over $X \times Y$, then all three of the integrals

$$\int_{X\times Y} f(x, y)\, dxdy, \quad \int_X dx \int_Y f(x, y)\, dy, \quad \int_Y dy \int_X f(x, y)\, dx.$$

Exist and are equal.

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