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Multiple Solutions for Partial Discrete Dirichlet boundary value problem with p-Laplacian

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بسم الله الرحمن الرحيم

شكر و عرفان

الحمد لله السميع العليم ذي العزة والفضل العظيم و الصلاة و السلام على المصطفى الهادي الكريم و على اله و صحبه اجمعين. و بعد قوله تعالى " و لئن شكرتم لأزيدنكم "، اشكر الله العلي القدير الذي انار لي درب العلم و المعرفة و أعاني على إنهاء هذا العمل.

كما نتقدم بالشكر الجزيل إلى الأستاذ المشرف " قفايفية رفيق " الذي رافقنا طيلة هذا البحث و أمدنا بالمعلومات و الإرشادات راجين من الله عز وجل أن يسدد خطاه و يحقق مناه فجزاه الله عنا كل خير.

و الشكر موجه أيضا للأساتذة أعضاء اللجنة المناقشة لقبولهم مناقشة هذا العمل و في الأخير لا يسعنا إلا ان ندعو الله عز وجل أن يرزقنا السداد و الرشاد و العفاف و الغنى و أن يجعلنا هداة مهتدين.

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إلى من قال الله عنهما
و إخفض لهما جناح الذل من الرحمة و قل ربي إرحمهما كما "
" ربياني صغيرا

إلى قدوتي رغم تعاظم الناس من حولي، اِلي نبعي الصافي، اِلي الشجرة التي لا تذبل، اِلي الظل الذي آوي اِليه في كل حين، اِلي ابي الغالي... رعاك المولى و جزاك من الثواب اجزاه.

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قواسمية هديل

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بوزنادة صورية

Abstract

By employing three critical points theorem, we investigate the existence of solutions to a boundary value problem for p-Laplacian partial difference equation with real parameter and give it accurate estimates to ensure that the studied problem has at least three solutions. Moreover, two positive solutions are obtained under some suitable assumptions for nonlinearity f depending on the strong maximum principle.

Keywords: Boundary value problem, Partial difference equation, Critical point Theory, p-Laplacian

Résumé

En employant le théorème de trois points critiques, nous étudions l’existence de solutions à un probléme de valeur limite pour l’équation de différence partielle p-laplacienne basée sur un paramétre réel et lui donnons des estimations exactes pour s’assurer que le probléme étudié a au moins trois solutions.

De plus, deus solutions positives sont obtenues sous certaines hypothéses adaptées pour la non-linéarité f selon le principe fort maximum.

Mots clés : problème de valeur aux limites, équation aux différences partielles, théorie des points critiques, p-laplacien.

ملخص

من خلال إستخدام نظرية الثلاث نقاط الحرجة، نقوم بالبحث عن وجود حلول مسألة ذات شروط حدية لمعادلة تفاضلية جزئية منفصلة تتضمن مؤثر p لابلاس في وجود معامل حقيقي و إعطائه تقديرات دقيقة للتأكد من ان المسألة المدروسة لها ثلاث حلول على الأقل. بالإضافة إلى ذلك، إعتمادا على مبدأ الحد الأعظمي القوي نتحصل على حلين موجبين وفقا لبعض الشروط المناسبة على للاخطية f.

الكلمات المفتاحية : مسألة ذات شروط حدية، معادلة تفاضلية منفصلة، نظرية النقطة الحرجة، مؤثر p لابلاس.

Contents

Introduction

Due to the demands of many fields and great interests in the study of partial differential equations involving two or more variables, which have also been continuously developed due to their theoretical background and realistic importance, for example they have been widely used as separate mathematical models describing real-life scenarios in Electrical circuit, analysis, economics, dynamical systems, physics and biology.

The importance of differential equations cannot be defined, but we also do not forget the important and main role that critical point theory plays in the study of various such equations and finding solutions to them. What should be mentioned is what many scientists have done on a large scale, most notably Guo and Yu [6] the first applicators of the critical point theory, in addition to the efforts of some other scientists in the process of solving this theory, such as the scientist Ji did use theorem (2) in [15] and got some new results with three solutions to the following problem:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)+|u|^{p(x)-2}u=\lambda f(x,u)+\mu g(x,u) \quad \text{in } \Omega, \\ Bu=0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Since then the critical point theorem has become a powerful tool for dealing with discrete nonlinear problems and excellent results have been obtained for periodic solutions [12, 18], homoclinic solutions [13, 17], heteroclinic solutions [2, 11], and boundary value problems.

In 2010, Galewski and Orpel in [5] used critical point theory, following some of the ideas from [9] to rewrite the problem $\left(E_\lambda^f\right)$ defined as follows:

$$\begin{cases} -\triangle_1^2 u(i-1,j)-\triangle_2^2 u(i,j-1)=\lambda f\left((i,j),u(i,j)\right), \quad (i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n), \\ u(i,0)=u(i,n+1), \quad i\in\mathbb{Z}(1,m), \\ u(0,j)=u(m+1,j), \quad j\in\mathbb{Z}(1,n). \end{cases}$$

As a non-linear algebraic system, they get at least a non-trivial solution. Similarly, Haidarkhani and Imbesi in [7] set sufficient conditions to ensure that the problem$\left(E_\lambda^f\right)$ has at least three distinct solutions. By making use of the same techniques as [5, 7], imbesi and bisci in [8] further studied the nonlinear algebraic system of the problem$\left(E_\lambda^f\right)$ and getting two kinds of results:

Either there is an infinite sequence of solutions or a sequence of non-zero paired solutions that converge to zero and this was in 2010.

As it was recently shown by Du and Zhou in [4] how to treat a class of partial discrete Dirichlet boundary value problem involving the p-Laplacian, namely problem $\left(S_{\lambda}^{f,q}\right)$:

$$\begin{cases} -\Delta_1\left[\phi_p\left(\Delta_1 x(i-1,j)\right)\right]-\Delta_2\left[\phi_p\left(\Delta_2 x(i,j-1)\right)\right]=\lambda f\left((i,j),x(i,j)\right), \\ x(i,0)=x(i,n+1), \quad i\in\mathbb{Z}(0,m+1), \\ x(0,j)=x(m+1,j), \quad j\in\mathbb{Z}(0,n+1). \end{cases}$$

when $q(i,j)=0$ for any $(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)$ this by exploiting the critical point a series of results are obtained.

Based on the results of the previous research, we noticed that the previously mentioned problem $\left(E_{\lambda}^{f}\right)$ is a special case of $\left(S_{\lambda}^{f,q}\right)$ when $q(i,j)=0$, as it differs in the main tools used to prove them [7]. Given the importance of this topic, we have done a scientific research on the multiple solutions for partial discrete Dirichlet boundary value problem with p-Laplacian and divided it into three chapters:

In the first chapter we used start with some basic concepts, starting with Lebesgue spaces and ending with critical point theory, which will help us understand the following chapters.

In the second chapter, we contructed the problem structure variable $\left(P_{\lambda}^{f,q}\right)$ defined by the equation:

$$\begin{cases} -\Delta_1\left[\phi_p\left(\Delta_1 x(i-1,j)\right)\right]-\Delta_2\left[\phi_p\left(\Delta_2 x(i,j-1)\right)\right]+q\left(i,j\right)\phi_p\left(x(i,j)\right)=\lambda f\left(\left(i,j\right),x\left(i,j\right)\right), \\ (i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n), \\ x(i,0)=x(i,n+1)=0, \quad i\in\mathbb{Z}(0,m+1), \\ x(0,j)=x(m+1,j)=0, \quad j\in\mathbb{Z}(0,n+1). \end{cases}$$

and we will also discuss the existence of three solutions to the problem posed using the variational method and the theorem of the three critical points of Bonanno and Marano.

In the last chapter, our main findings were generated. Moreover, under the appropriate assumptions about the nonlinearity f, two corollaries are obtained by employing the three critical points theorem and the strong maximum principle. And at last, a concrete example is provided to illustrate our results.

Chapter 1

Preliminary

1- Banach Space.

2- l^p Space.

3- Monotone operators.

4- Holder's Inequality.

5- Gateaux derivative.

6- Three critical points theorem.

1.1 Banach Space

Definition 1.1 *[16]Let X be a vector space over A real-valued function $\|.\|$ defined on X and satisfying the following conditions is called a norm:*

i) $\|u\| \geq 0, \quad \|u\| = 0$ *if and only if* $u = 0$.

ii) $\|\lambda u\| = |\lambda| \|u\|$, *for all* $u \in X$ *and* $\lambda \in \mathbb{R}$.

iii) $\|u + v\| \leq \|u\| + \|v\|, \forall u, v \in X$.

$(X, \|.\|)$, *is called a normed space equipped with the norm*$\|.\|$.

Definition 1.2 *[16]A normed space X is called a Banach space, if its every Cauchy sequence is convergent, that is $\|u_n - u_m\| \to 0$ as $n, m \to \infty$ $\forall u_n, u_m \in X$ implies that $\exists u \in X$ such that $\|u_n - u\| \to 0$ as $n \to \infty$.*

1.2 l^p Space

Definition 1.3 *[3]For $0 < p < \infty$, $l^p$is the subspace of $\mathbb{k}^{\mathbb{N}}$ consisting of all sequences $x = (x_n)_{n\in\mathbb{N}}$ satisfying;*

$$\sum_n |x_n| < \infty.$$

If $p \geq 1$, then the real-valued operation $\|.\|_p$ defined by;

$$\|x\|_p = \left(\sum_n |x_n|^p\right)^{\frac{1}{p}}$$

defines a norm on l^p. In fact, l^p is a complete metric space with respect to this norm, and therefore is a Banach space.

If $0 < p < 1$, then l^p does not carry a norm, but rather a metric defined by;

$$d(x, y) = \sum_n |x_n - y_n|^p.$$

If $p = \infty$, then l^∞ is defined to be the space of all bounded sequences endowed with the norm

$$\|x\|_\infty = \sup_n |x_n|,$$

l^∞ is also a Banach space.

1.3 Monotone operators

Definition 1.4 *[18]Let X be real Banach space, and let $A : X \to X^*$ be an operator.*

i) *A is called monotone iff*

$$\langle Au - Av, u - v\rangle \geq 0 \text{ for all } u, v \in X.$$

ii) *A is called strictly monotone iff*

$$\langle Au - Av, u - v\rangle > 0 \text{ for all } u, v \in X \text{ with } u \neq v.$$

iii) *A is called strongly monotone iff there is a $c > 0$ such that*

$$\langle Au - Av, u - v\rangle \geq c\,\|u - v\|^2 \text{ for all } u, v \in X.$$

iv) *A is called uniformly monotone iff*

$$\langle Au - Av, u - v\rangle \geq a\left(\|u - v\|\right)\|u - v\| \text{ for all } u, v \in X,$$

where the continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly monotone increasing with $a(0) = 0$ and $a(t) \to +\infty$ as $t \to +\infty$.

Definition 1.5 *[18]Let X be real Banach space, and let $A : X \to X^*$ be an operator. A is called hemicontinuous if for all $u,\ v \in X$, l'appliction $t \to \langle A\left(u + tv\right), v\rangle$ is continuous from $\mathbb{R}$ in $\mathbb{R}$.*

Definition 1.6 *[18]Let X be real Banach space, and let $A : X \to X^*$ be an operator. A is called coercive iff*

$$\lim_{\|u\| \to \infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty.$$

1.4 Holder's Inequality

Lemma 1.1 *If $p > 1$ and $q > 1$ are such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

then

$$\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}}.$$

Proof *We prove this relationship. Denote*

$$A = \sum_{i=1}^{n} a_i^p, \quad B = \sum_{i=1}^{n} b_i^q.$$

Then Hölder's inequality is written as follows:

$$\sum_{i=1}^{n} a_i b_i \le A^{\frac{1}{p}} B^{\frac{1}{q}}.$$

Next, we use Young's inequality in the form:

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{a}{p} + \frac{b}{q}.$$

Let

$$a = \frac{a_i^p}{A}, \quad b = \frac{b_i^q}{B}.$$

Applying Young's inequality to each pair of numbers a_i and b_i we obtain:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{a_i b_i}{A^{\frac{1}{p}} B^{\frac{1}{q}}} &\le \sum_{i=1}^{n} \left(\frac{a_i^p}{pA} + \frac{b_i^q}{qB}\right), \\
&\Rightarrow \frac{\sum_{i=1}^{n} a_i b_i}{A^{\frac{1}{p}} B^{\frac{1}{q}}} \le \frac{\sum_{i=1}^{n} a_i^p}{pA} + \frac{\sum_{i=1}^{n} b_i^q}{qB}, \\
&\Rightarrow \frac{\sum_{i=1}^{n} a_i b_i}{A^{\frac{1}{p}} B^{\frac{1}{q}}} \le \frac{A}{pA} + \frac{B}{qB} = \frac{1}{p} + \frac{1}{q} = 1, \\
&\Rightarrow \sum_{i=1}^{n} a_i b_i \le A^{\frac{1}{p}} B^{\frac{1}{q}}, \\
&\Rightarrow \sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}} \qquad (p > 1).
\end{aligned}$$

■

1.5 Gateaux derivative

Definition 1.7 *[10]Let ω be a part of a Banach space X and $J : \omega \to \mathbb{R}$. If $u \in \omega$ and $v \in X$ are such that for $t > 0$ quite small we have $u + tv \in \omega$ we say that J admits (at the point u) a derivative in the direction v if*

$$\lim_{t \to 0^+} \frac{J(u + tv) - J(u)}{t},$$

exist. We will denote this limit by $J'_v(u)$.

Definition 1.8 *[10]Let ω be a part of a Banach space X and $J : \omega \to \mathbb{R}$. If $u \in \omega$, we say that J is Gâteaux differentiable (or G-differentiable) at u, if there exists $l \in X'$ such that in each direction $v \in X$ where $J(u + tv)$ exists for $t > 0$ small enough, the directional derivative $J'_v(u)$ exists and we have*

$$\lim_{t \to 0^+} \frac{F(u + tv) - F(u)}{t} = \langle l, v \rangle .$$

We write $J'(u) = l$.

Definition 1.9 *[10]Let X be a Banach space, $\omega \in X$ an open space and $J \in C^1(\omega, \mathbb{R})$. We say that $u \in \omega$ is a critical point of J if $J'(u) = 0$ with $J'(u)$ is the G-differentiable of J at point. If u are not a critical point then we say that u is a regular point of J. If $c \in \mathbb{R}$, we say that c is a value critical of J, if there exists $u \in \omega$ such that $J(u) = c$ and $J'(u) = 0$. If c is not a critical value then we say that c is a regular value of J.*

1.6 Three critical points theorem

We present a critical point theorem due to Bonanno and Marano critical points theorems to prove the existence of at least three weak solutions.

Theorem 1.1 *[4]Let X be a separable and reflexive real Banach space. $\Phi : X \to \mathbb{R}$ is a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*. $J : X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$ and that* (**i**) $\lim_{\|x\|\to+\infty}[\Phi(x)-\lambda J(x)]=+\infty$ *for all* $\lambda\in[0,+\infty)$;

Further, assume that there are $r>0$, $x_1\in X$ *such that*

(**ii**) $r<\Phi(x_1)$;

(**iii**) $\sup_{x\in\overline{\Phi^{-1}(-\infty,r)}^w} J(x)<\frac{r}{r+\Phi(x_1)}J(x_1)$.

Then, for each

$$\lambda\in\Lambda_1=\left(\frac{\Phi(x_1)}{J(x_1)-\sup_{x\in\overline{\Phi^{-1}(-\infty,r)}^w} J(x)},\frac{r}{\sup_{x\in\overline{\Phi^{-1}(-\infty,r)}^w} J(x)}\right),$$

the equation

$$\Phi'(x)-\lambda J'(x)=0 \tag{1.1}$$

has at least three solutions in X *and, moreover, for each* $h>1$, *there exist an open interval*

$$\Lambda_2\subseteq\left[0,\frac{h}{\frac{rJ(x_1)}{\Phi(x_1)}-\sup_{x\in\overline{\Phi^{-1}(-\infty,r)}^w} J(x)}\right]$$

and a positive real number σ *such that, for each* $\lambda\in\Lambda_2$, *the equation* 1.1 *has at least three solutions in* X *whose norms are less than* σ.

Chapter 2

Three solutions for partial discrete Dirichlet boundary value problem with p-Laplacian

2.1 Introduction

In this chapter, we will study and discuss a three solutions for partial discrete Dirichlet boundary value problem with p-Laplacian, denoted by $(P_\lambda^{f,q})$ by the equation:

$$\begin{cases} -\Delta_1\left[\phi_p\left(\Delta_1 x(i-1,j)\right)\right]-\Delta_2\left[\phi_p\left(\Delta_2 x(i,j-1)\right)\right]+q\left(i,j\right)\phi_p\left(x(i,j)\right)=\lambda f\left(\left(i,j\right),x\left(i,j\right)\right), \\ (i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n), \\ x(i,0)=x(i,n+1)=0, \qquad i\in\mathbb{Z}(0,m+1), \\ x(0,j)=x(m+1,j)=0, \qquad j\in\mathbb{Z}(0,n+1). \end{cases}$$

This is done by using the variational method and the theory of the three critical points of Bonanno and Marano.

Where that Δ_1 and Δ_2 denote the forward difference operators defined by $\Delta_1 x(i,j) = x(i+1,j)-x(i,j)$ and $\Delta_2 x(i,j)=x(i,j+1)-x(i,j)$, $\Delta_1^2 x(i,j)=\Delta_1(\Delta_1 x(i,j))$ and $\Delta_2^2 x(i,j) = \Delta_2(\Delta_2 x(i,j))$, λ is nonnegative parameter, ϕ_p denotes the p-Laplacian operator, that is, $\phi_p = |s|^{p-2}s$, $p>1$, $q\left(i,j\right)\geq 0$ for all $(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)$, and $f\left(\left(i,j\right),\cdot\right)\in C(\mathbb{R},\mathbb{R})$ for each $(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)$.

2.2 The energy function of the problem $(P_\lambda^{f,q})$

We consider the mn-dimensional Banach space:

$$X=\left\{\begin{array}{c} x:\mathbb{Z}(0,m+1)\times\mathbb{Z}(0,n+1)\to\mathbb{R} \text{ such that } x(i,0)=x(i,n+1)=0, \\ i\in\mathbb{Z}(0,m+1) \text{ and } x(0,j)=x(m+1,j)=0, j\in\mathbb{Z}(0,n+1) \end{array}\right\},$$

Endowed with the norm:

$$\|x\|=\left(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^p+\sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^p+\sum_{j=1}^{n}\sum_{i=1}^{m}q(i,j)\,|x(i,j)|^p\right)^{\frac{1}{p}}$$

For all $x\in X$.

Moreover, define:

$$\Phi\left(x\right)=\frac{\|x\|^p}{p} \text{ and } J\left(x\right)=\sum_{j=1}^{n}\sum_{i=1}^{m}F\left(\left(i,j\right),x\left(i,j\right)\right)\ , \quad \forall x\in X\ , \tag{2.1}$$

Where;

$$F((i,j),\xi)=\int_0^{\xi} f((i,j),\tau)d\tau, \;\; \forall((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times\mathbb{R}.$$

Clearly, Φ and J are two functionals of class $C^1(X,\mathbb{R})$ and, for all $x,z\in X$,

$$\begin{aligned}
\Phi'(x)(z) &= \lim_{t\to 0}\frac{\Phi(x+tz)-\Phi(x)}{t}\\
&= \sum_{j=1}^{n}\sum_{i=1}^{m+1}\phi_p(\Delta_1 x(i-1,j))\,\Delta_1 z(i-1,j)+\sum_{i=1}^{m}\sum_{j=1}^{n+1}\phi_p(\Delta_2 x(i,j-1))\,\Delta_2 z(i,j-1)\\
&\quad+\sum_{j=1}^{n}\sum_{i=1}^{m}\phi_p q(i,j)x(i,j)z(i,j)\\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\phi_p(\Delta_1 x(i-1,j))\,\Delta_1 z(i-1,j)-\sum_{j=1}^{n}\phi_p(\Delta_1 x(m,j))\,z(m,j)\\
&\quad+\sum_{i=1}^{m}\sum_{j=1}^{n}\phi_p(\Delta_2 x(i,j-1))\,\Delta_2 z(i,j-1)-\sum_{i=1}^{m}\phi_p(\Delta_2 x(i,n))\,z(i,n)\\
&\quad+\sum_{j=1}^{n}\sum_{i=1}^{m}\phi_p q(i,j)x(i,j)z(i,j)\\
&= -\sum_{j=1}^{n}\sum_{i=1}^{m}\Delta_1\phi_p(\Delta_1 x(i-1,j))\,z(i,j)-\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta_2\phi_p(\Delta_2 x(i,j-1))\,z(i,j)\\
&\quad+\sum_{j=1}^{n}\sum_{i=1}^{m}q(i,j)\phi_p(x(i,j))\,z(i,j)\\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\{-\Delta_1\left[\phi_p(\Delta_1 x(i-1,j))\right]-\Delta_2\left[\phi_p(\Delta_2 x(i,j-1))\right]+q(i,j)\,\phi_p(x(i,j))\}z(i,j)\,.
\end{aligned}$$

and

$$J'(x)(z)=\lim_{t\to 0}\frac{J(x+tz)-J(x)}{t}=\sum_{j=1}^{n}\sum_{i=1}^{m}f((i,j),x(i,j))z(i,j),$$

For all $x,z\in X$.

Taken together, we have,

$$\begin{aligned}
\left[\Phi'(x)-\lambda J'(x)\right](z) &= \sum_{j=1}^{n}\sum_{i=1}^{m}\{-\Delta_1\left[\phi_p(\Delta_1 x(i-1,j))\right]-\Delta_2\left[\phi_p(\Delta_2 x(i,j-1))\right]\\
&\quad+q(i,j)\,\phi_p(x(i,j))-\lambda f((i,j),x(i,j))\}z(i,j)\,.
\end{aligned}$$

Remark 2.1 *Consequently, the critical points of the functional $\Phi - \lambda J$ in X are exactly the solution of problem $(P_\lambda^{f,q})$. Then we transform the problem of seeking the solutions of $(P_\lambda^{f,q})$ into looking for the critical points of $\Phi - \lambda J$ in X.*

Put

$$q_* = \min_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{q(i,j)\}, \quad q^* = \max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{q(i,j)\}$$

Lemma 2.1 *[4]For any $x \in X$, we have*

$$\max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{x(i,j)\} \leq \frac{(m+n+2)^{\frac{p-1}{p}}}{4} \left(\sum_{j=1}^{n} \sum_{i=1}^{m+1} |\Delta_1 x(i-1,j)|^p + \sum_{i=1}^{m} \sum_{j=1}^{n+1} |\Delta_2 x(i,j-1)|^p \right)^{\frac{1}{p}}. \quad (2.2)$$

Proof [4]For any given $x \in X$, there exist $s \in \mathbb{Z}(1,m)$ and $\tau \in \mathbb{Z}(1,n)$ such that;

$$x(s,\tau) = \max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{x(i,j)\}$$

Since $x(i,0) = x(i,n+1) = 0$, $i \in \mathbb{Z}(0,m+1)$ and $x(0,j) = x(m+1,j) = 0$, $j \in \mathbb{Z}(0,n+1)$, we can obtain:

$$\begin{aligned} |x(s,\tau)| &= \frac{1}{2} \left| \sum_{i=1}^{s} \Delta_1 x(i-1,\tau) + \sum_{j=1}^{\tau} \Delta_2 x(s,j-1) \right| \\ &\leq \frac{1}{2} \sum_{i=1}^{s} |\Delta_1 x(i-1,\tau)| + \sum_{j=1}^{\tau} |\Delta_2 x(s,j-1)| \\ &\leq \frac{1}{2} \cdot (s+\tau)^{\frac{1}{q}} \left(\sum_{i=1}^{s} |\Delta_1 x(i-1,\tau)|^p + \sum_{j=1}^{\tau} |\Delta_2 x(s,j-1)|^p \right)^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} |x(s,\tau)| &= \frac{1}{2} \left| \sum_{i=s+1}^{m+1} \Delta_1 x(i-1,\tau) + \sum_{j=\tau+1}^{n+1} \Delta_2 x(s,j-1) \right| \\ &\leq \frac{1}{2} \sum_{i=s+1}^{m+1} |\Delta_1 x(i-1,\tau)| + \sum_{j=\tau+1}^{n+1} |\Delta_2 x(s,j-1)| \\ &\leq \frac{1}{2} (m+n-s-\tau+2)^{\frac{1}{q}} \left(\sum_{i=s+1}^{m+1} |\Delta_1 x(i-1,\tau)|^p + \sum_{j=\tau+1}^{n+1} |\Delta_2 x(s,j-1)|^p \right)^{\frac{1}{p}}, \end{aligned}$$

where q is the conjugative number of p, that is, $\frac{1}{p}+\frac{1}{q}=1$ If

$$\begin{aligned}&\sum_{i=1}^{s}|\Delta_1 x(i-1,\tau)|^p+\sum_{j=1}^{\tau}|\Delta_2 x(s,j-1)|^p\\ \leq\ & \frac{(m+n+2)^{p-1}}{2^p.\,(s+\tau)^{p-1}}\left(\sum_{i=1}^{m+1}|\Delta_1 x(i-1,\tau)|^p\right)+\frac{(m+n+2)^{p-1}}{2^p.\,(s+\tau)^{p-1}}\left(\sum_{j=1}^{n+1}|\Delta_2 x(s,j-1)|^p\right),\end{aligned}$$

then we can get;

$$\max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}}\{x(i,j)\}\leq\frac{(m+n+2)^{\frac{p-1}{p}}}{4}\left(\sum_{i=1}^{m+1}|\Delta_1 x(i-1,\tau)|^p+\sum_{j=1}^{n+1}|\Delta_2 x(s,j-1)|^p\right)^{\frac{1}{p}}.$$

So, we obtain the required relation 2.2. If, on the contrary,

$$\begin{aligned}\sum_{i=1}^{s}|\Delta_1 x(i-1,\tau)|^p+\sum_{j=1}^{\tau}|\Delta_2 x(s,j-1)|^p\ >\ & \frac{(m+n+2)^{p-1}}{2^p.\,(s+\tau)^{p-1}}\left(\sum_{i=1}^{m+1}|\Delta_1 x(i-1,\tau)|^p\right)\\ &+\frac{(m+n+2)^{p-1}}{2^p.\,(s+\tau)^{p-1}}\left(\sum_{j=1}^{n+1}|\Delta_2 x(s,j-1)|^p\right),\end{aligned}$$

Then we have:

$$\begin{aligned}&\sum_{i=s+1}^{m+1}|\Delta_1 x(i-1,\tau)|^p+\sum_{j=\tau+1}^{n+1}|\Delta_2 x(s,j-1)|^p\\ =\ & \sum_{i=1}^{m+1}|\Delta_1 x(i-1,\tau)|^p+\sum_{j=1}^{n+1}|\Delta_2 x(s,j-1)|^p-\left(\sum_{i=1}^{s}|\Delta_1 x(i-1,\tau)|^p+\sum_{j=1}^{\tau}|\Delta_2 x(s,j-1)|^p\right)\\ <\ & \left(1-\frac{(m+n+2)^{p-1}}{2^p.\,(s+\tau)^{p-1}}\right)\left(\sum_{i=1}^{m+1}|\Delta_1 x(i-1,\tau)|^p\right)+\left(1-\frac{(m+n+2)^{p-1}}{2^p.\,(s+\tau)^{p-1}}\right)\left(\sum_{j=1}^{n+1}|\Delta_2 x(s,j-1)|^p\right).\end{aligned}$$

Moreover, we have;

$$\begin{aligned}&|x(s,\tau)|\\ <\ & \frac{1}{2}(m+n-s-\tau+2)^{\frac{1}{q}}\left(1-\frac{(m+n+2)^{p-1}}{2^p.\,(s+\tau)^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m+1}|\Delta_1 x(i-1,\tau)|^p+\sum_{j=1}^{n+1}|\Delta_2 x(s,j-1)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

We claim that inequality

$$\frac{1}{2}.\,(m+n-s-\tau+2)^{\frac{1}{q}}\left(1-\frac{(m+n+2)^{p-1}}{2^p.\,(s+\tau)^{p-1}}\right)^{\frac{1}{p}}\leq\frac{(m+n+2)^{\frac{1}{q}}}{4}\tag{2.3}$$

holds. In fact, we define a function $g : (0, m+n+2) \to \mathbb{R}$ by;

$$g(t) = \frac{1}{(m+n-t+2)^{p-1}} + \frac{1}{t^{p-1}}.$$

The function g can attain its minimum $\frac{2^p}{(m+n+2)^{p-1}}$ at $t = \frac{m+n+2}{2}$. Since $s \in \mathbb{Z}(1,m)$, $t \in \mathbb{Z}(1,n)$, we can get $g(s+\tau) \geq \frac{2^p}{(m+n+2)^{p-1}}$, that is,

$$\frac{1}{(m+n-s-\tau+2)^{p-1}} + \frac{1}{(s+\tau)^{p-1}} \geq \frac{2^p}{(m+n+2)^{p-1}}.$$

This implies assertion 2.3 and we can obtain the required inequality 2.2. ∎

Lemma 2.2 *For all $x \in X$, the inequality*

$$\max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{|x(i,j)|\} \leq \frac{(m+n+2)^{\frac{p-1}{p}}}{\left[4^p + q_*(m+n+2)^{p-1}\right]^{\frac{1}{p}}} \|x\| \tag{2.4}$$

holds.

Proof Owing to 2.2, we infer

$$\begin{aligned}
\|x\|^p &= \sum_{j=1}^{n}\sum_{i=1}^{m+1} |\Delta_1 x(i-1,j)|^p + \sum_{i=1}^{m}\sum_{j=1}^{n+1} |\Delta_2 x(i,j-1)|^p + \sum_{j=1}^{n}\sum_{i=1}^{m} q(i,j)\,|x(i,j)|^p \\
&\geq \frac{4^p}{(m+n+2)^{p-1}} \left(\max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{|x(i,j)|\} \right)^p + q_* \sum_{j=1}^{n}\sum_{i=1}^{m} |x(i,j)|^p \\
&\geq \frac{4^p}{(m+n+2)^{p-1}} \left(\max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{|x(i,j)|\} \right)^p + q_* \left(\max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{|x(i,j)|\} \right)^p \\
&= \frac{4^p + q_*(m+n+2)^{p-1}}{(m+n+2)^{p-1}} \left(\max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{|x(i,j)|\} \right)^p .
\end{aligned}$$

Therefore,

$$\max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{|x(i,j)|\} \leq \frac{(m+n+2)^{\frac{p-1}{p}}}{\left[4^p + q_*(m+n+2)^{p-1}\right]^{\frac{1}{p}}} \|x\| .$$

For later convenience, we define another norm:

$$\|x\|_p = \left(\sum_{j=1}^{n} \sum_{i=1}^{m} |x(i,j)|^p \right)^{\frac{1}{p}}, \forall x \in X.$$

Since X is an mn-dimensional space, the norms $\|.\|$ and $\|.\|_p$are equivalent. To be specific, we have the following numerical estimation. ■

Lemma 2.3 *For all $x \in X$, one has*

$$\frac{[4^p + q_* mn(m+n+2)^{p-1}]^{\frac{1}{p}}}{(mn)^{\frac{1}{p}} (m+n+2)^{\frac{p-1}{p}}} \|x\|_p \leq \|x\| \leq (2^{p+1} + q^*)^{\frac{1}{p}}_{p} \|x\|_p . \tag{2.5}$$

Proof On the one hand, from 2.2 we have

$$\begin{aligned} \|x\|^p &= \sum_{j=1}^{n} \sum_{i=1}^{m+1} |\Delta_1 x(i-1,j)|^p + \sum_{i=1}^{m} \sum_{j=1}^{n+1} |\Delta_2 x(i,j-1)|^p + \sum_{j=1}^{n} \sum_{i=1}^{m} q(i,j) |x(i,j)|^p \\ &\geq \frac{4^p}{(m+n+2)^{p-1}} \left(\max_{\substack{i \in \mathbb{Z}(1,m) \\ j \in \mathbb{Z}(1,n)}} \{|x(i,j)|\} \right)^p + q_* \|x\|_p^p \\ &\geq \frac{4^p}{(m+n+2)^{p-1}} |x(i,j)|^p + q_* \|x\|_p^p \end{aligned}$$

for any $(i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n)$. This implies that

$$|x(i,j)|^p \leq \frac{(m+n+2)^{p-1}}{4^p} \left[\|x\|^p - q_* \|x\|_p^p \right], \forall (i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n).$$

Hence,

$$\begin{aligned} \|x\|_p^p &= \sum_{j=1}^{n} \sum_{i=1}^{m} |x(i,j)|^p \\ &\leq \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{(m+n+2)^{p-1}}{4^p} \left[\|x\|^p - q_* \|x\|_p^p \right] \\ &= \frac{mn(m+n+2)^{p-1}}{4^p} \left[\|x\|^p - q_* \|x\|_p^p \right] \\ &= \frac{mn(m+n+2)^{p-1}}{4^p} \|x\|^p - \frac{q_* mn(m+n+2)^{p-1}}{4^p} \|x\|_p^p , \end{aligned}$$

that is,

$$\left[1 + \frac{q_* mn(m+n+2)^{p-1}}{4^p} \right] \|x\|_p^p \leq \frac{mn(m+n+2)^{p-1}}{4^p} \|x\|^p .$$

Therefore,

$$\frac{[4^p + q_* mn(m+n+2)^{p-1}]^{\frac{1}{p}}}{(mn)^{\frac{1}{p}}(m+n+2)^{\frac{p-1}{p}}} \|x\|_p \leq \|x\|.$$

On the other hand, for every $(i,j) \in \mathbb{Z}(1, m+1) \times \mathbb{Z}(1, n)$, we infer

$$|\Delta_1 x(i-1,j)|^p \leq (|x(i,j)| + |x(i-1,j)|)^p \leq 2^{p-1}(|x(i,j)|^p + |x(i-1,j)|^p),$$

where the last inequality is due to the convexity property of the function $\phi(t) = t^p$ $(t \geq 0)$.

Thus,

$$\begin{aligned}
|\Delta_1 x(i-1,j)|^p &\leq 2^{p-1} \sum_{j=1}^{n} \sum_{i=1}^{m+1} (|x(i,j)|^p + |x(i-1,j)|^p) \\
&= 2^{p-1} \left(\sum_{j=1}^{n} \sum_{i=1}^{m+1} |x(i,j)|^p + \sum_{j=1}^{n} \sum_{i=1}^{m+1} |x(i-1,j)|^p \right) \\
&= 2^{p-1} \left(\sum_{j=1}^{n} \sum_{i=1}^{m} |x(i,j)|^p + \sum_{j=1}^{n} \sum_{i=1}^{m} |x(i,j)|^p \right) \\
&= 2^p \sum_{j=1}^{n} \sum_{i=1}^{m} |x(i,j)|^p \\
&= 2^p \|x\|_p^p .
\end{aligned}$$

In the same way we get

$$\sum_{j=1}^{n+1} \sum_{i=1}^{m} |\Delta_2 x(i,j-1)|^p \leq 2^p \|x\|_p^p .$$

Besides,

$$\sum_{j=1}^{n} \sum_{i=1}^{m} q(i,j) |x(i,j)|^p \leq q^* \sum_{j=1}^{n} \sum_{i=1}^{m} |x(i,j)|^p = q^* \|x\|_p^p .$$

Summarizing,

$$\|x\|^p \leq 2^p \|x\|_p^p + 2^p \|x\|_p^p + q^* \|x\|_p^p = (2^{p+1} + q^*) \|x\|_p^p ,$$

that is,

$$\|x\| \leq (2^{p+1} + q^*)^{\frac{1}{P}} \|x\|_p$$

which yields our conclusion. ■

2.3 Main results

Denote

$$Q = \sum_{j=1}^{n} \sum_{i=1}^{m} q(i,j)$$

Our main result is the following.

Theorem 2.1 *Assume that there exist four positive constants c, d, μ, α satisfying $\alpha < p$ and*

$$d^p > \frac{\left[4^p + q_*(m+n+2)^{p-1}\right] c^p}{(2m+2n+Q)(m+n+2)^{p-1}}$$

such that

(A_1) $\max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi) < \frac{\left[4^p+q_*(m+n+2)^{p-1}\right]c^p \sum_{j=1}^{n}\sum_{i=1}^{m} F((i,j),d)}{mn\{[4^p+q_*(m+n+2)^{p-1}]c^p+(2m+2n+Q)(m+n+2)^{p-1}d^p\}}$;

(A_2) $F((i,j),\xi) \le \mu(1+|\xi|^\alpha), \forall((i,j),\xi) \in \mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times\mathbb{R}$.

Furthermore, put

$$\lambda_1 = \frac{pmn(m+n+2)^{p-1}\max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi)}{\left[4^p+q_*(m+n+2)^{p-1}\right]c^p},$$

$$\lambda_2 = \frac{p\left[\sum_{j=1}^{n}\sum_{i=1}^{m} F((i,j),d) - mn\max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi)\right]}{(2m+2n+Q)d^p}.$$

Then, for each $\lambda \in \Lambda_1 = \left(\frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right)$, problem $(P_\lambda^{f,q})$ possesses at least three solutions in X.

Moreover, put

$$\begin{aligned}
a &= (2m+2n+Q)\left[4^p+q_*(m+n+2)^{p-1}\right](cd)^p,\\
b &= p\left[4^p+q_*(m+n+2)^{p-1}\right]c^p\sum_{j=1}^{n}\sum_{i=1}^{m} F((i,j),d)\\
&\quad - pmn(2m+2n+Q)(m+n+2)^{p-1}d^p \max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi).
\end{aligned}$$

Then, for any $h>1$, there exist an open interval $\Lambda_2 \subseteq [0, \frac{a}{b}h]$ and a real number $\sigma > 0$ such that, for each $\lambda \in \Lambda_2$, problem $(P_\lambda^{f,q})$ possesses at least three solutions in X and their norms are all less than σ.

Proof Since X is a finite-dimensional real Banach space, X is separable and reflexive. From the definitions in 2.1 of Φ and J, we know that $\Phi : X \to \mathbb{R}$ is a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and $J : X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Choose $x_0(i,j) = 0$ for each $(i,j) \in \mathbb{Z}(0, m+1) \times \mathbb{Z}(0, n+1)$, it is clear that $x_0 \in X$ and $(x_0) = 0 = J(x_0)$.

According to the assumption (A_2) and Lemma 2.3, we deduce

$$\begin{aligned}
\Phi(x) - \lambda J(x) &= \frac{1}{p}\|x\|^p - \lambda \sum_{j=1}^{n}\sum_{i=1}^{m} F((i,j), x(i,j)) \\
&\geq \frac{4^p + q_* mn(m+n+2)^{p-1}}{pmn(m+n+2)^{p-1}} \|x\|_p^p - \lambda \sum_{j=1}^{n}\sum_{i=1}^{m} \mu(1 + |x(i,j)|^\alpha) \\
&= \frac{4^p + q_* mn(m+n+2)^{p-1}}{pmn(m+n+2)^{p-1}} \sum_{j=1}^{n}\sum_{i=1}^{m} |x\,(i,j)|^p - \lambda\mu \sum_{j=1}^{n}\sum_{i=1}^{m} (1 + |x\,(i,j)|^\alpha) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m} \left[\frac{4^p + q_* mn(m+n+2)^{p-1}}{pmn(m+n+2)^{p-1}} |x\,(i,j)|^p - \lambda\mu\, |x\,(i,j)|^\alpha - \lambda\mu \right],
\end{aligned}$$

for any $x \in X$ and $\lambda \geq 0$. Bearing in mind $\alpha < p$, one has

$$\lim_{\|x\| \to +\infty} [\Phi\,(x) - \lambda J\,(x)] = +\infty, \forall \lambda \in [0, +\infty),$$

namely, the condition (i) of Theorem 1.1 is fulfilled.

For the condition (ii), we put

$$r = \frac{[4^p + q_*(m+n+2)^{p-1}]c^p}{p(m+n+2)^{p-1}},$$

$$x_1(i,j) = \begin{cases} 0, \text{ if } i = 0, j \in \mathbb{Z}(0, n+1) \text{ or } i = m+1, j \in \mathbb{Z}(0, n+1), \\ d, \text{ if } (i,j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n), \\ 0, \text{ if } j = 0, i \in \mathbb{Z}(0, m+1) \text{ or } j = n+1, i \in \mathbb{Z}(0, m+1). \end{cases}$$

It follows that $x_1 \in X$ and

$$\begin{aligned}
\Phi(x_1) &= \frac{\|x_1\|^p}{p} = \frac{2m + 2n + Q}{p} d^p, \\
J(x_1) &= \sum_{j=1}^{n}\sum_{i=1}^{m} F\left((i,j), x_1\,(i,j)\right) = \sum_{j=1}^{n}\sum_{i=1}^{m} F\left((i,j), d\right).
\end{aligned}$$

In view of $d^p > \frac{[4^p+q_*(m+n+2)^{p-1}]c^p}{(2m+2n+Q)(m+n+2)^{p-1}}$, we have

$$\Phi(x_1) = \frac{2m+2n+Q}{p}d^p > \frac{[4^p+q_*(m+n+2)^{p-1}]c^p}{p(m+n+2)^{p-1}} = r,$$

which means that the condition (ii) of Theorem 1.1 is satisfied.

Next, we verify the condition (iii) of Theorem 1.1. By direct computation, we get

$$\frac{r}{r+\Phi(x_1)}J(x_1) = \frac{[4^p+q_*(m+n+2)^{p-1}]c^p \sum_{j=1}^{n}\sum_{i=1}^{m} F\left((i,j),d\right)}{[4^p+q_*(m+n+2)^{p-1}]c^p + (2m+2n+Q)(m+n+2)^{p-1}d^p}.$$

On the other hand, for any $x \in \Phi^{-1}(-\infty, r]$, i.e.,$\Phi(x) \le r$, we infer

$$\begin{aligned} |x(i,j)| &\le \max_{\substack{i\in\mathbb{Z}(1,m)\\ j\in\mathbb{Z}(1,n)}} \{|x(i,j)|\} \\ &\le \frac{(m+n+2)^{\frac{p-1}{p}}}{[4^p+q_*(m+n+2)^{p-1}]^{\frac{1}{p}}}\|x\| \le \frac{(m+n+2)^{\frac{p-1}{p}}(pr)^{\frac{1}{p}}}{[4^p+q_*(m+n+2)^{p-1}]^{\frac{1}{p}}} = c \end{aligned}$$

for every $(i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n)$. This leads to

$$\Phi^{-1}(-\infty, r] \subseteq \{x \in X : |x(i,j)| \le c, \forall (i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n)\}.$$

Hence, this along with assumption (A_1) yields

$$\begin{aligned} \sup_{x\in\overline{\Phi^{-1}(-\infty,r)}^{w}} J(x) &\le \sup_{x\in\{x\in X:|x(i,j)|\le c,\forall(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\}} \sum_{j=1}^{n}\sum_{i=1}^{m} F\left((i,j),x(i,j)\right) \\ &\le mn \max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F\left((i,j),\xi\right) \\ &< \frac{[4^p+q_*(m+n+2)^{p-1}]c^p \sum_{j=1}^{n}\sum_{i=1}^{m} F((i,j),d)}{[4^p+q_*(m+n+2)^{p-1}]c^p + (2m+2n+Q)(m+n+2)^{p-1}d^p} \\ &= \frac{r}{r+\Phi(x_1)}J(x_1) \end{aligned}$$

for any $x \in X$. The condition (iii) of Theorem 1.1 is verified.

Note that

$$\begin{aligned} &\frac{\Phi(x_1)}{J(x_1) - \sup_{x\in\overline{\Phi^{-1}(-\infty,r)}^{w}} J(x)} \\ &\le \frac{(2m+2n+Q)d^p}{p[\sum_{j=1}^{n}\sum_{i=1}^{m} F((i,j),d) - mn\ \max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi)]} = \frac{1}{\lambda_2}, \\ \frac{r}{\sup_{x\in\overline{\Phi^{-1}(-\infty,r)}^{w}} J(x)} &\ge \frac{[4^p+q_*(m+n+2)^{p-1}]c^p}{pmn(m+n+2)^{p-1}\max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi)} = \frac{1}{\lambda_1}. \end{aligned}$$

According to Theorem 1.1 and Remark 2.1, for any $\lambda \in \Lambda_1 = \left(\frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right)$, problem $(P_\lambda^{f,q})$ possesses at least three solutions in X.

Moreover, for any $h > 1$, it follows from the expressions of a and b that

$$\frac{hr}{\frac{rJ(x_1)}{\Phi(x1)} - \sup_{\overline{x\in\Phi^{-1}(-\infty,r)}^w} J(x)} \leq \frac{a}{b}h.$$

By Theorem 1.1 and Remark 2.1, for any $h > 1$, there exist an open interval $\Lambda_2 \subseteq [0, \frac{a}{b}h]$ and a real number $\sigma > 0$ such that, for each $\lambda \in \Lambda_2$, $(P_\lambda^{f,q})$ possesses at least three solutions in X and their norms all are less than σ. ■

Remark 2.2 *From (A_1) it follows that*

$$\begin{aligned} & mn(2m+2n+Q)(m+n+2)^{p-1}d^p \max_{((i,j),\xi)\epsilon\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi) \\ < \; & \left[4^p + q_*(m+n+2)^{p-1}\right] \\ & \times \left[\sum_{j=1}^{n}\sum_{i=1}^{m} F((i,j),d) - mn \max_{((i,j),\xi)\epsilon\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi)\right]. \end{aligned}$$

Then

$$\begin{aligned} & \frac{pmn(m+n+2)^{p-1}\max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi)}{\left[4^p + q_*(m+n+2)^{p-1}\right]c^p} \\ < \; & \frac{p\left[\sum_{j=1}^{n}\sum_{i=1}^{m} F((i,j),d) - mn\max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi)\right]}{(2m+2n+Q)d^p}. \end{aligned}$$

That is, $\lambda_1 < \lambda_2$, which indicates that the interval $(\frac{1}{\lambda_2}, \frac{1}{\lambda_1})$ is well-defined.

Remark 2.3 *In view of assumption (A_1), we infer*

$$\begin{aligned} & \left[4^p + q_*(m+n+2)^{p-1}\right]c^p \sum_{j=1}^{n}\sum_{i=1}^{m} F((i,j),d) \\ > \; & mn(2m+2n+Q)(m+n+2)^{p-1}d^p \max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[-c,c]} F((i,j),\xi), \end{aligned}$$

so $b > 0$ and $[0, \frac{a}{b}h]$ is a well-defined interval.

Remark 2.4 *If it was $q\,(i,j) = 0$ and $p = 2$ the problem P_λ^f is seen as discrete nonlinear Laplace equation in dimension 2. As pointed out by Galewski and Orpel in [5], problem P_λ^f serves as the* *discrete counterpart of the following problem;*

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda f\left((x,y), u(x,y)\right) = 0, \\ u(x,0) = u(x,n+1), \quad x \in \mathbb{Z}(0,m+1), \\ u(0,y) = u(m+1,y), \quad y \in \mathbb{Z}(0,n+1). \end{cases}$$

Remark 2.5 *We can write the problem $P_\lambda^{f,q}$ for $q(i,j) = 0$ and $p = 2$ as the nonlinear algebraic system of from;*

$$Aw = \lambda g(w),$$

Let it be $w_k = u(h^{-1}(k))$ and $g_k(w_k) = f(h^{-1}(k), w_k)$ for all $k \in [1, mn]$.

As you know A as follows

$$A = (a_{ij}) = \begin{pmatrix} L & -I_m & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -I_m & L & -I_m & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -I_m & L & -I_m & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & L & \cdots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & L & -I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -I_m & L & -I_m & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -I_m & L & -I_m \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -I_m & L \end{pmatrix} \in M_{mn \times mn}(\mathbb{R}),$$

L is defined by

$$L = \begin{pmatrix} 4 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 4 \end{pmatrix} \in M_{mn \times mn}(\mathbb{R}),$$

$I_m \in M_{mn\times mn}(\mathbb{R})$ is the identity matrix and $g(w) = (g_1(w_1), \ldots, g_{mn}(w_{mn}))^t$, for every $w \in X$.Moreover, a direct computation shows that

$$\sum_{i,j=1}^{mn} a_{ij} = 2(m+1)n - 2(n-1)m = 2(m+n).$$

Based in the above and based on that $q(i,j) \neq 0$, $p > 1$ it appears to us that

$$\sum_{i,j=1}^{mn} a_{ij} = 2(m+n) + Q.$$

Chapter 3

Two positive solutions for partial discrete Dirichlet boundary value problem with p-Laplacian

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1- Introduction.

2- The strong maximum principle.

3- Main results.

4- An example.

3.1 Introduction

In the third chapter, we are concerned a two positive solutions for partial discrete Dirichlet boundary value problem with p-Laplacian, denoted by $(P_\lambda^{f,q})$, based on the previously mentioned which is the three critical points theorem and the strong maximum principle.

3.2 The strong maximum principle

In order to obtain positive solutions of problem $(P_\lambda^{f,q})$, we establish the following strong maximum principle:

Lemma 3.1 *Fix $x \in X$ such that, for any $(i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n)$, either*

$$x(i,j) > 0 \ \text{ or } -\Delta_1\left[\phi_p\left(\Delta_1 x(i-1,j)\right)\right] - \Delta_2\left[\phi_p\left(\Delta_2 x(i,j-1)\right)\right] + q(i,j)\phi_p\left(x\left(i,j\right)\right) \geq 0. \quad (3.1)$$

Then either $x(i,j) > 0$ for all $(i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n)$ or $x \equiv 0$.

Proof Fix $x \in X$ satisfying 3.1.

Let $\theta \in \mathbb{Z}(1,m), \omega \in \mathbb{Z}(1,n)$ such that

$$x(\theta,\omega) = \min\left\{x(i,j) : i \in \mathbb{Z}(1,m), j \in \mathbb{Z}(1,n)\right\}.$$

If $x(\theta,\omega) > 0$,then $x(i,j) > 0$ for all $i \in \mathbb{Z}(1,m), j \in \mathbb{Z}(1,n)$, and the proof is finished.

If $x(\theta,\omega) \leq 0$,then $x(\theta,\omega) = \min\left\{x(i,j) : i \in \mathbb{Z}(0,m+1), j \in \mathbb{Z}(0,n+1)\right\}$. At this point, it is easy to see that $\Delta_1 x(\theta-1,\omega) = x(\theta,\omega) - x(\theta-1,\omega) \leq 0$ and $\Delta_1 x(\theta,\omega) = x(\theta+1,\omega) - x(\theta,\omega) \geq 0$. Since $\phi_p(s)$ is increasing in s, and $\phi_p(0) = 0$, one has

$$\phi_p\left(\Delta_1 x(\theta-1,\omega)\right) \leq 0 \leq \phi_p\left(\Delta_1 x(\theta,\omega)\right),$$

which implies that

$$\Delta_1\left[\phi_p\left(\Delta_1 x(\theta-1,\omega)\right)\right] \geq 0.$$

Similarly,

$$\Delta_2\left[\phi_p\left(\Delta_2 x(\theta,\omega-1)\right)\right] \geq 0.$$

Thus,

$$\Delta_1\left[\phi_p\left(\Delta_1 x(\theta-1,\omega)\right)\right]+\Delta_2\left[\phi_p\left(\Delta_2 x(\theta,\omega-1)\right)\right]\geq 0. \tag{3.2}$$

On the other hand, in view of 3.1 , we infer

$$\Delta_1\left[\phi_p\left(\Delta_1 x(\theta-1,\omega)\right)\right]+\Delta_2\left[\phi_p\left(\Delta_2 x(\theta,\omega-1)\right)\right]\leq q(\theta,\omega)\phi_p\left(x(\theta,\omega)\right)\leq 0. \tag{3.3}$$

Combining 3.2 and 3.3, we have

$$\Delta_1\left[\phi_p\left(\Delta_1 x(\theta-1,\omega)\right)\right]+\Delta_2\left[\phi_p\left(\Delta_2 x(\theta,\omega-1)\right)\right]=0,$$

which yields

$$\Delta_1\left[\phi_p\left(\Delta_1 x(\theta-1,\omega)\right)\right]=\Delta_2\left[\phi_p\left(\Delta_2 x(\theta,\omega-1)\right)\right]=0,$$

namely,

$$\begin{cases}\phi_p\left(\Delta_1 x(\theta,\omega)\right)=\phi_p\left(\Delta_1 x(\theta-1,\omega)\right)=0,\\ \phi_p\left(\Delta_2 x(\theta,\omega)\right)=\phi_p\left(\Delta_2 x(\theta,\omega-1)\right)=0.\end{cases}$$

Therefore,

$$\begin{cases}x(\theta+1,\omega)=x(\theta,\omega)=x(\theta-1,\omega),\\ x(\theta,\omega+1)=x(\theta,\omega)=x(\theta,\omega-1).\end{cases}$$

If $\theta+1=m+1$, we get $x(\theta,\omega)=0$. Otherwise, $\theta+1\in\mathbb{Z}(1,m)$. Replacing θ by $\theta+1$, we have $x(\theta+2,\omega)=x(\theta+1,\omega)$. Continuing this process $m+1-\theta$ times, we obtain $x(\theta,\omega)=x(\theta+1,\omega)=x(\theta+2,\omega)=\cdots=x(m,\omega)=x(m+1,\omega)=0$. Analogously, we have $x(\theta,\omega)=x(\theta-1,\omega)=x(\theta-2,\omega)=\cdots=x(1,\omega)=x(0,\omega)=0$. Hence, $x(i,\omega)=0$ for each $i\in\mathbb{Z}(1,m)$. In the same way we can prove that $x\equiv 0$ and the conclusion of Lemma 3.1 holds.

Remark 3.1 *When $f:\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times\mathbb{R}\to\mathbb{R}$ is a nonnegative function, Lemma 2.3 guarantees that every solution mentioned in Theorem 2.1 is either positive or zero.*

■

3.3 Main results

Corollary 3.1 *If $f((i,j),0) \geq 0$ for all $(i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n)$, and there exist four positive constants c, d, μ, α with $\alpha < p$ and*

$$d^p > \frac{[4^p + q_*(m+n+2)^{p-1}]c^p}{(2m+2n+Q)(m+n+2)^{p-1}}$$

such that

(A_1^*)

$$\max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[0,c]} \int_0^\xi f((i,j),\tau)d\tau < \frac{[4^p + q_*(m+n+2)^{p-1}]c^p \sum_{j=1}^n \sum_{i=1}^m f((i,j),\tau)d\tau}{mn\{[4^p + q_*(m+n+2)^{p-1}]c^p + (2m+2n+Q)(m+n+2)^{p-1}d^p\}};$$

(A_2^*)

$$\int_0^\xi f((i,j),\tau)d\tau \leq \mu\left(1 + |\xi|^\alpha\right), \forall((i,j),\xi) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n) \times (0,+\infty).$$

Furthermore, denote

$$\lambda_1 = \frac{pmn(m+n+2)^{p-1}\max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[0,c]} \int_0^\xi f((i,j),\tau)d\tau}{[4^p + q_*(m+n+2)^{p-1}]c^p},$$

$$\lambda_2 = \frac{p\left[\sum_{j=1}^n \sum_{i=1}^m \int_0^d f((i,j),\tau)d\tau - mn\ \max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[0,c]} \int_0^\xi f((i,j),\tau)d\tau\right]}{(2m+2n+Q)d^p}.$$

Then, for any $\lambda \in \Lambda_1 = (\frac{1}{\lambda_2}, \frac{1}{\lambda_1})$, problem $(P_\lambda^{f,q})$ has at least two positive solutions in X.

Moreover, denote

$$a = (2m+2n+Q)\left[4^p + q_*(m+n+2)^{p-1}\right](cd)^p,$$

$$b = p\left[4^p + q_*(m+n+2)^{p-1}\right]c^p \sum_{j=1}^n \sum_{i=1}^m \int_0^d f((i,j),\tau)d\tau - pmn(2m+2n+Q)(m+n+2)^{p-1}d^p \max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[0,c]} \int_0^\xi f((i,j),\tau)d\tau.$$

Then, for any $h > 1$, there exist an open interval $\Lambda_2 \subseteq [0, \frac{a}{b}h]$ and a positive real number σ such that, for each $\lambda \in \Lambda_2$, problem $(P_\lambda^{f,q})$ has at least two positive solutions in X and their norms are all less than σ .

Proof For any $(i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n)$ and $t \in \mathbb{R}$, we put

$$\begin{aligned} f^*((i,j),t) &= \begin{cases} f((i,j),t), & t > 0, \\ f((i,j),0), & t \leq 0, \end{cases} \\ F^*((i,j),t) &= \int_0^t f^*(i,j),\tau)d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} \max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[0,c]} F^*((i,j),\xi) &= \max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[0,c]} \int_0^\xi f((i,j),\tau)d\tau, \\ \sum_{j=1}^{n}\sum_{i=1}^{m} F^*((i,j),d) &= \sum_{j=1}^{n}\sum_{i=1}^{m} \int_0^d f((i,j),\tau)d\tau. \end{aligned}$$

In view of hypotheses (A_1^*) and (A_2^*), the conclusion of Theorem 2.1 holds for problem $(P_\lambda^{f^*,q})$. Further, by applying Lemma 3.1, we find that problem $(P_\lambda^{f^*,q})$ admits at least two positive solutions when λ belongs to intervals Λ_1 and Λ_2, respectively, which are exactly positive solutions of problem $(P_\lambda^{f,q})$. ■

Next, we study a special case in which f has separated variables. Specifically, we consider the following problem, namely $(P_\lambda^{\omega g,q})$:

$$\begin{cases} -\Delta_1\left[\phi_p\left(\Delta_1 x\left(i-1,j\right)\right)\right] - \Delta_2\left[\phi_p\left(\Delta_2 x\left(i,j-1\right)\right)\right] + q(i,j)\phi_p\left(x(i,j)\right) = \lambda\omega(i,j)g\left(x(i,j)\right), \\ (i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n), \end{cases}$$

with Dirichlet boundary conditions

$$\begin{aligned} x(i,0) &= x(i,n+1), \quad i \in \mathbb{Z}(0,m+1) \\ x(0,j) &= x(m+1,j), \quad j \in \mathbb{Z}(0,n+1) \end{aligned}$$

where $\omega : \mathbb{Z}(1,m) \times \mathbb{Z}(1,n) \to \mathbb{R}$ is nonnegative and non-zero, and $g : [0,+\infty) \to \mathbb{R}$ is a nonnegative continuous function.

Define

$$W = \sum_{j=1}^{n}\sum_{i=1}^{m} \omega(i,j), \; G(\xi) = \int_0^\xi g(s)ds.$$

Then we have the following result.

Corollary 3.2 *Assume that there exist four positive constants c, d, η, α satisfying $\alpha < p$ and*

$$d^p > \frac{[4^p + q_*(m+n+2)^{p-1}]c^p}{(2m+2n+Q)(m+n+2)^{p-1}}$$

such that

(A_1') $\max_{(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)} \omega(i,j) < \frac{[4^p+q_*(m+n+2)^{p-1}]c^pWG(d)}{mn\{[4^p+q_*(m+n+2)^{p-1}]c^p+(2m+2n+Q)(m+n+2)^{p-1}d^p\}G(c)}$;

(A_2') $G(\xi) \le \eta(1+|\xi|^\alpha), \forall \xi > 0.$

Furthermore, denote

$$\lambda_1 = \frac{pmn(m+n+2)^{p-1}G(c)\max_{(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)}\omega(i,j)}{[4^p+q_*(m+n+2)^{p-1}]c^p},$$

$$\lambda_2 = \frac{p[WG(d) - mnG(c)\max_{(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)}\omega(i,j)]}{(2m+2n+Q)d^p}.$$

Then, for any $\lambda \in \Lambda_1 = (\frac{1}{\lambda_2}, \frac{1}{\lambda_1})$, problem $(P_\lambda^{\omega g,q})$ has at least two positive solutions in X.

Moreover, denote

$$\begin{aligned} a &= (2m+2n+Q)\left[4^p + q_*(m+n+2)^{p-1}\right](cd)^p, \\ b &= p\left[4^p + q_*(m+n+2)^{p-1}\right]c^pWG(d) \\ &\quad -pmn(2m+2n+Q)(m+n+2)^{p-1}d^pG(c)\max_{(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)}\omega(i,j). \end{aligned}$$

Then, for any $h > 1$, there exist an open interval $\Lambda_2 \subseteq [0, \frac{a}{b}h]$ and a positive real number σ such that, for each $\lambda \in \Lambda_2$, problem $(P_\lambda^{\omega g,q})$ has at least two positive solutions in X and their norms are all less than σ.

Proof Set

$$f((i,j),s) = \begin{cases} \omega(i,j)g(s), & s \ge 0, \\ \omega(i,j)g(0), & s < 0, \end{cases} \tag{3.4}$$

for any $(i,j) \in \mathbb{Z}(1,m)\times\mathbb{Z}(1,n)$ and $s \in \mathbb{R}$. It is easy to verify that

$$f((i,j),0) = \omega(i,j)g(0) \ge 0, \forall (i,j) \in \mathbb{Z}(1,m)\times\mathbb{Z}(1,n),$$

$$\max_{((i,j),\xi)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)\times[0,c]} \int_0^\xi f((i,j),\tau)\,d\tau = G(c)\max_{(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)}\omega(i,j),$$

$$\sum_{j=1}^{n}\sum_{i=1}^{m}\int_0^d f((i,j),\tau)\,d\tau = WG(d).$$

Besides, we take $\mu = \eta \max_{(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)} \omega(i,j)$. The conclusion follows from Corollary 3.1 and taking into account 3.4. ■

3.4 An example

To illustrate our results, we present a concrete example.

Example 3.1 *Consider the problem* $(P_\lambda^{\omega g,q})$ *and take* $p=4$, $m=2$, $n=2$, $c=1$, $d=10$, $\eta=e^{12}$, $\alpha=1$ *and*

$$\begin{aligned} q(i,j) &= ij,\ \forall (i,j)\in\mathbb{Z}(1,2)\times\mathbb{Z}(1,2),\\ \omega(i,j) &= i+j,\ \forall (i,j)\in\mathbb{Z}(1,2)\times\mathbb{Z}(1,2),\\ g(s) &= \begin{cases} se^s, & 0\le s\le 9,\\ 9e^9, & s>9. \end{cases} \end{aligned}$$

Then we get $Q=9$, $W=12$, $q_*=1$, $\max_{(i,j)\in\mathbb{Z}(1,2)\times\mathbb{Z}(1,2)} \omega(i,j)=4$, *and*

$$G(\xi) = \begin{cases} (\xi-1)e^\xi+1, & 0\le\xi\le 9,\\ 9e^9\xi-73e^9+1, & \xi>9. \end{cases} \tag{3.5}$$

So $G(c)=1$, $G(d)=17e^9+1$. *Furthermore,*

$$\frac{[4^p+q_*(m+n+2)^{p-1}]c^p}{(2m+2n+Q)(m+n+2)^{p-1}} = \frac{472}{3672} < 104 = d^p$$

and

$$\frac{[4^p+q_*(m+n+2)^{p-1}]c^pWG(d)}{mn\{[4^p+q_*(m+n+2)^{p-1}]c^p+(2m+2n+Q)(m+n+2)^{p-1}d^p\}G(c)} = \frac{177(17e^9+1)}{4,590,059}.$$

Then the condition (A_1') *of Corollary* 3.2 *holds.*

Due to 3.5, *we have*

$$\begin{aligned} G(\xi) &= (\xi-1)e^\xi+1 \le 8e^9+1 < e121+|\xi| = \eta\left(1+|\xi|^\alpha\right),\ \forall 0<\xi\le 9;\\ G(\xi) &= 9e^9\xi-73e^9+1 < e^{12}\xi+e^{12} = e^{12}\left(1+|\xi|\right) = \eta\left(1+|\xi|^\alpha\right), \forall\xi>9, \end{aligned}$$

which indicate

$$G(\xi) \leq \eta\left(1+|\xi|^{\alpha}\right), \ \forall \xi > 0,$$

that is, the condition (A_2') *of Corollary* 3.2 *is fulfilled.*

Moreover,

$$\begin{aligned}
\lambda_1 &= \frac{pmn(m+n+2)^{p-1}G(c)\max_{(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)}\omega(i,j)}{[4^p+q_*(m+n+2)^{p-1}]c^p} = \frac{1728}{59}, \\
\lambda_2 &= \frac{p[WG(d)-mnG(c)\max_{(i,j)\in\mathbb{Z}(1,m)\times\mathbb{Z}(1,n)}\omega(i,j)]}{(2m+2n+Q)d^p} = \frac{51e^9-1}{10,625}.
\end{aligned}$$

By Corollary 3.2*, for any* $\lambda \in \Lambda_1 = \left(\frac{10,625}{51e^9-1}, \frac{59}{1728}\right)$*, the considered problem possesses at least two positive solutions in* X.

Besides, a *and* b *in Corollary* 3.2 *are*

$$a = 80,240,000, \quad b = 385,152e^9 - 2,350,057,344,$$

respectively. Therefore, for any $h > 1$*, there exist anopeninterval* $\Lambda_2 \subseteq [0, \frac{626,875}{3009e^9-18,359,823}h]$ *and a positive real number* σ *such that, for each* $\lambda \in \Lambda_2$*, the considered problem has at least two positive solutions in* X *and their norms are all less than* σ *.*

Conclusion

Through this study, it was confirmed and proven that there are multiple solutions for partial discrete Dirichlet boundary value problem with p-Laplacian that we discussed to achieve them gradually and in detail with the statement of the critical points theory relationship to solve such a problem. Moreover, based on the strong maximum principle, we come up with two positive solutions under some suitable non-linear assumptions, and then solve such complex equations that require deep and proven studies to reach the confirmed results mentioned above.

At last, further researches are recommended to enlarge this study and prove the existence of one solution for partial discrete Dirichlet boundary value problem with p-Laplacian for the investigated problem.

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