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Theme

Bifurcations in economic growth model transformed to ODE

Presented by:
Abdallah HABHOUB   Laila GATTAL

Dissertation jury members:

Elhadj ZERAOULIA	Pr	Larbi Tébessi University	President
Abderrazak NABTI	MCA	Larbi Tébessi University	Examiner
Khaled BERRAH	MCA	Larbi Tébessi University	Supervisor

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بسم الله الرحمن الرحيم

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Today, the journey is ending; it will have been long, strewn with pitfalls and often painful, but also bounded with love. However, this is only a step on the path of life.

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ملخص

يركز عملنا على دراسة مفصلة لورقتين علميتين لإحدى نماذج النمو اقتصادي، قمنا بدراسة ومناقشة المقال العلمي لـ Guerrini et al و الذي كان الجزء الرئيسي من مذكرتنا، يعتبر هذا العمل تعميما لعملKrawiecوSzydlowskiحيثبشكل مستقل حسب الفرضية الذي درس فيه الباحثون أن النموذج الديناميكية المعدل لـ Kaldorنتج عنها أن سلوك دورة النهاية مستقل مع وجود فرضية أن دالة الإستثمار تكون ذات شكل-sو خطية، كما إعتبروا أن وقت التأخير هو الوسيط الأساسي في دراسة التشعب و من نتائج الدراسة تحت هذه الفرضيات إنشاء دورة النهاية و توقع زمن الحرج T لظهور أول تشعب.

لما يكون T في تزايد، النظام يتشعب إلى سلوك دورة النهاية ثم دورة متعددة الدورية و الغير دورية أو يميل في النهاية نحو السلوك الفوضوي عبر فترة مضاعفة المسار التعاقبي للاضطراب.

Guerrini et al درسوا التشعب المتوقع لتغيير قيم الوسائط لنموذج نمو Kaldor-Kaleckiمع تأخير زمني في شكل دالة توزيع Gamma حول هذا النموذج لأنظمة ديناميكية مختارة ذات 3 و 4 أبعاد (تحصلنا على هذه الأنظمة بإستخدامlinearchain trick) أخذنا كوسيطفي دراستنا تأخير الزمن و نمو الإستثمار من أجل تحقيق إثبات وجود شروط تشعب Hopf . في الأخير ناقشنا النتائج العددية للدراستين و قمنا بربطها بالجانب التحليلي للدراسة.

كلمات مفتاحية: نموذج النمو الاقتصادي، نموذج Kaldor–Kalecki ، وسيط التأخير الزمن، وسيط دالة توزيع تأخير الزمن، تشعب Hopf

RÉSUMÉ

Notre travail se concentre sur l'étude détaillée de deux articles, où nous examinons et discutons l'article principal de Guerrini et al [?], qui est considéré comme une généralisation des travaux de Krawiec et Szydlowski [?]. Par la suite, les auteurs ont étudié la dynamique du modèle de Kaldor modifié et ont conclu en leur donnant le résultat que le comportement du cycle limite est indépendant de l'hypothèse que la fonction d'investissement est en s-shaped. Dans cette étude, on suppose que la fonction linéaire $I(Y)$, et seulement le paramètre de temps avec retard jouent un rôle essentiel dans la création du cycle limite. Ils ont également montré que pour un petit paramètre de retard, dans l'approximation linéaire, le modèle de Kaldor-Kalecki prend la forme de l'équation de Liénard. Une analyse de stabilité linéaire comprenant le temps de retard s'avère être un prédicteur précis du T critique pour la première bifurcation. Lorsque T est augmenté, le système bifurque vers un comportement de cycle limite, puis vers des cycles multiples périodiques et apériodiques ou tend finalement vers un comportement chaotique via la cascade de doublement de période vers la turbulence.
Guerrini et al ont étudié une bifurcation possible due à un changement des valeurs des paramètres du modèle de croissance de Kaldor-Kalecki, dans lesquels, ils considèrent deux cas les plus simples de systèmes dynamiques à trois et quatre dimensions, qui ont été obtenus par le mécanisme linear chain trick à partir du modèle de croissance de Kaldor-Kalecki avec un temps de retard distribué. Pour les deux modèles, les "conditions d'existence" de la bifurcation de Hopf par rapport au paramètre de temps de retard et au paramètre de taux de croissance sont établies. Enfin, nous discutons l'étude numérique dans les articles précédents.
Mots clés: Modèle de croissance économique, modèle de Kaldor-Kalecki, fonction d'investissement, paramètre de temps de retard, temps de retard distribué, bifurcation de Hopf.

ABSTRACT

Our work focuses on studying two papers deeply, where we scrutinize and discuss the main paper of Guerrini et al. [**?**], which is considered a generalization of Krawiec and Szydlowski work [**?**]. Whither, the authors investigated the dynamics of the modified Kaldor model and concluded giving them the result that the limit cycle behavior is independent of the assumption that the investment function is s-shaped. In this study, as an assumption, supposed the linear function $I(Y)$, and only the time-delay parameter play a crucial role in creating the limit cycle. Also, they showed that for a small time-delay parameter, in the linear approximation, the Kaldor–Kalecki model assumes the form of the Liénard equation. A linear stability analysis including the time delay is shown to be an accurate predictor of the critical T for the first bifurcation. As T is increased, the system bifurcates to the limit cycle behavior, then to multiple periodic and aperiodic cycles or eventually tends towards the chaotic behavior via the period doubling cascade route to the turbulence.
Guerrini et al studied a possible bifurcation expected to a change of the parameter values of the Kaldor-Kalecki growth model, which, they consider two simplest cases of three and four-dimensional dynamical systems, that were obtained through the linear chain trick from the Kaldor-Kalecki growth model with distributed delay. For both models, the existence conditions' of the Hopf bifurcation with respect to the time delay parameter and the rate of growth parameter are established. Finally, we discuss the numerical study in the previous papers.
Keywords: Economic growth model, Kaldor–Kalecki model, Investment function, time-delay parameter, distributed time delay, Hopf bifurcation.

CONTENTS

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GENERAL INTRODUCTION

Towards economic growth, how does determine the investment decisions?

In the economic theory, Investment is considered a subject of great importance, for this reason, we should make the best way to determine the investment decisions which lead to formulating a business cycle model and economic growth theories that follow the tradition established by Kalecki and Kaldor. As though a prototype of a dynamic system with cyclic, nominate Kaldor trade cycle model which their behavior nonlinearity plays a crucial role to generate the endogenous cycles. Around, the investment delay was assumed to be the average time of making an investment as it was proposed by Kalecki [4]. In view of the fact that, is natural to use the time delay differential equations to model economic phenomena. Since, in economics, many processes depend on past events. Many studies with good augmentations on The Kaldor-Kalecki business cycle model were realized. To modify This model used incorporating the exponential trend to characterize the growth of an economy [5]. in the same way, the new Kaldor-Kalecki growth model was formulated as the Kaldor growth model was obtained from the Kaldor business cycle model [6]. As more benefit secures in the research we are interested in our memoir on the models with distributed delays which are a more realistic description of economic systems with time delay. There are some examples of such models in the context of economic growth.
Our main study is on the paper of Guerrini et al. [2]. which is considered a generalization of the work of [3]. Whither, the authors investigated the dynamics of the modified Kaldor model and concluded giving them the result that the limit cycle behavior is independent of the assumption that the investment function is s-shaped. In this study, as an assumption, supposed the linear function $I(Y)$, and only the time-delay parameter play a crucial role in creating the limit cycle. Also, they showed that for a small time-delay parameter, in the linear approximation, the Kaldor–Kalecki model assumes the form of the Liénard equation. A linear stability analysis including the time delay is shown to be an accurate

predictor of the critical T for the first bifurcation. As T is increased, the system bifurcates to the limit cycle behavior, then to multiple periodic and aperiodic cycles or eventually tends towards the chaotic behavior via the period doubling cascade route to the turbulence.
Guerrini et al studied a possible bifurcation expected to a change of the parameter values of the Kaldor-Kalecki growth model. which, they consider two simplest cases of three and four-dimensional dynamical systems, that were obtained through the linear chain trick from the Kaldor-Kalecki growth model with distributed delay. For both models, the existence conditions' of the Hopf bifurcation with respect to the time delay parameter and the rate of growth parameter are established. also, they showed that both parameters play role in a plot leading to the Hopf bifurcation and arising cyclic behavior.
Our memoir consists of three chapters besides a general introduction and conclusion. In the first chapter, we present some concepts and preliminaries that we need in the following chapter, where we are interested in concepts such as existence and uniqueness of solutions, equilibria, stability, bifurcation ...
We provide the essentials of our work in the second chapter, whither, we focus our study with detail on the view analytic of two papers about Hopf bifurcation of the economic growth model with time delay.
Finally, we discuss the numerical study in the previous papers in the last chapter.

CHAPTER 1

GENERAL NOTIONS

In the first chapter, as in the usual memoir, we introduce some basic notions, and preliminaries wish will be used throughout this work. The first section is devoted to the linear system of differential equations notions employed in this work. The Bifurcation is presented in section 2. Finally, the Limit cycle introduced in section 3.

1.1 Linear system of differential equations

[7] Consider linear two-dimensional autonomous systems of the form:

$$\frac{dx}{dt} = \dot{x} = a_{11}x + a_{12}y, \qquad \frac{dy}{dt} = \dot{y} = a_{21}x + a_{22}y \tag{1.1}$$

where the a_{ij} are constants. The system is linear as the terms in $x, y, \dot{x}$, and $\dot{y}$ are all linear. System (1.1) can be written in the equivalent matrix form as

$$\dot{x} = Ax \tag{1.2}$$

where $x \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Definition 1.1 *Every solution of (1.1) and (1.2), say,$\Phi(t) = (x(t), y(t))$, can be represented as a curve in the plane. The solution curves are called trajectories or orbits.*

1.1.1 Existence and uniqueness of solutions

Theorem 1.1 *Existence and uniqueness[8]*

i) *If $f \in \mathbb{C}^0(\mathbb{R}, \mathbb{R})$, then, for any $x_0 \in \mathbb{R}$, there is an interval (possibly infinite) $I_{x_0} = (\alpha_{x_0}, \beta_{x_0})$ containing $t_0 = 0$ and a solution $\varphi(t, x_0)$ of the initial-value problem*

$$\dot{x} = f(x), x(0) = x_0,$$

defined for all $t \in I_{x_0}$ satisfying the initial condition $\varphi(0, x_0) = x_0$. Also, if $\alpha^+_{x_0}$ is finite, then

$$\lim_{t \to \alpha^+_{x_0}} \varphi(t, x_0) = +\infty.$$

or, if $\beta^-_{x_0}$ is finite, then

$$\lim_{t \to \beta^-_{x_0}} \varphi(t, x_0) = +\infty.$$

ii) *If, in addition,$f \in \mathbb{C}^1(\mathbb{R}, \mathbb{R})$, then $\varphi(t, x_0)$ is unique on I_{x_0} and $\varphi(t, x_0)$ is continuous in (t, x_0) together with its first partial derivatives, that is, $\varphi(t, x_0)$ is $\mathbb{C}^1$ function*

1.1.2 Equilibria

[9] The location of the equilibria is a relatively simple element of ODEs that can be observed without much analysis, equilibria for a system of the form

$$\begin{aligned} \dot{x_1} &= F_1(x_1(t), x_2(t)) \\ \dot{x_2} &= F_2(x_1(t), x_2(t)) \end{aligned} \tag{1.3}$$

witch are sometimes also referred as critical points of the system, are points $X \in \mathbb{R}^2$ where

$$F_1(X) = F_2(X) = 0$$

It is obvious that if $X \in \mathbb{R}^2$ is the system's equilibrium solution (1.3), and so the constant functions $x_1(t) = x_1$ and $x_2(t) = X_2$ define a solution $x(t) = (X_1, X_2)$. to the equation system. To solve the non linear algebraic equation F(x)=0, find these equilibrium solutions. This procedure begins by forming a mental image of the solution to a system such as (1.3)

1.1.3 Stability

[9] Most of the studies' analyses of the system of ODEs will be focused on whether or not the systems have stable equilibria . Accorded on the behavior of solutions whose initial conditions are in the neighborhood of the equilibrium, on the condition that the solution near a critical point of a system stays close to the critical point as t approaches infinity, then we classify an equilibrium as stable or unstable. The critical point is considered to be stable; nevertheless, if

this condition is not valid, so, the critical point becomes unstable. Furthermore, he term a stable critical point asymptotically stable if the solution approaches the critical point with time rather than merely remaining constant:

Definition 1.2 *Let $X \in \mathbb{R}^2$ be a critical point in an ODEs system of the form (1.3)*

- *The critical point X is stable if ,for any $\epsilon > 0$,there is a $\delta > 0$ such that if a solution $x = \phi(t)$ satisfies $\|\phi(0) - X\| < \delta$, then $\|\phi(t) - X\| < \epsilon$ for all $t > 0$ Here*

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$

The euclidean norm on $\mathbb{R}^2$ is represented by

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$

- *If the critical point X is not stable as defined above, it is unstable.*
- *If there is a $\delta > 0$ such that a solution $x = \phi(t)$ satisfies $\|\phi(0) - X\| < \delta$, then $\lim_{t\to\infty} \phi(t) = X$, then the critical point X is asymptotically stable.*

1.1.4 Classification of singular points of linear systems

[10] Recall that a point x_0 is called ordinary if $P(x)$ and $Q(x)$ from the equation

$$\ddot{y} + P(x)\dot{y} + Q(x)y = 0 \tag{1.4}$$

are both analytic at x_0. A point x_0 which is not ordinary is called singular point. In this portion we discuss the various phase portraits that are possible for the linear system . $\dot{x} = Ax$ When $x \in R^2$ and A is $2 * 2$ matrix
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = a + d, D = ad - bc,$

$$P(\lambda) = \lambda^2 - T\lambda + D \tag{1.5}$$

Case A
$T^2 - 4D > 0$ There are sub cases of case A

1. if D<0, saddle solely
2. D>0 there are four cases

i) $T > 0$ source.

ii) $T < 0$ sink.

iii) $T^2 > 4D$ node.

iv) $T^2 < 4D$ spiral(focus).

Border line case

i) If $T = 0$ and $D > 0$ center.

ii) If $T \neq 0$ and $T^2 > 4D$ saddle -node.

iii) If $T > 0$ unstable.

iv) If $T < 0$ stable.

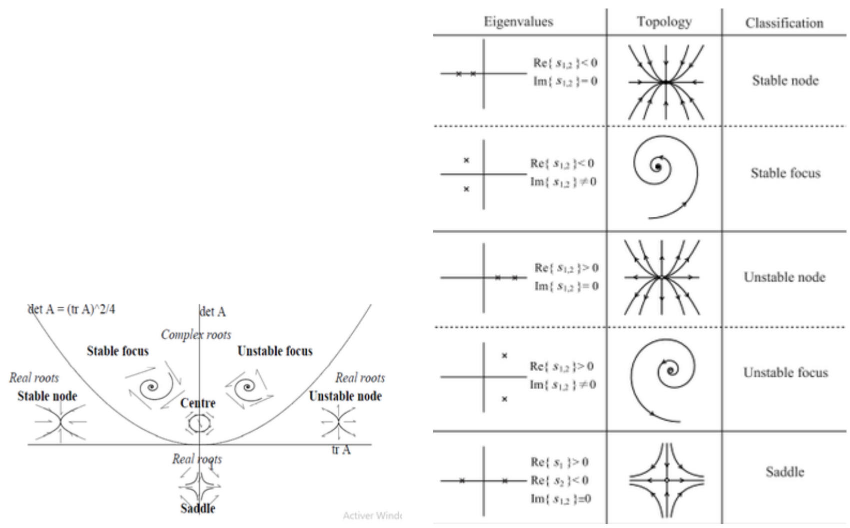

Figure 1.1: Classification of singular points of linear systems

Example 1.1 *Determine the linear system $Ax = x$ has saddle-node ,center at the origin and determine the stability of each node or focus.*

$A = \begin{pmatrix} 8 & 5 \\ -10 & -7 \end{pmatrix}$, $D = -6 < 0$ *the system is saddle at the origin,*

$A = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}$, $D = 2 > 0, T = -3 < 0$, *the system is node sink at the origin*

$A = \begin{pmatrix} -10 & -25 \\ 5 & 10 \end{pmatrix}$, $D = 25 > 0, T = 0$, *center and counter clock wise direction of rotation*

$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, $D = 8 > 0$,$T = 6 > 0$

case A

$T^2 - 4D$, *gives the real distinct eigenvalues*

λ_1 *and* $\lambda_2 = \dfrac{T \pm \sqrt{T^2 - 4D}}{2}$

Sub cases of case *A*

$\lambda_1 > 0 > \lambda_2$, *saddle* $\lambda_1 \geqslant \lambda_2 > 0$ *node saddle* $\lambda_1 \leqslant \lambda_2 < 0$ *sink*

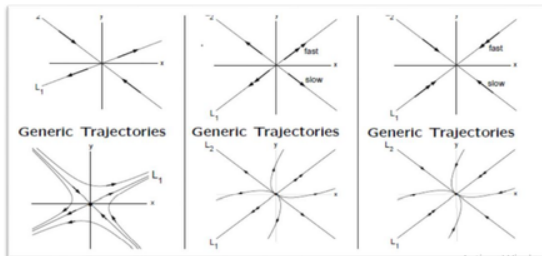

Figure 1.2: Classification of trajectory

1.1.5 Delay Differential Equations

Delay differential equation (DDE) is one of the mathematical models that commonly possess the result in differential equations with time delay. In general, the unknown function of this derivative equation not only depends on the current value but also depends on the past value .

Definition 1.3 *[7] A DDE subject to constant time delays is of the form :* $dx/dt = f(x(t), x(t-\tau_1), x(t-\tau_2), ..., x(t-\tau_n))$*, where* $x \in \mathbb{R}^n$ *and the delays*τ_i *are positive constants In order to solve DDEs it is necessary to define an initial history function which determines the behavior of the dynamical system* $x(t)$ *defined on the interval* $[-\tau, 0]$*, assuming that the systems start at* $t = 0$*. The simplest method for solving some systems of DDEs has been labeled as the method of steps. DDEs differ from ODEs in that the solution for the DDE can be thought of as a mapping from functions on an interval* $[t-\tau, t]$ *onto functions on an interval* $[t, \tau + t]$*. In some very simple cases, it is possible to work out an analytical solution to this problem as the following example demonstrates.*

There is a solution to this problem, as shown in the following example.

Example 1.2 *solve the simple linear DDE given by*

$$\frac{dx}{dt} = -x(t-1),$$

with initial history function $x(t) = 1$*, on* $[-1, 0]$.

solution. Suppose that $x(t) = \Phi_{i-1}(t)$ on the interval $[t_i - 1, t_i]$. Then using separation of variable, on the interval $[t_i, t_i + 1]$

$$\int_{\Phi_{i-1}}^{x(t)} \Phi_{i-1} d\dot{x} = -\int_{t_i}^{t} \Phi_{i-1}(\dot{t} - 1) d\dot{t}$$

and

$$x(t) = \Phi_i(t) = \Phi_{i-1}(t_i) - \int_{t_i}^{t} \Phi_{i-1}(\dot{t} - 1) d\dot{t}$$

Therefore, in the interval $[0, 1]$, equation gives $x(t) = 1 - \int_0^t d\dot{t} = 1 - t$ and in the interval[1,2] $x(t) = -2t + t^2/2 + 3/2$
One could continue to calculate the solution on further intervals by hand but the process can be easily implemented in Python.

1.2 Bifurcation

The practically common feature of the bifurcation point is that all the branches have represented equilibria that is, that signifies, that both branches intersecting in a bifurcation point have consisted of (stationary) solutions of the equation

$$\dot{y} = 0 = f(y, \lambda).$$

The term equilibrium characterizes a physical situation. Mathematically speaking, we say that the solutions on the emanating branch remain in the same "space" namely, furthermore, we have called a bifurcation that is characterized by intersecting branches of equilibrium points stationary bifurcation (also called steady-state bifurcation,[11], see Figure 1.3.

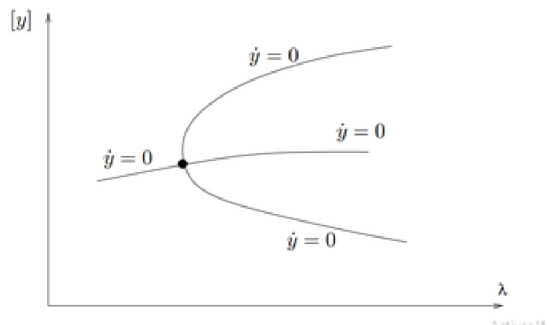

Figure 1.3: Stationary bifurcation

1.2.1 Poincare-Andronov-Hopf bifurcation

Now, We present a general characterization of Hopf bifurcation. Poincaré is initiated the basic results and known for him. After that, in 1929, Andronov is developed the planar case. Because of these early results, bifurcation from equilibria to limit cycles is also called Poincaré–Andronov–Hopf bifurcation. The commonly used name "Hopf bifurcation" may be seen as an abbreviation. It was Hopf who proved the following theorem for the n-dimensional case in 1942.

Theorem 1.2 *[11] Assume for* $f \in \mathbb{C}^2$

(1) $f(y_0, \lambda_0) = 0$,

(2) $f_y(y_0, \lambda_0)$ *has a simple pair of purely imaginary eigenvalues* $\mu(\lambda_0) = \pm i\beta$ *and no other eigenvalue with zero real part, and*

(3) $d(Re(\mu(\lambda_0)/d\lambda)) \neq 0$

Then there is a birth of limit cycle at (y_0, λ_0)*. The initial period (of the zero -amplitude oscillation) is* $T_0 = \frac{2\pi}{\beta}$

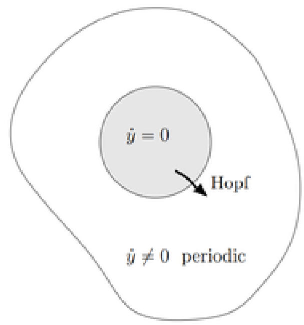

Figure 1.4: Complex plane of eigenvalues; path of eigenvalues $\mu(\lambda)$ related to Hopf bifurcation

Hypotheses (1), (2), and (3) can be viewed as a formal definition of Hopf bifurcation. Hypothesis (2) can be relaxed in that no eigenvalue is allowed that is an integer multiple of $\pm i\beta$. Condition (3) is the *transversality* hypothesis, it is " usually " satisfied.
The Hopf bifurcation is a critical point where a system's stability switches and a periodic solution arises. More exactly, it is a local bifurcation in which a fixed point of a dynamical system loses stability. As a simple definition for Hopf bifurcation, we can also give.

Definition 1.4 *[11] A bifurcation from a branch of equilibria to a branch of periodic oscillations is called Hopf bifurcation.*

Theorem 1.3 *Poincare-Andronov-Hopf bifurcation*
[7] Let $\dot{x} = A(\lambda)x+F(\lambda,x)$ be a C^k , with $k \geqslant 3$, planar vector field depending on a scalar parameter λ such that $F(\lambda,0) = 0$ and $D_xF(\lambda,0) = 0$ for all sufficiently small $|\lambda|$. Assume that the linear part $A(\lambda)$ at the origin has the eigenvalues $\alpha(\lambda) \pm i\beta(\lambda)$ with $\alpha(0) = 0$ and $\beta(0) \neq 0$. Furthermore, suppose that the eigenvalues cross the imaginary axis with non zero speed, that is,

$$\frac{d(\alpha)}{d\lambda}(0) \neq 0$$

Then,in any neighborhood U of the origin in $\mathbb{R}^2$. and any given $\lambda_0 > 0$ there is a X with $|\overline{\lambda}| < \lambda_0$ such that the differential equation $\dot{x} = A(\overline{\lambda})x + F(\overline{\lambda}, x)$ has a non trivial periodic orbit in U. It is remarkable that the essential hypotheses of the theorem concern only the linear part of the vector field. The requirement that the vector field vanish at the origin is inconsequential since it can always be satisfied with a change of variables around an arbitrary equilibrium point. To uncover some of the finer details of this bifurcation, such as the stability of the resulting periodic orbit, one must investigate the effects of the non linear terms.

1.2.2 Periodic Solution

In the following, we give the different conditions pending a differential equation that has a periodic solution [12].

Definition 1.5 *A system*

$$\dot{x} = A(t)x \tag{1.6}$$

with $A(t+T) = A(t)$ *for all and some* $T > 0$ *is called non critical relative to* T *if* (1.6) *has no periodic solution of period* T *other than the zero solution.*

Theorem 1.4 *Let*

$$\dot{x} = A(t)x + P(t) \tag{1.7}$$

where $p : (-\infty, \infty) \longrightarrow \mathbb{R}^n$, A *and* p *are continuous on* $(-\infty, \infty)$, *both are periodic of period* T,*and let* $\dot{x} = A(t)x$ *be non critical relative to* T. *then* $\dot{x} = A(t)x + P(t)$

1.2.3 Routh-Hurwitz Criterion and Bifurcation method

Routh-Hurwitz Criterion is a method to show the system stability by taking the coefficients of an equation characteristic without counting the roots, suppose the equation characteristic[13].

$$p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + K + a_{n-1}\lambda + a^n = 0$$

with $a_j$is a coefficient in a real number $j = 1, 2, K, n$. Under condition the value of $a_0 > 0$, is can be said that $p(\lambda)$ stable if all the roots have the parts real negative. Hurwitz matrix from the equation is:

$$H = \begin{bmatrix} a_1 & a_0 & 0 & 0 & K & 0 \\ a_3 & a_2 & a_1 & a_1 & K & 0 \\ a_5 & a_4 & a_3 & a_2 & K & 0 \\ a_7 & a_6 & a_5 & a_4 & K & 0 \\ M & M & M & M & M & M \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & K & K & a_n \end{bmatrix}$$

All element with subscript more than n or less than 0 can be replaced by zero.The principle of the essentials and must be found on stability is about the principle of sub determinant polynomial .
$D_1 = a_1$,

$$D_2 = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}$$

$$D_n = \begin{bmatrix} a_1 & a_0 & 0 & 0 & K & 0 \\ a_3 & a_2 & a_1 & a_1 & K & 0 \\ a_5 & a_4 & a_3 & a_2 & K & 0 \\ a_7 & a_6 & a_5 & a_4 & K & 0 \\ M & M & M & M & M & M \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & K & K & a_n \end{bmatrix}$$

Of the some explanation above,can be used each square , cubic, and quartic. Then it can be started from a quadratic equation,that is $p(\lambda) = a_0\lambda^2 + a_1\lambda + a_2$ obtained $D_1 = a_1 > 0$,

$$D_2 = \begin{vmatrix} a_1 & a_0 \\ 0 & a_2 \end{vmatrix} = a_1 a_2 > 0.$$

Quadratic equation $p(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ Routh-Huirwitz Criterion apply, i.e $D_1 = a_1 > 0$,

$$D_2 = \begin{vmatrix} a_1 & a_0 \\ 0 & a_2 \end{vmatrix} = a_1 a_2 > 0$$

. and

$$D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} = a_3 D_2 > 0$$

1.3 Limit cycle

A limit cycle is an isolated periodic solution limit cycles in planar differential systems commonly occur when modeling both the technological and natural sciences. Most of the early history in the theory of limit cycles in the plane was stimulated by practical problems.
For example the differential equation derived by Rayleigh, is given by:

$$\ddot{x} + \varepsilon(\frac{1}{3}(x)^2 - 1)(x) + x = 0 \tag{1.8}$$

Where $\ddot{x} = \frac{d^2x}{dt^2}$ *and* $\dot{x} = \frac{dx}{dt}$ Let $x = y$ then this differential equation can be written as a system of first order autonomous differential initial equation in the plan.

$$x = y, x = -x - \varepsilon(\frac{y^2}{3} - 1)y, \tag{1.9}$$

periodic behavior in the Rayleigh system (1.9) when $\in= 1$ following the invention of the triod vacuum, which was able to produce following differential equation to describe this phenomena

$$\ddot{x} = \varepsilon(x^2 - 1)\dot{x} + x = 0 \tag{1.10}$$

which can be written as planar system of the form (1.9) periodic behavior for system (1.10) when $\in = 5$ Class of differential equation that generalize(1.10)are those first investigated

$$\ddot{x} + f(x)x + g(x) = 0, \tag{1.11}$$

or in the phase plan

$$x = y, y = -g(x) - f(x)y \tag{1.12}$$

This system can be used to model mechanical system, where $f(x)$ is known as the damping term and $g(x)$ is called the restoring force or stiffness equation (1.12) is also used to model resistor inductor Capacitor circuits with non linear circuit elements, limit cycle of Liénard systems possible physical interpretations for the limit cycle behavior of certain dynamical systems limit cycles are common solution for all types times it becomes necessary to prove the existence and uniqueness.

1.3.1 Limit cycle problems for Liénard systems:

In [10]. El-Nour et al. provided a new contribution to this subject which can be also applied to Liénard differential systems with some kind of discontinuities. We consider for $x \in [a, b]$, where $-\infty < a < 0 < b < \infty$, the Liénard differential equation,

$$\ddot{x} - f(x)\dot{x} + g(x) = 0, \tag{1.13}$$

$$f(x) = \begin{pmatrix} f_1(x) & if x < 0, & g_1(x) & if x < 0, \\ f_2(x) & if x > 0, & g_2(x) & if x > 0, \end{pmatrix}. \tag{1.14}$$

Starting f_1, g_1 continuously differentiable in $[a, 0]$, and f_2 , g_2 continuously differentiable in $[0, b]$. Note that the functions f and g are not defined at $x = 0$, allowing for a jump discontinuity at the origin if we later define $f(0)$ and $g(0)$. Note that the functions f and g are not defined at $x = 0$ so that, if we eventually define $f(0)$ and $g(0)$, they are allowed to have a jump discontinuity at the origin. By using the classical Liénard plane we can obtain the equivalent differential system .

$$\dot{x} = F(x) - y, \dot{y} = g(x), \text{where } F(x) = \int_0^x f(s)ds, \tag{1.15}$$

and it understood that $F(0) = 0$ while $g(0)$ is not defined by now. This system has associated the vector filed

$$X(x) = (X_1(x) \quad if x < 0, X_2(x) \quad if x > 0,) \text{ where } X(x) = (F(x) - y g_i(x)) \tag{1.16}$$

with $x = (x, y)^t$ and standing $i = 1$ for $x \leq 0$, and $i = 2$ for $x \geq 0$.

1.3.2 Existence and uniqueness of limit cycles in the plan

Indispensable for defining some features of phase portraits for understanding the existence and uniqueness theorem. Assume that the existence and uniqueness theorem holds for all solutions considered, since their periodic orbits in the plane are special in that they divide the plane into a region inside the orbit and a region outside, it this makes it possible to obtain criteria for detecting the presence or absence of periodic orbits for second order systems, which have no generalizations to higher order systems theorem (Poincare – Bendixson criterion)[10].
Consider the system, $\dot{x} = f(x)$ and let M be a closed bounded subset of the plane such that M contains an equilibrium points, or contains only one equilibrium point such that Jacobean matrix $[\partial f/\partial x]$ at this points has eigenvalues with positive real parts. Every trajectory starting in M remains in M for all future time. Then M contains a periodic orbit of $\dot{x} = f(x)$.

Theorem 1.5 *Negative poincare-Bendixson If on simply connected region D of the plan, the expression $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ is not identically zero and dose not change sigh then the system $\dot{x} = f(x)$ has no periodic orbit lying entirely in D.*

Definition 1.6 *A limit cycle, say Γ is*

i) *The limit cycle is stable if $A^+(x) = \Gamma$ for every x in a neighborhood, Thus, nearby trajectories are attracted to the limit cycle.*

ii) *An unstable limit cycle if $A^-(x) = \Gamma$ for all x some neighborhood, Thus, nearby trajectories are repelled away from the limit cycle.*

iii) *A semi stable limit cycle if it is attracting on one side and repelling on the other.*

.

1.3.3 Limit cycle and bifurcation

Oscillation are one of the most important phenomena that occur in dynamical systems, A system oscillation when it has a non trivial periodic solution, $x(t + T) = x(t), \forall t \geq 0$ In a phase portrait an oscillation or periodic solution, like a closed curve [10].

i) $T > 0$ source,

ii) $T < 0$ sink,

iii) $T^2 > 4D$ node,

iv) $T^2 < 4D$ spiral(focus),

Example 1.3 *Vander pol oscillator Let* $\dot{x_1} = x_2$ *where,*

$$N_{x_0} = X \in \mathbb{R}^N |x - x_0| < \in \tag{1.17}$$

In the case $\in = 0$ *we have a continuum of periodic solutions, while in the second case* $\in \neq 0$ *there is only one. An isolated periodic orbit is called a limit cycle.*

CHAPTER 2

ANALYSIS OF ECONOMIC GROWTH MODEL

2.1 Introduction

In this chapter, we served and detailed two papers [3, 2] that were studied on the Hopf bifurcation of the Kaldor–Kalecki business cycle model. The former assumed the time delay was fixed, which allowed them to predict the occurrence of a limit cycle bifurcation for the time-delay parameter $T = T_{bif}$. The latter makes the study more generalized for the time delay distributed where the time delay parameter is a mean time delay of the gamma distribution. The authors reduce the system with distribution delay to both three and four-dimensional ODEs. Then, they studied the Hopf bifurcation in these systems for two parameters: the time delay parameter and the rate of growth parameter

2.2 Bifurcations of the Kaldor-Kalecki business cycle model with time delay

The authors apply the Hopf theorem for demonstrate that its conditions are fulfilled for a generic class of the parameters in the Kaldor–Kalecki model. Authors' analysis of the dynamics of the modified Kaldor model given that the limit cycle behavior is independent of the assumption that the investment function is s-shaped. In Their study, They assumed the linear function $I(Y)$, and only the time-delay parameter plays a crucial role in creating the limit cycle. The study highlights the linear stability including the time delay to be an accurate predictor of the critical T for the first bifurcation. As T increases, the system bifurcates into limit cycle behavior, then into multiple periodic and non-periodic cycles, or eventually approaches chaotic behavior through the period-doubling the waterfall path to turbulence. It is noteworthy that all these types of behavior (without limit cycles) are accompanied by Kalecki's time-delay parameter. We infer that the business cycle model has been reformulated on the basis of the Kaldor model and the time-delay parameter of Kalecki related to investment decisions generates limit cycles in the phase space. An important role in the creation of the limit cycle is the Kalecki time-delay parameter, rather than assuming the s-shaped investment function $I(Y)$.

There are three subsections in this section. We start by reconstruct the new model of business cycles based on the Kalecki theory in conjunction with the Kaldor two-dimensional model on the phase plane (Y, K). Afterward, the bifurcation theory does approve the investigation of the limit cycle behavior in The model.

In next section we investigate the Hopf Bifurcations of the Kaldor–Kalecki model of trade cycle with time delay.

2.3 The Kaldor–Kalecki model of trade cycle with delayed investment

Let the Kaldor macroeconomic business cycle model in the form of a two-dimensional autonomous dynamical system.

$$\frac{dY}{dt} = \alpha[I(Y,K) - S(Y,K)], \tag{2.1}$$

$$\frac{dK}{dt} = I(Y,K) - \delta K, \tag{2.2}$$

where I and S is the investment function and the savings function, Y is gross product, K is capital stock, is the goods market adjustment coefficient, and is the capital stock depreciation rate. Kaldor assumed that I is a nonlinear (s-shaped) investment function on Y. From the aspect of the existence and persistence of cycles in this model, the system (2.1)–(2.2) has been extensively investigated [1, 14]. The time delay parameter T influence on the investment decisions plays a crucial role in the Kalecki [9] theory. The three investment stages are identified as follows: investment orders I, production of capital goods A and deliveries of finished capital goods D. The change in the capital stock is due to the past investment orders,

$$\frac{dK(t)}{dt} = D(t) - U = I(t - T) - U,$$

where U denotes the capital depreciation

For formulating a new business cycle model, the authors assumed Keynes' proposition that the economy reaches the level of activity where savings and investment are equal. Then they apply Kalecki's idea of a time lag in the capital accumulation equation. After that, they couple the dynamic multiplier approach of the Kaldor model with the Kaleckian time delay in investment which plays a primary role in the capital accumulation process. Even the length of the lag varies between different types of equipment. Kalecki assumed an average lag between decision and implementation. Finally, they treat the lag as a constant. The Kaldor-Kalecki model is formulated as a time-delay differential equation system as follows

$$\frac{dY}{dt} = \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \tag{2.3}$$

$$\frac{dK}{dt} = I(Y(t-T), K(t)) - \delta K(t) \tag{2.4}$$

where the time delay T is const.

Investment depends on income at the time investment decisions are taken and on capital stock at the time investment is finished. The latter is a consequence of the fact that at time $t-T$, there are some investments which will be finished between $t-T$ and T. suppose that the capital stock produced in this period is taken into consideration when new investments are planned.

The savings function S depends only on Y and is linear such that $S_Y = \gamma \in (0,1)$. Additionally, we assume that the investment function $I(Y,K)$ separates with respect to its two arguments and $I_Y > 0$, $I(K)$ is linear such that $I_K = \beta < 0$, then $I(Y,K) = I(Y) + \beta K$. We have the Kaldor–Kalecki dynamical system after replacing the given assumption.

$$\dot{Y} \equiv \frac{dY}{dt} = \alpha I(Y(t)) + \alpha\beta K(t) - \alpha\gamma Y(t), \tag{2.5}$$

$$\dot{K} \equiv \frac{dK}{dt} = I(Y(t-T)) + (\beta - \delta)K(t), \tag{2.6}$$

and

$$\begin{aligned}
\ddot{Y} \equiv \frac{d^2Y}{dt^2} &= \alpha\frac{\partial I}{\partial Y}\dot{Y} + \alpha\beta\dot{K} - \alpha\gamma\dot{Y} \\
&= \alpha\frac{\partial I}{\partial Y}\dot{Y} + \alpha\beta(I(Y(t-T)) + (\beta-\delta)K(t)) - \alpha\gamma\dot{Y} \\
&= \alpha\frac{\partial I}{\partial Y}\dot{Y} + \alpha\beta I(Y(t-T)) + (\beta-\delta)(\dot{Y} - \alpha I(Y(t)) + \alpha\gamma Y(t)) - \alpha\gamma\dot{Y} \\
&= \alpha\frac{\partial I}{\partial Y)}\dot{Y} + \alpha\beta I(Y(t-T)) + (\beta-\delta)\dot{Y} - \alpha(\beta-\delta)I(Y(t)) + \alpha\gamma(\beta-\delta)Y(t) - \alpha\gamma\dot{Y} \\
&= \left(-\alpha\gamma + (\beta-\delta) + \alpha\frac{\partial I}{\partial Y}\right)\dot{Y} + \alpha\beta I(Y(t-T)) - \alpha(\beta-\delta)I(Y(t)) + \alpha\gamma(\beta-\delta)Y(t)
\end{aligned}$$

Equivalently

$$\ddot{Y}(t) + f(Y)\dot{Y} + g(Y) = 0 \tag{2.7}$$

where

$$f(Y(t)) = \alpha\gamma - \beta + \delta - \alpha\frac{\partial I}{\partial Y}.$$
$$g(Y(t), Y(t-T)) = -\alpha\beta I(Y(t-T)) + \alpha(\beta - \delta)I(Y(t)) - \alpha\gamma(\beta - \delta)Y(t).$$

2.3.1 The Liénard form of the Kaldor–Kalecki model

For the form of the Liénard equation in the system, we take a small value of the time-to-build parameter T, i.e. where $T \ll 1$. Assuming the nonlinear terms in (2.7), i.e.

$$\begin{aligned} g(Y(t)) &= -\alpha\delta I(Y(t)) - \alpha\gamma(\beta - \delta)Y(t) + \alpha\beta T\frac{I(Y(t)) - I(Y(t-T))}{T}. \\ &= -\alpha\delta I(Y(t)) - \alpha\gamma(\beta - \delta)Y(t) + \Delta h(Y(t)), \end{aligned}$$

where

$$\Delta h(Y(t)) = \alpha\beta T\frac{\Delta I}{\Delta t}$$

When T is small, $\Delta h(Y(t))$ can be stated in terms of a differential quotient and we obtain

$$\Delta h(Y(t)) \to dh(Y) = \alpha\beta T\frac{dI}{dt}, \qquad T \ll 1 \tag{2.8}$$

We replacing that $dI/dt = \dot{Y}\partial I/\partial Y$ for small T, in the linear approximation we obtain

$$\ddot{Y} + \alpha(\gamma + \beta T I_Y(Y) - I_Y(Y))\dot{Y} - \alpha\delta I(Y) - \alpha\gamma(\beta - \delta)Y = 0 \tag{2.9}$$

Equation (2.9) can be reduced to a two-dimensional autonomous dynamical system of the Liénard type. This equation will be called the Kaldor–Kalecki equation in the linear approximation for a small time-delay parameter.
The Kaldor–Kalecki equation (2.9) can be represented in the Liénard form

$$\dot{x} = y - \mathcal{F}(x), \qquad \dot{y} = -g(x),$$

where $\mathcal{F}(x) = \int_0^x f(u)du$, with modified function $f(Y)$, and consequently $\mathcal{F}(x)$

$$\begin{aligned} f(Y) &\to f(Y) + \alpha\beta T I_Y, \\ \mathcal{F}(Y) &\to \mathcal{F}(Y) + \alpha\beta I(x) \end{aligned}$$

In the general case of any finite time-delay parameter T, the dynamics is described by second-order time-delay differential equations which are equivalent to an infinite set of ordinary differential equations of first order.

If we write $\Delta I(Y)$ as a series

$$\Delta I(Y) = I(Y(t)) - I(Y(t-T)) = -\sum_{n\geq 1} \frac{(-1)^n}{n!} T^n \frac{d^n I(Y(t))}{dt^n},$$

Thus, The associated Kaldor–Kalecki equations have the form of a dynamical system with infinite dimensions. To see this fact, it is sufficient to define the subsequent derivatives of Y as phase variables.
Let us note that in the case of the Liénard system, if the investment function $I(Y)$ is nonlinear, the existence of a limit cycle can be deduced on the basis of the standard theorems.

2.3.2 Time delay parameter dependence in the Kaldor–Kalecki model

Because the oscillatory and chaotic tendencies of the Kaldor–Kalecki model should have some observationally interesting consequences. It's interesting looking more closely at the parameter connections.. The investment function in the Kaldor model was nonlinear with respect to gross product. The investment function becomes when We give up this assumption.

$$I(Y, K) = \eta Y + \beta K,$$

where $\eta > 0$. Assume that a solution of (2.7) in the form $Y(t) = e^{\lambda t}$, we derive the eigenvalue equation

$$\begin{aligned}
&\lambda^2 e^{\lambda t} + \lambda(-\alpha\eta + \alpha\gamma - (\beta - \delta))e^{\lambda t} - \alpha\gamma(\beta - \delta)e^{\lambda t} + \alpha\eta(\beta - \delta)e^{\lambda t} - \alpha\beta\eta e^{\lambda(t-T)} \\
=&\lambda^2 + \lambda(-\alpha\eta + \alpha\gamma - (\beta - \delta)) - \alpha\gamma(\beta - \delta) + \alpha\eta(\beta - \delta) - \alpha\beta\eta e^{-\lambda T}
\end{aligned}$$

$$\Leftrightarrow \lambda^2 + A\lambda + B + De^{-\lambda T} \tag{2.10}$$

where A, B and D are constant:

$$\begin{aligned}
A &= -\alpha\eta + \alpha\gamma - (\beta - \delta) \\
B &= -\alpha\gamma(\beta - \delta) + \alpha\eta(\beta - \delta) \\
D &= -\alpha\beta\eta
\end{aligned}$$

The exponential term in (2.10), As usual for time-delay equations, makes the equation transcendental. The first consequence is that usually we cannot solve the general eigenvalue equation analytically, and the second one is that there is an indefinite number of eigenvalues. The latter is a reflection of the fact that, like partial differential equations, the differential time-delay equations are equivalent to an indefinite set of ordinary differential equations.

Except for direct numerical integration, we have two essential tools: first, linear stability analysis, especially in the case $T \ll 1.$, and secondly, the formal results of the bifurcation theory that have been extended to time-delay equations, notably including the Hopf bifurcation theorem.

Linear stability analysis

Considering the most important limit case when $T \ll 1$. Then, $\exp(-\lambda T) \approx 1 - \lambda T$ and the eigenvalue equation becomes

$$\lambda^2 + (A - DT)\lambda + B + D = 0 \tag{2.11}$$

Now this is a simple quadratic equation in λ. We obtain from this expression the real parts of eigenvalues are negative at very small values of T (if $A > 0$), but above a critical value of T_{bif}, at least one real part becomes positive. However, because the linearization theorem does not hold for time-delay differential equations, the utility of the small T expression is severely constrained. Any eigenvalue may be written in terms of its real and imaginary parts,$\lambda = \sigma + i\omega$, and the real and imaginary parts of the general eigenvalue equation can be written as well. For small $T \ll 1$,we have

$$\begin{aligned} &(\sigma + i\omega)^2 + (A - DT)(\sigma + i\omega) + B + D \\ =&\sigma^2 - \omega^2 + 2i\sigma\omega + \sigma(A - DT) + i\omega(A+ \\ &DT) + B + D = 0. \end{aligned}$$

Equivalently

$$\sigma^2 - \omega^2 + \sigma(A - DT) + B + D = 0, \tag{2.12}$$

$$2\sigma\omega + \omega(A - DT) = 0, \tag{2.13}$$

then

$$\begin{aligned} \sigma =& \frac{DT - A}{2}, \\ T = T_{bif} \Leftrightarrow& \sigma = 0, \\ T_{bif} =& \frac{A}{D} = \frac{-\alpha\eta + \alpha\gamma - (\beta - \delta)}{-\alpha\beta\eta}, \end{aligned}$$

Therefore, if $A > 0$, then there is a bifurcation parameter $T = T_{bif}$ for which the real part of the eigenvalue changes sign. Let us note that $\sigma(T)$ is an increasing function of its argument T such that $\partial\lambda/\partial T|_{T=T_{bif}} > 0$. The imaginary part of the eigenvalues takes the form

$$\omega = \sqrt{B + D}, \tag{2.14}$$

where the condition $B + D \geq 0$ is equivalent to

$$\frac{\gamma(\delta - \beta)}{\delta} \geq \eta,\ \delta \neq 0.$$

Obviously, the value of ω for $T = T_{bif}$ is positive. This fact ensures that the bifurcation to a limit cycle when $T = T_{bif} = A/D$. The function $\sigma(T)$ is increasing and linear with respect to the argument. In next section we investigate the Hopf Bifurcations in economic growth model with distributed time delay.

2.4 Bifurcations in economic growth model with distributed time delay

In this section, the authors consider the economic growth model with a time-delayed investment function. They have assumed the investment is time distributed and used the linear chain trick technique to transform the delay differential equation system into an equivalent system of the ordinary differential system (ODE). The time delay parameter is the mean time delay of the gamma distribution. Furthermore, reduce the system with distribution delay to three- and four-dimensional ODEs. Then, they have studied the Hopf bifurcation in these systems concerning the time delay parameter and the rate of growth parameter. They have obtained sufficient criteria for the existence and stability of a limit cycle solution through the Hopf bifurcation.

2.4.1 Model

Krawiec and Szydlowski [5] developed an economic growth cycle model driven by investment delay based on the Kaldor business cycle model with two modifications: exponential growth introduced by Dana and Malgrange [6] and Kaleckian investment time delay [3]. This model of economic growth is described by the following system of differential equations with time delay $\tau \geq 0$

$$\dot{y}(t) = \alpha[I(y(t), k(t)) - \gamma y(t) + G_0] - gy(t) \tag{2.15}$$

$$\dot{k}(t) = I(y(t-\tau), k(t)) - (g+\delta)k(t) \tag{2.16}$$

where $I(y(t), k(t)) = k(t)\Phi(y(t)/k(t))$, with

$$\Phi(y(t)/k(t)) = c + \frac{d}{1 + e^{-a(vy/k-1)}}$$

where I is the investment and S is the saving function, y is gross product, k is capital stock α is the adjustment coefficient in the goods market, and δ is the depreciation rate of capital stock, and $\gamma \in (0, 1)$. It can be found that the system has a unique fixed point (y^*, K^*) where with positive coordinates, where

$$y^* = x^* k^* \wedge k^* = \frac{\alpha G_0}{gx^* + \alpha[\gamma x^* - (g+\delta)]}.$$

So, we can calculate it as follow,

$$\alpha[I(y^*, k^*) - \gamma y^* + G_0] - gy^* = 0 \tag{2.17}$$

$$I(y^*, k^*) - (g+\delta)k^* = 0 \tag{2.18}$$

Let x^* be the unique solution of $\Phi(y(t)/k(t)) = g + \delta$ we have

$$\begin{aligned} (2) &\Leftrightarrow k^*\Phi(y^*/k^*) - (g+\delta)k^* = 0 \\ &\Leftrightarrow k^*(\Phi(y^*/k^*) - (g+\delta)) = 0 \\ &\Leftrightarrow \Phi(y^*/k^*) - (g+\delta) = 0 \\ &\Leftrightarrow \Phi(y^*/k^*) = (g+\delta) \end{aligned}$$

the $y^*/k^* = x^*$, hence $y^* = x^*k^*$ using (2.17) in (2.18) we get:

$$\begin{aligned}
&\alpha[k^*(\Phi(y^*/k^*) - \gamma y^* + G_0] - gy^* = 0\\
\Rightarrow &\alpha[k^*(g+\delta) - \gamma x^*k^* + G_0] - gx^*k^* = 0\\
\Rightarrow &k^*[\alpha((g+\delta) - \gamma x^*) - gx^*] + \alpha G_0 = 0\\
\Rightarrow &k^* = \frac{-\alpha G_0}{\alpha((g+\delta) - \gamma x^*) + gx^*} = \frac{\alpha G_0}{gx^* + \alpha[\gamma x^* - (g+\delta)]}
\end{aligned}$$

Because of the S-shape of function $\Phi(x^*)$, we have that x^* always exists and the values of y^* and k^* depend only on x^* (in our case, $c < g + \delta < c + d$). Notice that, for economic considerations, the investment function $I(y(t), k(t))$ is such that

$$I_y^* = I_y(y^*, k^*) = \frac{adve^{-a(vx^*-1)}}{(1 + e^{-a(vx^*-1)})^2} \geq 0 \tag{2.19}$$

$$I_k^* = I_k(y^*, k^*) = g + \delta - x^* I_y^* \leq 0. \tag{2.20}$$

Furthermore, with a simple calculation we get:

$$\begin{aligned}
I_y^* &= I_y(y^*, k^*)\\
I(y(t), k(t)) &= k(t)(\Phi(y(t)/k(t)) = k(t)\left(c + \frac{d}{1 + e^{-a(vy/k-1)}}\right)\\
I_y(y(t), k(t)) &= k(t)(\Phi_y(y(t)/k(t)) = k(t)\left(\frac{\frac{dav}{k(t)}e^{-a(v\frac{y(t)}{k(t)}-1)}}{(1 + e^{-a(vy(t)/k(t)-1)})^2}\right)\\
I_y^* &= \frac{dave^{-a(vx^*-1)}}{(1 + e^{-a(vx^*-1)})^2} > 0\\
I_y^* &= -av(\Phi(x^*) - c)(\Phi(x^*) - (d + c))\\
I_y^* &> 0 \Rightarrow c < g + \delta < c + d
\end{aligned}$$

The previous model generalized by replacing the time delay in (2.16) with a distributed delay as follows,

$$\dot{y}(t) = \alpha[I(y(t), k(t)) - \gamma y(t) + G_0] + gy(t) \tag{2.21}$$

$$\dot{k}(t) = I\left(\int_{-\infty}^{t} y(r)\kappa(t - r)dr, k(t)\right) - (g + \delta)k(t), \tag{2.22}$$

where $\kappa(.)$ is a gamma distribution, i.e

$$\kappa(\xi) = \left(\frac{m}{T}\right)_m \frac{\xi^{m-1}e^{-\frac{m}{T}\xi}}{(m-1)!},$$

for m a positive integer defines the form of the weighting function. $T \geqslant 0$ is a parameter related to the mean time delay of the distribution. Notice that when $T \to 0$ the distribution function approximates the Dirac distribution, and thus, one recovers the time-delay case. Henceforth, we will consider only the cases $m = 1$ (weak delay kernel) and $m = 2$ (strong delay kernel). Using the so-called linear chain trick technique [12], system (2.21)-(2.22) can be transformed into equivalent systems of ODEs. More precisely, defining the new variable

$$
\begin{aligned}
u(t) &= \int_{-\infty}^{t} y(r)\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}dr \\
\dot{u}(t) &= \frac{d}{dt}\left(\int_{-\infty}^{t} y(r)\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}dr\right) \\
&= \int_{-\infty}^{t} y(r)\left(\frac{1}{T}\right)\frac{d}{dt}e^{-\frac{1}{T}(t-r)}dr + e^{-\frac{1}{T}(t)}\frac{d}{dt}\left(\int_{-\infty}^{t} y(r)\left(\frac{1}{T}\right)e^{\frac{r}{T}}dr\right) \\
&= -\frac{1}{T}\int_{-\infty}^{t} y(r)\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}dr + \frac{1}{T}y(t) \\
&= \frac{1}{T}[y(t) - u(t)]
\end{aligned}
$$

one has the system (case $m = 1$)

$$\dot{y}(t) = \alpha[I(y(t), k(t)) - \gamma y(t) + G_0] + gy(t), \tag{2.23}$$

$$\dot{u}(t) = \frac{1}{T}[y(t) - u(t)], \tag{2.24}$$

$$\dot{k}(t) = I(u(t), k(t)) - (g + \delta)k(t), \tag{2.25}$$

while defining the new variables

$$p(t) = \int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right)^2 (t-r)e^{-\frac{2}{T}(t-r)}dr$$

and

$$w(t) = \int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right)e^{-\frac{2}{T}(t-r)}dr$$

Let's calculate $\dot{p}(t)$and$\dot{w}(t)$

$$\begin{aligned}
\dot{p}(t) &= \frac{d}{dt}\int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right)^2 (t-r)e^{-\frac{2}{T}(t-r)}dr \\
&= \int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right)^2 \frac{d}{dt}\left((t-r)e^{-\frac{2}{T}(t-r)}\right) dr + te^{\frac{-2t}{T}}\frac{d}{dt}\int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right)^2 e^{\frac{2r}{T}}dr \\
&- e^{\frac{-2t}{T}}\frac{d}{dt}\int_{-\infty}^{t} ry(r)\left(\frac{2}{T}\right)^2 e^{\frac{2r}{T}}dr \\
&= \int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right)^2 e^{-\frac{2}{T}(t-r)}dr - \frac{2}{T}\int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right)^2 (t-r)e^{-\frac{2}{T}(t-r)}dr \\
&+ \left(\frac{2}{T}\right)^2 ty(t) - \left(\frac{2}{T}\right)^2 ty(t) \\
&= \frac{2}{T}[w(t)-p(t)]. \\
\dot{w}(t) &= \frac{d}{dt}\int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right) e^{-\frac{2}{T}(t-r)}dr \\
&= \frac{-2}{T}\int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right) e^{-\frac{2}{T}(t-r)}dr + e^{\frac{-2t}{T}}\frac{d}{dt}\int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right) e^{\frac{2r}{T}}dr \\
&= \frac{-2}{T}\int_{-\infty}^{t} y(r)\left(\frac{2}{T}\right) e^{-\frac{2}{T}(t-r)}dr + y(t)\left(\frac{2}{T}\right) \\
&= \frac{2}{T}[y(t)-w(t)]
\end{aligned}$$

one obtains the system (case $m = 2$)

$$\dot{y}(t) = \alpha[I(y(t),k(t)) - \gamma y(t) + G_0] + gy(t), \tag{2.26}$$

$$\dot{p}(t) = \frac{2}{T}[w(t)-p(t)], \tag{2.27}$$

$$\dot{w}(t) = \frac{2}{T}[y(t)-w(t)], \tag{2.28}$$

$$\dot{k}(t) = I(p(t),k(t)) - (g+\delta)k(t), \tag{2.29}$$

In the following, stability and Hopf bifurcation of systems $(2.23)-(2.25)$ and $(2.26)-(2.29)$ are analyzed by assigning the eigenvalues of linear systems around the critical point (y^*, y^*, k^*) and (y^*, y^*, y^*, k^*) respectively.

2.4.2 The time delay bifurcation analysis

Case $m = 1$

The characteristic equation of the linearised system (2.23)−(2.25) at the critical point $(y^*, u^*, k^*)$where $u^* = y^*$, is given by

$$\begin{vmatrix} \alpha I_y^* - \alpha\gamma - g - \lambda & 0 & \alpha I_k^* \\ \frac{1}{T} & -\frac{1}{T} - \lambda & 0 \\ 0 & I_y^* & I_k^* - (g+\delta) - \lambda \end{vmatrix} = 0, \tag{2.30}$$

After calculating and simplifying (2.30) where λ is the denotes a characteristic root , we get

$$\lambda^3 + a_1(T)\lambda^2 + a_2(T)\lambda + a_3(T) = 0, \tag{2.31}$$

where

$$a_1(T) = \frac{1}{T} - A, \qquad a_2(T) = \frac{A}{T} - B, \qquad a_3(T) = \frac{1}{T}(B - \alpha I_k^* I_y^*).$$

with

$$A = \alpha(I_y^* - \gamma) - g - x^* I_y^* \qquad \text{and} \qquad B = \left[\alpha(I_y^* - \gamma) - g\right] x^* I_y^*.$$

A necessary and sufficient condition for the local stability of the equilibrium point is that all the real parts of the characteristic roots of (2.31) are negative, when Routh-Hurwitz condition, is equivalent to $a_1(T) > 0$, $a_3(T) > 0$ and $a_1(T)a_2(T) > a_3(T)$. So, $a_2(T) > 0$ is necessarily satisfied. Let us examine whether these inequalities hold. First, we notice that $A < 0$. In fact, by contradiction, if $A = 0$, then $a_2(T) = -\left[x^* I_y^*\right]^2 < 0$. On the other hand, if $A > 0$, then $B > 0$, and so $a_2(T) < 0$. The fact $A < 0$ implies that $a_1(T) > 0$ holds always true, while the inequality $a_3(T) > 0$ is valid if and only if $B + \alpha I_k^* I_y^* < 0$. Thus, $a_3(T) > 0$ is always satisfied when $B \geq 0$, and it is verified for $(g + \delta) - (g + \alpha\gamma)x^* < 0$ when $B > 0$. Finally, let us consider $a_1(T)a_2(T) > a_3(T)$. Since

$$\begin{aligned} a_1(T)a_2(T) - a_3(T) &= \left(\frac{1}{T} - A\right)\left(-\frac{A}{T} - B\right) - \frac{1}{T}\left(-B - \alpha I_k^* I_y^*\right) \\ &= -A\left(\left(\frac{1}{T}\right)^2 + \left(\frac{B}{A} - A\right)\frac{1}{T} - B\right) - \frac{1}{T}\left(-B - \alpha I_k^* I_y^*\right) \\ &= -A\left(\frac{1}{T}\right)^2 + \left(A^2 + \alpha I_k^* I_y^*\right)\frac{1}{T} + AB \\ &= \frac{(AB)T^2 + \left(A^2 + \alpha I_k^* I_y^*\right)T - A}{T^2} \end{aligned}$$

the sign of $a_1(T)a_2(T) - a_3(T)$ depends on the sign of $(AB)T^2 + \left(A^2 + \alpha I_k^* I_y^*\right)T - A$, which is a quadratic polynomial in T. We have now several cases.

i) If $B = 0$, then $a_1(T)a_2(T) - a_3(T) > 0$ holds true if $A^2 + \alpha I_k^* I_y^* \geq 0$ or if $A^2 + \alpha I_k^* I_y^* < 0$ and $T < A/A^2 + \alpha I_k^* I_y^*).T = T_0^*$.

ii) If $B > 0$, then $AB < 0$ and $-A > 0$. By Descartes' rule of signs, we find that the polynomial $(AB)T^2 + \left(A^2 + \alpha I_k^* I_y^*\right) T - A$ has exactly one positive root $T = T_1^*$. Hence, $a_1(T)a_2(T) - a_3(T) > 0$ if $0 < T < T_1^*$.

iii) If $B < 0$, then $AB > 0$ and $-A > 0$. Applying again the Descartes' rule of signs we see that $(AB)T^2 + \left(A^2 + \alpha I_k^* I_y^*\right) T - A$ has (two) sign changes only if $A^2 + \alpha I_k^* I_y^* < 0$, meaning that this polynomial may have two positive roots $T_2^* < T_3^*$. If this happens, then $a_1(T)a_2(T) - a_3(T) > 0$ if $0 < T < T_2^*$ and $T > T_3^*$.

Let $T = T_*$ such that $a_1(T_*)a_2(T^*) - a_3(T_*) = 0$, namely $T_* = T_j^* (j = 0, 1, 2, 3)$. The curve $T = T_*$ divides the parameter space into stable and unstable parts. Choosing T as a bifurcation parameter, we apply the Hopf bifurcation theorem to establish the existence of a cyclical movement. This theorem asserts the existence of the closed orbit if the characteristic equation (2.31) has a pair of purely imaginary roots and a non-zero real root and if the real part of the imaginary roots is not stationary concerning the parameter changes T. At the critical value $T = T_*$, Eq. (2.31) factors as $[\lambda + a_1(T^*)] \left[\lambda^2 + a_2(T^*)\right] = 0$. Thus, we have the following three roots $\lambda_{1,2} = \pm i\sqrt{a_2(T^*)} = \pm i\omega_*$ and $\lambda_3 = -a_1(T^*) < 0$. Subsequently, let us examine the sign of the real parts of these roots as T varies. A differentiation of (2.31) concerning T, we get

$$\left[3\lambda^2 + 2\lambda a_1(T) + a_2(T)\right] \frac{d\lambda}{dT} = -\left[a_1^{'}(T)\lambda^2 + a_2^{'}(T)\lambda + a_3^{'}(T)\right], \tag{2.32}$$

where

$$\begin{aligned} a_1^{'}(T) &= -\frac{1}{T^2} < 0, \\ a_2^{'}(T) &= \frac{A}{T^2} < 0, \\ a_3^{'}(T) &= -\frac{1}{T^2}(-B - \alpha I_k^* I_y^*) = -\frac{a_3(T)}{T} < 0, \end{aligned}$$

Then, from (2.32), we get

$$
\begin{aligned}
\left(\frac{d\lambda}{dT}\right)_{T=T_*} &= -\frac{a_1^{'}(T)(i\sqrt{a_2(T_*)})^2 + a_2^{'}(T)(i\sqrt{a_2(T_*)}) + a_3^{'}(T)}{3(i\sqrt{a_2(T_*)})^2 + 2(a_1(T)i\sqrt{a_2(T_*)}) + a_2(T)} \\
&= -\frac{-a_1^{'}(T)a_2(T_*) + a_2^{'}(T)(i\sqrt{a_2(T_*)}) + a_3^{'}(T)}{-3a_2(T_*) + 2i(a_1(T)\sqrt{a_2(T_*)}) + a_2(T)} \\
&= -\frac{\left(-a_1^{'}(T_*)a_2(T_*) + a_3^{'}(T_*) + ia_2^{'}(T_*)\sqrt{a_2(T_*)}\right)\left(-a_2(T_*) - ia_1(T_*)\sqrt{a_2(T_*)}\right)}{2\left(-a_2(T_*) + ia_1(T_*)\sqrt{a_2(T_*)}\right)\left(-a_2(T_*) - ia_1(T_*)\sqrt{a_2(T_*)}\right)} \\
Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} &= -\frac{a_1^{'}(T_*)a_2^2(T_*) - a_2(T_*)a_3^{'}(T_*) + a_1(T_*)a_2^{'}(T_*)a_2(T_*)}{2\left(a_2^2(T_*) + a_1^2(T_*)a_2(T_*)\right)} \\
&= -\frac{a_1^{'}(T_*)a_2(T_*) + a_1(T_*)a_2^{'}(T_*) - a_3^{'}(T_*)}{2\left(a_2(T_*) + a_1^2(T_*)\right)}.
\end{aligned}
$$

Since

$$
\begin{aligned}
-a_1^{'}(T_*)a_2(T_*) - a_1(T_*)a_2^{'}(T_*) + a_3^{'}(T_*) &= \frac{1}{T^2}\left(\frac{-A}{T} - B\right) - \frac{A}{T^2}\left(\frac{1}{T} - A\right) - \frac{1}{T^2}(-B - \alpha I_k^* I_y^*) \\
&= \frac{1}{T^2}\left(-\frac{A}{T} - \frac{A}{T} + A^2 + \alpha I_k^* I_y^*\right) \\
&= \frac{1}{T^2}\left(-\frac{A}{T} + \frac{-A + \left(A^2 + \alpha I_k^* I_y^*\right)T}{T}\right) \\
&= \frac{1}{T^2}\left(-\frac{A}{T} - \frac{(AB)T^2}{T}\right),
\end{aligned}
$$

we obtain

$$
\text{sign}\left[Re\left(\frac{d\lambda}{dT}\right)_{T=T_*}\right] = \text{sign}\left(BT_*^2 + 1\right).
$$

If $B \geq 0$, we observe that $Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} > 0$ (with $T_* = T_0^*, T_1^*$) holds always true, whether if $B < 0$, then $Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} > 0$ (with $T_* = T_2^*, T_3^*$) if $0 < T < \frac{1}{\sqrt{-B}}$, and $Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} < 0$ if $T > \frac{1}{\sqrt{-B}}$. The results of the previous analysis are as follows.

Theorem 2.1 *Let $A < 0$, with A defined as in (2.31) .*

1) *If $B = 0$ and $A^2+\alpha I_k^* I_y^* < 0$ or if $B > 0$ and $B+\alpha I_k^* I_y^* < 0$, then there exists $T = T_* > 0$ such that the equilibrium point (y^*, y^*, k^*) of (2.23) – (2.25) is locally asymptotically stable for all $T < T_*$ and unstable for $T > T_*$. System (2.23) – (2.25) undergoes a Hopf bifurcation at (y^*, y^*, k^*) when $T = T_*$.*

2) *If $B < 0$, then there exists $0 < T_2^* < T_3^*$ such that the equilibrium point (y^*, y^*, k^*) of (2.23) – (2.25) is locally asymptotically stable for all $T < T_2^*$*

and $T < T_3^*$, and unstable for all $T_2^* < T < T_3^*$. A comparison of $\frac{1}{\sqrt{-B}}$ with T_2^* and T_3^* yields that system $(2.23) - (2.25)$ undergoes a Hopf bifurcation at (y^*, y^*, k^*) when $T = T_2^*$ or $T = T_3^*$ or $T = T_2^*$ and $T = T_3^*$.

Figure 1 shows the bifurcation diagram for the time delay parameter T of system $(2.23) - (2.25)$.

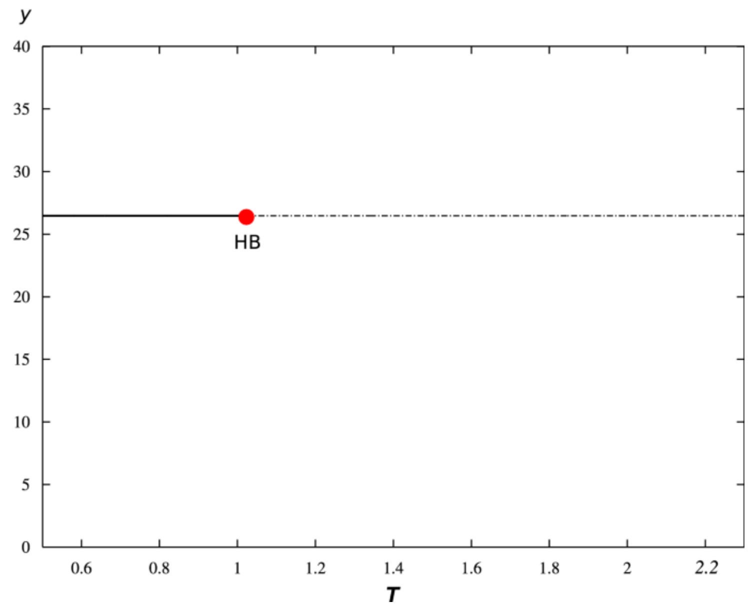

Figure 2.1: The bifurcation diagram for model for system $(2.23) - (2.25)$ $m = 1$ with investment function (2.31) for delay parameter T. The solid line indicates a critical point with asymptotic stability and the dot-dash line corresponds to an unstable critical point with a limit cycle around it.

Case $m = 2$

The characteristic equation of the linearised system $(2.26) - (2.29)$ at the critical point (y^*, p^*, w^*, k^*), where $p^* = w^* = y^*$, takes the form

$$\begin{vmatrix} \alpha I_y^* - \alpha\gamma - g - \lambda & 0 & 0 & \alpha I_k^* \\ 0 & -\frac{2}{T} - \lambda & \frac{2}{T} & 0 \\ \frac{2}{T} & 0 & -\frac{2}{T} - \lambda & 0 \\ 0 & I_y^* & 0 & I_k^* - (g+\delta) - \lambda \end{vmatrix} = 0, \qquad (2.33)$$

where I_y^* and I_k^* are defined as in (2.19) and (2.20), resulting in the fourth-order algebraic equation in *lambda* as follows:

$$\lambda^4 + a_1(T)\lambda^3 + a_2(T)\lambda^2 + a_3(T)\lambda + a_4(T) = 0 \qquad (2.34)$$

$$a_1(T) = \frac{4}{T} - (M+N), a_2(T) = \frac{4}{T^2} - \frac{4(M+N)}{T} + MN,$$

and

$$a_3(T) = \frac{4}{T}\left[MN - \frac{M+N}{T}\right], a_4(T) = \frac{4(MN+P)}{T^2},$$

with

$$M = \alpha(I_y^* - \gamma) - g \qquad N = I_k^* - (g+\delta) < 0 \qquad P = -\alpha I_k^* I_y^* > 0 \qquad (2.35)$$

The equilibrium point (y^*, y^*, k^*) of the system (2.30) is locally asymptotically stable under the routh-hurwitz conditions for stable roots if $a_1(T) > 0, a_3(T) > 0, a_4(T) > 0$ and $a_1(T)a_2(T)a_3(T) > a_3^2(T) + a_1^2(T)a_4(T)$, namely if

$$\frac{4}{T} - (M+N) > 0, \qquad MN - \frac{M+N}{T} > 0, \qquad MN + P > 0$$

$$\begin{aligned}
& a_1(T)a_2(T)a_3(T) > a_3^2(T) + a_1^2(T)a_4(T) \\
\Rightarrow & a_3^2(T) + a_1^2(T)a_4(T) - a_1(T)a_2(T)a_3(T) < 0 \\
\Rightarrow & \frac{16}{T^2}\left[MN - \frac{M+N}{T}\right]^2 + \left(\frac{4}{T} - (M+N)\right)^2 \frac{4(MN+P)}{T^2} \\
& - \left(\frac{4}{T} - (M+N)\right)\left(\frac{4}{T^2} - \frac{4(M+N)}{T} + MN\right)\frac{4}{T}\left[MN - \frac{M+N}{T}\right] < 0 \\
\Rightarrow & \frac{16}{T^4}\left((MN)T - (M+N)\right)^2 + \frac{4}{T^2}\left((MN)+P\right)\frac{1}{T^2}\left(4 - (M+N)T\right)^2 - \frac{1}{T}\left(4 - (M+N)T\right) \\
& \frac{1}{T^2}\left(4 - 4(M+N)T + (MN)T^2\right)\frac{4}{T^2}\left((MN)T - (M+N)\right) < 0 \\
\Rightarrow & \frac{4}{T^4}(4(MN)^2T^2 + 4(M+N)^2 - 8(M+N)(MN)T + 16(MN) + (M+N)^2(MN)T^2 - 8(M+N) \\
& (MN)T + 16P + (M+N)^2 PT^2 - 8(M+N)PT) - \frac{4}{T^5}(16(MN)T - 16(M+N) \\
& - 8(M+N)(MN)T^2 + 8(M+N)^2 T + 4(M+N)^2(MN)T^3 \\
& - 4(M+N)^3 T^2 + 4(MN)^2 T^3 - 4(M+N)(MN)T^2 \\
& - (M+N)(MN)^2 T^4 + (M+N)^2(MN)T^3) < 0 \\
\Rightarrow & (M+N)(MN)^2 T^4 + ((M+N)^2 P - 4(M+N)^2(MN))T^3 + (-8(M+N)P + 8(M+N) \\
& (MN) + 4(M+N)^3)T^2 + \left(16P - 16(M+N)^2\right)T + 16(M+N)(MN) < 0.
\end{aligned}$$

Knowing that $N < 0$ and $P > 0$, we derive these conditions (2.19) are verified always true if $M \leq 0$. On the other hand, when $M > 0$, they are true if $M+N < 0$, $MN + P > 0$ and $T < (M+N)/(MN)$. It is difficult to handle the case $\varphi(T) < 0$, unless $M = 0$. Indeed, in this case,

$$\varphi(T) = \left(N^2P\right)T^3 + \left(-8NP + 4N^3\right)T^2 + \left(16P - 16N^2\right)T + 16N,$$

is such that $\varphi(0) < 0$ and $\varphi(+\infty) = +\infty$. Hence, there exists at least a positive value of T, say T_0^*, such that $\varphi(T) = 0$. We need now to recall all Descartes' rule of signs and its corollary. That state "the number of positive roots of the polynomial $\varphi(T)$ is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number" and "the number of negative roots is the number of sign changes after multiplying the coefficients of odd-power terms by -1, or fewer than it by an even number", respectively. Applying these rules to the polynomial $\varphi(T)$, we get that $\varphi(T)$ has one positive zero and the number of negative zeros must be either 2 or 0. Therefore, $\varphi(T) < 0$ if $T < T_0^*$. When $M = 0$, the equilibrium point (y^*, y^*, y^*, k^*) of (2.30) is locally asymptotically stable for $T < T_0^*$. Assume there exists $T^* > 0$ such that $\varphi(T^*) = 0$, i.e
In this case, We can reformulate the characteristic equation (2.34) as follows.

$$a_1(T_*)a_2(T_*)a_3(T_*) - a_3^2(T_*) - a_1^2(T_*)a_4(T_*) = 0$$

Multiply the equation (2.34) by $a_1(T_*)$

$$\begin{aligned}
&a_1^2\lambda^4 + a_1^2a_1\lambda^3 + a_1^2a_2\lambda^2 + a_1^2a_3\lambda + a_1^2a_4\\
&a_1^2\lambda^4 + a_1^2a_1\lambda^3 + a_1^2a_2\lambda^2 + a_1^2a_3\lambda + a_1a_2a_3 - a_3^2\\
&a_1^2\lambda^4 + a_1^2a_1\lambda^3 + a_1^2a_2\lambda^2 + a_1^2a_3\lambda + a_1a_2a_3 - a_3^2 + a_1a_3\lambda^2 - a_1a_3\lambda^2\\
=&[a_1\lambda^2 + a_3][a_1\lambda^2 + a_1^2\lambda + a_1a_2 - a_3]
\end{aligned}$$

$$\left[a_1(T_*)\lambda^2 + a_3(T_*)\right]\left[a_1(T_*)\lambda^2 + a_1^2(T_*)\lambda + a_1(T_*)a_2(T_*) - a_3(T_*)\right] = 0$$

Thus, we have two purely imaginary roots

$$\lambda_{1,2} = \pm i\sqrt{\frac{a_3(T_*)}{a_1(T_*)}} = \pm i\omega$$

and two other roots,

$$\lambda_{3,4} = \frac{-a_1^2(T_*) \pm \sqrt{a_1^4(T_*) - 4a_1(T_*)\left[a_1(T_*)a_2(T_*) - a_3(T_*)\right]}}{2a_1(T_*)},$$

which have real parts different from zero since

$$\lambda_3 + \lambda_4 = -a_1(T_*) < 0$$

and

$$\lambda_3\lambda_4 = \frac{[a_1(T_*)a_2(T_*) - a_3(T_*)]}{a_1(T_*)} > 0$$

When we differentiate the characteristic equation (2.34) with respect to T, we get

$$\left[4\lambda^3 + 3a_1(T)\lambda^2 + 2a_2(T)\lambda + a_3(T)\right]\frac{d\lambda}{dT} = -\left[a_1'(T)\lambda^3 + a_2'(T)\lambda^2 + a_3'(T)\lambda + a_4'(T)\right] \tag{2.36}$$

i.e.

$$\frac{d\lambda}{dT} = -\frac{a_1'(T)\lambda^3 + a_2'(T)\lambda^2 + a_3'(T)\lambda + a_4'(T)}{4\lambda^3 + 3a_1(T)\lambda^2 + 2a_2(T)\lambda + a_3(T)} \tag{2.37}$$

where

$$a_1'(T) = -\frac{4}{T^2} \qquad a_2'(T) = -\frac{8}{T^3} + \frac{4\,(M+N)}{T^2}$$

and

$$a_3'(T) = -\frac{4MN}{T^2} + \frac{8\,(M+N)}{T^3}, \qquad a_4'(T) = -\frac{8\,(MN+P)}{T^3}$$

Substituting $\lambda = i\omega_*$ in (2.37),

$$\begin{aligned}
\frac{d\lambda}{dT}_{T=T_*} &= -\frac{a_1'(T_*)(i\omega_*)^3 + a_2'(T_*)(i\omega_*)^2 + a_3'(T_*)(i\omega_*) + a_4'(T_*)}{4(i\omega_*)^3 + 3a_1(T_*)(i\omega_*)^2 + 2a_2(T_*)(i\omega_*) + a_3(T_*)} \\
&= -\frac{-a_2'(T_*)\omega_*^2 + a_4'(T_*) + (-a_1'(T_*)\omega_*^3 + a_3'(T_*)\omega_*)i}{-2a_3(T_*) + (2a_2(T_*)\omega_* - 4\omega_*^3)i} \\
&= -\frac{(-a_2'(T_*)\omega_*^2 + a_4'(T_*) + (-a_1'(T_*)\omega_*^3 + a_3'(T_*)\omega_*)i)(-a_3(T_*) - (a_2(T_*)\omega_* - 2\omega_*^3)i)}{2(-a_3(T_*) + (a_2(T_*)\omega_* - 2\omega_*^3)i)(-a_3(T_*) - (a_2(T_*)\omega_* - 2\omega_*^3)i)} \\
Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} &= \frac{a_1(T_*)\varphi'(T_*)}{2(a_1^3(T_*)a_3(T_*) + [a_1(T_*)a_2(T_*) - 2a_3(T_*)]^2)}
\end{aligned}$$

where $\varphi'(T_*) = a_1'(T_*)a_2(T_*)a_3(T_*) + a_1(T_*)a_2'(T_*)a_3(T_*) + a_1(T_*)a_2(T_*)a_3'(T_*) - 2a_3(T_*)a_3'(T_*) - 2a_1'(T_*)a_1(T_*)a_4(T_*) - a_1^2(T_*)a_4'(T_*)$

Let us notice that sign $[Re(d\lambda/dT)_{T=T_*}] =$ sign $[-\varphi'(T_*)]$ and recall that sign $[Re(d\lambda/dT)_{T=T_*}] > 0$ and sign $[Re(d\lambda/dT)_{T=T_*}] < 0$ correspond to crossings of the imaginary axis from right to left, and from left to right, respectively. Below are the conclusions of all of the previous studies.

Theorem 2.2 *Let M be defined as in* (2.35).

1) *Let $M = 0$. There exists $T_0^* > 0$ such that the equilibrium point (y^*, y^*, y^*, k^*) of* (2.30) *is locally asymptotically stable for $T < T_0^*$, unstable for $T > T_0^*$, and bifurcates to a limit cycle through a Hopf bifurcation at the equilibrium point when $T = T_0^*$.*

2) *Let $M \neq 0$. The equilibrium point (y^*, y^*, y^*, k^*) of* (2.31) *is locally asymptotically stable if $M < 0$ and $\varphi(T) < 0$ or if $M > 0, M+N < 0, MN+P > 0, T < (M+N)/(MN)$ and $\varphi(T) < 0$. If there exists $T = T_*$ such that $\varphi(T_*) = 0$ and $\varphi'(T_*) \neq 0$, then a Hopf bifurcation may occurs at the equilibrium point as T passes through T_*.*

2.4.3 The rate of growth bifurcation analysis

Considering the dynamics of the system (2.23) − (2.25) with respect to the change of the parameter g (the rate of economic growth).

Proposition 2.1 *The critical point of system* (2.23) − (2.25) *(and equivalently system* (2.15) − (2.16) *always exists for the rate of growth parameter g in the interval $c - \delta < g < c + d - \delta$.*

As previously noted, the system (2.15) − (2.16) has a unique fixed point for $c < g + \delta < c + d$. The economy with the investment function $I(y, k) = k\varphi(y, k)$ has a fixed point or a limit cycle solution only for some rates of growth within the interval $(c - \delta, c + d - \delta)$. The investment function's parameters c and d (as well as capital stock depreciation) provide a restriction on the minimum and maximum growth rates. Let us consider the characteristic equation of the linearised system (2.26) − (2.29) at the critical point (y^*, u^*, k^*), in the form

$$\begin{vmatrix} \alpha I_y^*(g) - \alpha\gamma - g - \lambda & 0 & \alpha(g + \delta - x^*(g)I_y^*(g)) \\ \frac{1}{T} & -\frac{1}{T} - \lambda & 0 \\ 0 & I_y^*(g) & -x^*(g)I_y^*(g) - \lambda \end{vmatrix} = 0, \qquad (2.38)$$

or

$$\lambda^3 + a_1(g)\lambda^2 + a_2(g)\lambda + a_3(g) = 0 \qquad (2.39)$$

where

$$a_1(g) = \frac{1}{T} - (\alpha(I_y^*(g) - \gamma) - g - x^*(g)I_y^*(g)),$$
$$a_2(g) = -\frac{\alpha(I_y^*(g) - \gamma) - g - x^*(g)I_y^*(g)}{T} - \left[\alpha(I_y^*(g) - \gamma) - g\right] x^*(g)I_y^*(g),$$
$$a_3(g) = \frac{1}{T}\left[- \left[\alpha(I_y^*(g) - \gamma) - g\right] x^*(g)I_y^*(g)) - \alpha I_k^*(g)I_y^*(g)\right]$$

where λ is a root of the characteristic equation. The discriminant of the characteristic equation is

$$\Delta = 18a_1a_2a_3 - 4a_2^3a_3 + a_2^2a_3^2 - 4a_3^3 - 27a_3^2. \qquad (2.40)$$

Proposition 2.2 *If the expression* (2.40) *is positive than the all eigenvalues are real and if it is negative there is one real, one pair of conjugate complex eigenvalues. For the zero value of the expression* (2.40) *the critical point is non-hyperbolic.*

We have the following proposition for real eigenvalues:

Proposition 2.3 *In the interval $c - \delta < g < c + d - \delta$, there are two sub-intervals with the positive values of the discriminant* (2.40) *there two cases for the values of rate of growth parameter g. So, there are three negative real eigenvalues in these sub-intervals .*

The sub-intervals of the parameter g with two negative and one positive eigenvalues are non-physical regions as the critical point (y^*, u^*, k^*) isn't in the positive quadrant.

We have the following proposition for complex eigenvalues:

Proposition 2.4 *In the interval $c - \delta < g < c + d - \delta$ and negative values of the discriminant* (2.40) *for the increasing value of the rate of growth parameter g there are two super critical Hopf bifurcations. For the value $g = g_{1,Hopf}$ the limit cycle is created, and for the value $g = g_{2,Hopf}$, the limit cycle is destroyed* $(g_{1,Hopf} < g_{2,Hopf})$.

Therefore, as the the rate of growth parameter is increasing in the interval $g_{\min} = c - \delta < g < c + d - \delta = g_{\max}$ the eigenvalues change as follows. In the first sub interval $(g_{\min}; g_1)$ there are three real eigenvalues (two negative, one positive). In the second sub interval $(g_1; g + 1, Hopf)$ there are three real eigenvalues (three negative). In the third sub-interval $(g_{1,Hopf}; g_{2,Hopf})$ there are one real eigenvalue (negative) and one conjugate complex eigenvalue (positive real parts). In the fourth sub-interval $(g_{2,Hopf}; g_2)$ there are three real eigenvalues (three negative). And finally, in the fifth sub-interval $g_2, (g + \max)$ there are three real eigenvalues (two negative, one positive).

We some example values of parameters we can determine the values of the rate of growth parameter for which the eigenvalues change their character or sign. We assume the values of investment function parameters obtained by Dana and Malgrange, namely,$c = 0.01, d = 0.026, a = 9, v = 4.23$. We fix also the following model parameters $\alpha = 1, \gamma = 0.15, \delta = 0.007, G_0 = 2$ and $T = 1$. The rate of growth parameter g is taken within the interval $g_{\min} = c - \delta < g < c + d - \delta = g_{\max}$. The results are presented in table 1.

Figure 3.4 shows the bifurcation diagram for the rate of growth parameter g in systems $(2.23) - (2.25)$. **Table 1.** shows the ranges of values for the rate-of-growth parameter g, as well as the signs of the eigenvalues of the characteristic equation (2.38). The investment function is supposed to be $c = 0.01, d = 0.026, a = 9, v = 4.23$, $\alpha = 1, \gamma = 0.15, \delta = 0.007, G_0 = 2$, and $T = 1$. (rest model parameters).

real eigenvalues	complex rigenvalues	rate of growth parameter
1 negative	pair with negative real part	(0.003, 0.0101198)
1 negative	pair with postive real part	(0.0101199,0.0203258)
1 negative	pair with negative real part	(0.0203259, 0.029)

CHAPTER 3

NUMERICAL ANALYSIS

This chapter consecrates on numerical analysis of our subject where the results may be driven by the analytical as well as numerical investigations. furthermore, the numerical analyses provided some results before was it established in theory. In the first section, we present a numerical analysis of the Hopf bifurcation with the time delay of Krawiec and Szydlowski work [3]. In the second section, we discuss a numerical analysis of the Hopf bifurcation with distributed time delay of Guerrini et al. [2], The authors use in this numerical study the investment function of Dana and Malgrange, and they obtained two Hopf bifurcations concerning the rate growth parameter and they detected the existence of stable long-period cycles in the economy.

This result is concerned with the time delay and adjustment speed parameters, the range of acceptable values of the rate of growth parameter breaks down into three intervals. First, we have a stable focus, then the limit cycle, and again the stable solution with two Hopf bifurcations. Such behavior appears for some middle interval of the acceptable range of values of the rate of growth parameter.

3.1 Numerical analysis of the Hopf bifurcation with time delay

It is useful to write any eigenvalue in terms of its real and imaginary parts, $\lambda = \sigma + i\omega$, and write the real and imaginary parts of the general eigenvalue equation, We obtain the following system:

$$\lambda = \sigma + i\omega \Rightarrow e^{-\lambda T} = e^{-\sigma T}(\cos(\omega T) - i\sin(\omega T))$$

we obtain

$$\sigma^2 - \omega^2 + 2i\sigma\omega + A(\sigma + i\omega) + B + De^{-\sigma T)}(\cos(\omega T) - i\sin(\omega T))$$

Equivalently

$$\sigma^2 - \omega^2 + A\sigma + B + De^{-\sigma T}\cos(\omega T) = 0 \tag{3.1}$$

$$2\sigma\omega + A\omega - De^{-\sigma T}\sin(\omega T) = 0 \tag{3.2}$$

In the Kaldor–Kalecki model, the generalized eigenvalue formula (3.1)–(3.2) can be used to predict limit cycles. Only numerically can this be easily solved. The limit cycle bifurcation occurs when the real part of a complex conjugate pair of eingenvalues changes its sign from negative to positive. We set $\sigma = 0$ to determine the initial bifurcation point. We might make the naive assumption that further bifurcations are caused by additional eigenvalue pairs acquiring a positive real part. We can see from the eigenvalue equations that this happens when the limit cycle period and the time delay

$$\omega T = \omega_{bif}T_{bif} + 2\pi n, \; n = 1, 2, n....$$

In fact, it is better to use the new parametric variable $y = \omega T$, since this is the argument of the trigonometric functions at (3.1) – (3.2). The imaginary part can be solved for ω, as follow

$$\omega = \frac{D\sin y}{A} \tag{3.3}$$

This result is in turn substituted into the real part, equation (3.1). Then, with the help of the trigonometric identity $\sin^2 y = 1 - \cos^2 y$, we obtain

$$\begin{aligned} &-\frac{D^2\sin y^2}{A^2} + B + D\cos y \\ =& - D^2(1 - cos^2 y) + A^2B + AD\cos y \\ =& - D^2 + D^2 cos^2 y + A^2B + AD\cos y \end{aligned}$$

Let $E = BA^2 - D^2$.

$$D^2 cos^2 y + A^2 D\cos y + E = 0 \tag{3.4}$$

And this is easy to solve for for y ,is then solved for ω, and finally

$$T_{bif} = \frac{y}{\omega} = \frac{A}{D}\frac{y}{\sin y} = \frac{A}{D}\frac{}{(\sin y)/y}. \tag{3.5}$$

From the last formula, we can see that as the time-delay parameter is small $T \ll 1$ (or $\sin y/y \approx 1), T = T_{bif} = A/D$ as in the previous linearized case.

The dependence of T_{bif} on the parameters of our model is determined from the following relation

$$T_{bif} = \frac{A}{D}\frac{y}{\sin y} = \frac{A}{D}\frac{\arccos z}{\sin(\arccos z)}$$

where

$$D^2 cos^2 y + A^2 D \cos y - D^2 + A^2 B = 0$$
$$D^2 z^2 + ADz - D^2 + A^2 B = 0$$
$$\Delta = A^4 D^2 - 4D^2(-D^2 + A^2 B)$$
$$z_{1,2} = \frac{1}{2D}\left(-A^2 \mp \sqrt{A^4 - 4(-D^2 + A^2 B)}\right)$$

In figure 3.1, the dependence of the bifurcation parameter on z is shown. The amplitude of this relation is equal to A/D and corresponds to the value of $z = 0$. Figure 3.2 shows that for small y, the derivative of the real part of the eigenvalue with respect to T is positive.

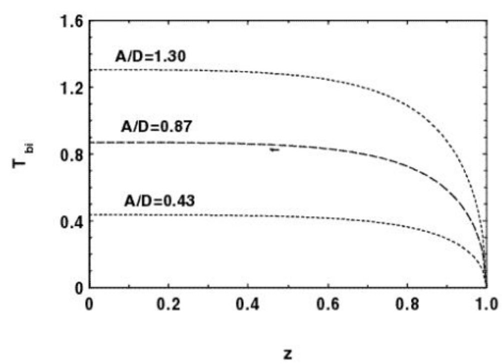

Figure 3.1: Diagram to illustrate the dependence of bifurcation parameter T_{bif} on $z = \cos y$. When $z = 0$, $T_{bif} = A/D$, as in the linear approximation case.

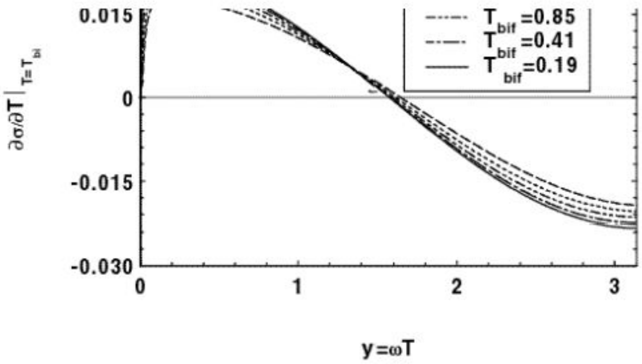

Figure 3.2: Dependence of $\frac{\partial \sigma}{\partial T}|_{T_{dif}}$ on y.

The Hopf theorem predicts a bifurcation to a limit cycle when (a) the real part σ of a pair of complex conjugate (with non zero imaginary part) eigenvalues changes its sign from negative to positive as parameter T is varied, and (b) the derivative of the real part with respect to parameter T is positive as σ passes through zero. We saw that condition (a) occurs in the Kaldor–Kalecki model as the time-delay parameter is increased. We can verify condition (b) by differentiating the eigenvalue equation (2.10), which yields

$$\frac{\partial \lambda}{\partial T} = \frac{\lambda D e^{-\lambda T}}{2\lambda + A - DTe^{-\lambda T}},$$

and it follows that

$$\frac{\partial\lambda}{\partial T} = \frac{\lambda D e^{-\lambda T}}{2\lambda + A - DTe^{-\lambda T}}$$

$$\frac{\partial\lambda}{\partial T} = \frac{(\sigma + i\omega)De^{-(\sigma+i\omega)T}}{2(\sigma+i\omega) + A - DTe^{-(\sigma+i\omega)T}}$$

$$= \frac{(\sigma + i\omega)De^{-\sigma T}(\cos\omega T - i\sin\omega T)}{2(\sigma+i\omega) + A - DTe^{-\sigma T}(\cos\omega T - i\sin\omega T)}$$

$$= \frac{\sigma\cos\omega T + \omega\sin\omega T + (\omega\cos\omega T - \sigma\sin\omega T)i}{2\sigma + A - DTe^{-\sigma T}\cos\omega T + i(2\omega + DTe^{-\sigma T}\sin\omega T)} De^{-\sigma T}$$

$$= \frac{(\sigma\cos\omega T + \omega\sin\omega T + (\omega\cos\omega T - \sigma\sin\omega T)i)\,(2\sigma + A - DTe^{-\sigma T}\cos\omega T}{(2\sigma + A - DTe^{-\sigma T}\cos\omega T)^2 + (2\omega + DTe^{-\sigma T}\sin\omega T)^2}$$

$$\frac{-i(2\omega + DTe^{-\sigma T}\sin\omega T)e^{-\sigma T})}{} De^{-\sigma T}$$

$$\frac{\partial\sigma}{\partial T} = Re\frac{\partial\lambda}{\partial T} = \frac{(\sigma\cos\omega T + \omega\sin\omega T)(2\sigma + A - DTe^{-\sigma T}\cos\omega T) + (\omega\cos\omega T - \sigma\sin\omega T)}{(2\sigma + A - DTe^{-\sigma T}\cos\omega T)^2 + (2\omega + DTe^{-\sigma T}\sin\omega T)^2}$$

$$\frac{(2\omega + DTe^{-\sigma T}\sin\omega T)e^{-\sigma T})}{} De^{-\sigma T}$$

$$= \frac{[A(\sigma\cos\omega T + \omega\sin\omega T) + 2\cos\omega T(\sigma^2 + \omega^2) - TD\sigma e^{-\sigma T}]}{(2\sigma + A - DTe^{-\sigma T}\cos\omega T)^2 + (2\omega + DTe^{-\sigma T}\sin\omega T)^2} De^{-\sigma T}$$

or

$$\left.\frac{\partial\sigma}{\partial T}\right|_{T=T_{bif}} = \frac{D(A\omega\sin\omega T + 2\omega^2\cos\omega T)}{(A - DT\cos\omega T)^2 + (2\omega + DT\sin\omega T)^2}$$

If we assume that $A/D \geq 0D$ is always positive, then $\partial\sigma/\partial T|_{T=T_{bif}}$ is positive for the argument $y = \omega T \in [-\pi/2, \pi/2]$. In summary, the study of linear stability analysis shows that the establishment of a limit cycle is due to the Hopf bifurcation (that is, the real part of a complex conjugate pair changes its sign from negative to positive), and, as usual in Hopf bifurcation, the limit cycle approximates the ellipse in the vicinity of the bifurcation. In the general case, one can make sure that for ,

$$\begin{aligned}
&A\omega \sin \omega T + 2\omega^2 \cos \omega T = 0 \\
&A\omega \tan \omega T + 2\omega^2 = 0 \\
&\omega(A \tan \omega T + 2\omega) = 0 \\
&A \tan \omega T = -2\omega \\
&\tan \frac{y}{T} T = -\frac{2y}{TA} \\
&\tan^{-1} \frac{y}{T} T = -\frac{TA}{2y} \\
&\frac{y}{T} T = \operatorname{arctan}^{-1} -\frac{TA}{2y}
\end{aligned}$$

or equivalently $T < T_{crit} = (T_{bif}/y) \operatorname{arctan}^{-1}(-AT_{bif}/(2y))$, we have $\partial\sigma/\partial T > 0$, which means that $\sigma(T)$ is an increasing function of its amplitude
Knowing also that, the orbital period can be determined by

$$P = \frac{2\pi}{|\lambda(T_{bif})|} = \frac{2\pi T_{bif}}{|\lambda|}$$

and the radius of an orbit by $R \propto \sqrt{T}$.

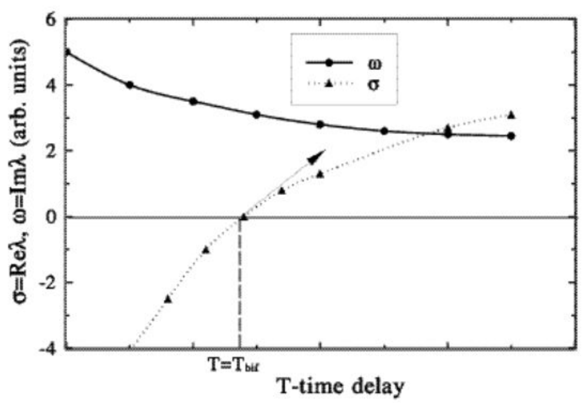

Figure 3.3: Real s (triangles) and imaginary w (circles) part of the eigenvalue as a function of the time delay T in the region of the limit cycle bifurcation T

Figure 3.3 shows a sequence $\sigma, \omega(\mod 2\pi T)$ values as T increases, for a different typical parameter set. It was obtained by numerical experiments. The change of the sign of σ from negative to positive is confirmed by the linear stability analysis

3.2 Numerical analysis of the Hopf bifurcation with distributed time delay

Due to the Hopf bifurcation created by increasing the value, the original Kaldor model showed limit cycle behavior. α value [7]. Next, both the investment lag T and the exogenous growth trend g have been contributed to it. The limit cycle is also generated by increasing the investment time delay parameter value. [3]. However, the dependence the Hopf bifurcation on the rate of growth parameter was not elaborated so far. Both Chang and Smyth in the Kaldor model [7] as well as Dana and Malgrange in the Kaldor model with exogenous growth trend investigated the parameter α as the bifurcation parameter [6]. We conduct the numerical analysis of both models: $m = 1$ and $m = 2$. We assume the investment function with the following parameter values: $c = 0.01, d = 0.026, a = 9$ and $v = 4.23$ [6]

$$I(y,k) = k\varphi(y,k) = 0.01 + \frac{0.026}{1 + e^{-9(4.23y/k-1)}}. \tag{3.6}$$

The following model parameters were assumed: $\gamma = 0.15, \delta = 0.007, G_0 = 2$. $T \in (0,5)$, Then, suppose the three parameters at the following intervals: $\alpha \in (0.5, 1.0)$, and $g \in (0.01, 0.02)$.

3.2.1 The Hopf bifurcation *case* $m = 1$

We consider in this case, for the state variables (y, u, k) the three-dimensional model (2.23) – (2.25). In this model we investigate numerically the stability of the critical point $(y^* = u^*, k^*)$ to find the values of parameter T for which the critical point loses the stability and the limit cycle is created through the Hopf bifurcation mechanism.

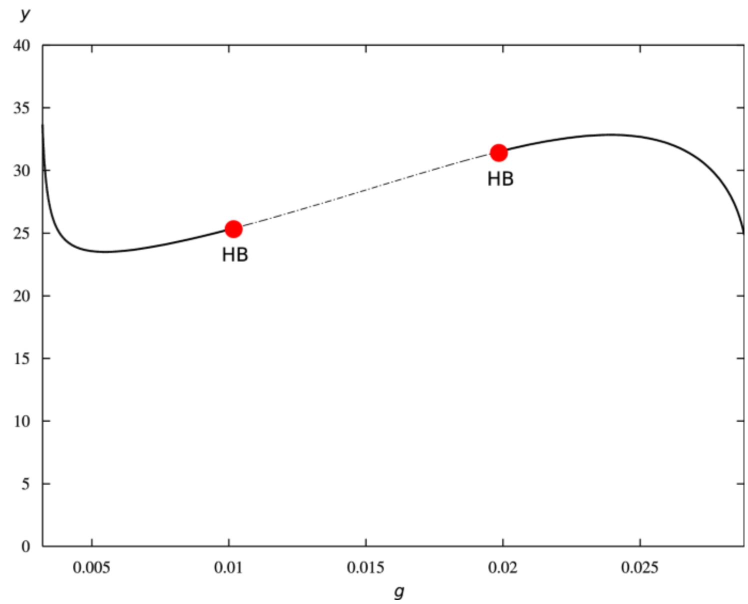

Figure 3.4: bifurcation diagram for model for system (2.23) − (2.25) $m = 1$ with investment function (3.6) for delay parameter T. The solid line indicates critical point with asymptotic stability and the dot-dash line corresponds to the unstable critical with a limit cycle around it.

Also, the bifurcation value of T and the bifurcation value of g on the model parameters α and T are studied in depth. The bifurcation surface in the parameter space (a, g, T) is presented in figure 3.4. The region below the surface corresponds to the asymptotic stability of the critical point (y^*, u^*, k^*). The parameter values for which system (2.20) has an unstable critical point with a limit cycle around it correspond to the region inside. Let conduct more detailed analysis and consider relations between two parameters with a third parameter fixed. First, the relation of T on α for $g = 0.016$ is shown in figure 3.5. We find that the asymptotic stability region exist only if $\alpha < 0.7644$ (with $g = 0.016$). In the interval of $\alpha \in (0.6, 0.764)$

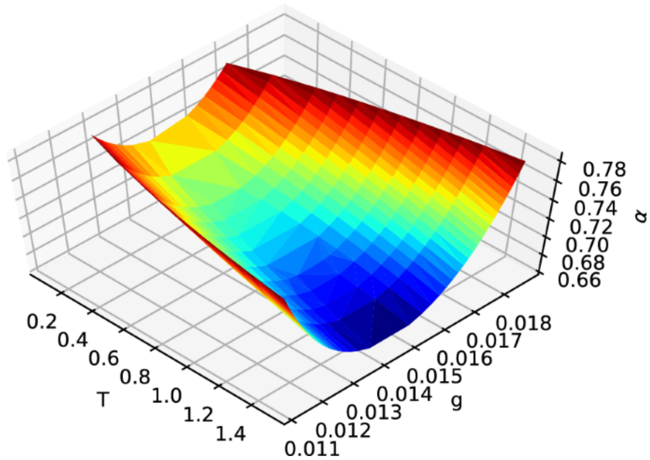

Figure 3.5: The Hopf bifurcation surface in the space of parameters (α, g, T) for system $m = 1$ and with investment function (3.6). Outside of the surface is the region of asymptotic stability, while inside of the surface is the region of parameters values for which a limit cycle solution exists.

The relation $T_{bi}(\alpha)$ is given by

$$T_{bi} = -11.137983 + \frac{8.512805}{\alpha}. \tag{3.7}$$

Ensuing, we analyze the dependence of the parameter T on the parameter g with the fixed value of α. We consider the three values of parameter α. The stability regions on the plane (g, T) are shown for $\alpha = 0.6$ and $\alpha = 0.9$ in *Figure 3.5* As the value of the parameter g increases the region B is rising. A quadratic equation describes the bifurcation line that separates the regions A and B.

$$T_{bi} = a_2 g^2 + a_1 g + a_0. \tag{3.8}$$

Now, Consider the model's time paths for the economic variable y for different values of the time delay parameter in respect of cycles' amplitude and period. In region I there is a stable equilibrium reached by trajectories in an oscillating manner. It is the region of asymptotic stability of the model.

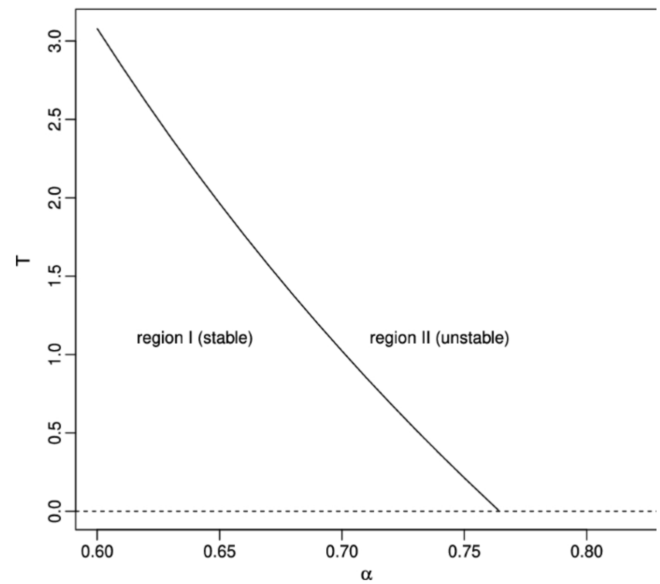

Figure 3.6: The plane of parameters (α, T) for system $m = 1$ and $g = 0.016$ with investment function (3.6). Region I is the region of asymptotic stability, while region II is the region of parameters value for which a limit cycle solution exists

In figure 3.6 there are the two cases for $\alpha = 0.6, 0.9$ with three solutions $y(t)$ obtained for given $T = 0.5, 1.5, 3$ and the same initial function $y(t) = 1, k(t) = 100$ where $t \in (-T, 0)$. As the time delay T increases, the dumping of oscillations becomes weaker.

3.2.2 The Hopf bifurcation *case* $m = 2$

Considering the four-dimensional model (2.26) – (2.29) for the state variables (y, p, w, k). In this model the critical point values of $(y^* = p^* = w^*, k^*)$ are the same as the critical point values of $(y^* = u^*, k^*)$ of three dimensional model presented in the previous section. We investigate the occurrence of the Hopf bifurcation for the parameter T depending on parameters α and g, in the same way that we did in the previous section. The relation of T on α for $g = 0.016$ is shown in Figure 3.6. The asymptotic stability region exist only if $\alpha < 0.7644$ (with $g = 0.016$). It is the same result as in the case of model $m = 1$.

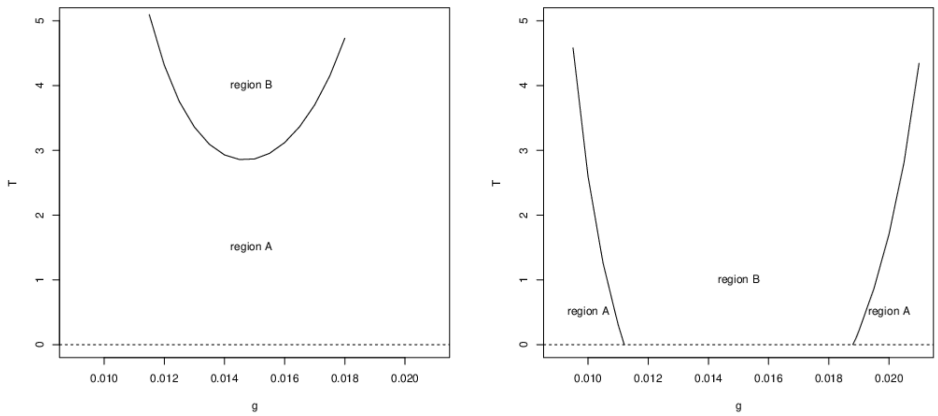

Figure 3.7: The plane of parameters (g, T) for system $m = 1$ with investment function (3.6). Here, it is assumed that $\alpha = 0.6$ (left panel) and $\alpha = 0.9$ (right panel). Region A is the region of asymptotic stability and region B is the region of limit cycle solution and their are separated by the bifurcation line $T_{bi}(g)$.

Now, we study the cycle characteristics for some values of the parameter α with different values of the delay parameter T. Figure 3.7 shows the solutions of y for two cases of $\alpha = 0.6, 0.9$ and three values of the parameter $T = 0.5, 1.5, 3.0$. The amplitude and period of cycles are decreasing as the parameter T increases for $\alpha = 0.6$.

3.2.3 Case study comparison $m = 1$ and *case* $m = 2$

The models under consideration can be treated as the approximation of the Kaldor-Kalecki growth model with delay. It's crucial to determine how accurate the subsequent approximations are. Therefore, we compare the bifurcation values of the parameter T in models $m = 1$ and $m = 2$. We start by considering the diagram of bifurcation in the parameter plane (α, T) presented in Figure 9. Thus, for the given value of parameter α, we observe that the Hopf bifurcation value of the parameter T_{bi} is lower for the model $m = 2$ and the difference is zero for $T = 0$ and then increases as the value T_{bi} increases for the fixed value of the parameter α.

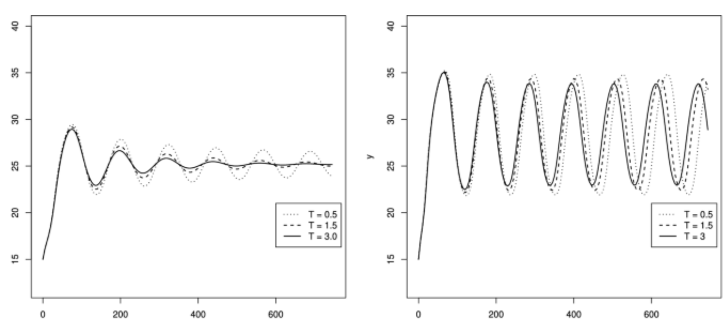

Figure 3.8: Trajectories of model m = 1 with investment function (3.6) for the parameter $g = 0.016$ and $\alpha = 0.6$ (left panel) and $\alpha = 0.9$ (right panel).

However, the bifurcation value of the parameter α_{bi} is greater for the fixed value of the parameter T. In the model $m = 2$, For the parameter $g = 0.016$ the difference is close zero at $\alpha = 0.7644$ and is equal 0.04 at $\alpha = 0.6$ while for the parameter $g = 0.011$ the difference is 0.234 at $\alpha = 0.6$. It is demonstrated in Figure 3.8 for $g = 0.016$ (left panel) and $g = 0.011$ (right panel).

To compare trajectories of $y(t)$ for systems $m = 1$ and $m = 2$ with the same initial conditions. In Figure 3.9, 3.10 there are trajectories $y(t)$ for assumed the parameter $g = 0.016$ and combinations of parameters α and T. We observe that for the same parameters α, g and T the period of cycles is smaller for model $m = 2$ and amplitude is also smaller, although the difference is very small. For example, for $\alpha = 0.9, g = 0.016$ and $T = 3$, the period of cycle in model $m = 1$ is 114.85 and in model $m = 2$ is 116.45 while amplitudes are 12.9555 and 12.966, respectively. When Take models with greater m we should obtain the cycles with longer periods and amplitudes. We can explore further approximations of the model and compare values of the bifurcation parameter g for successive m in table 2.

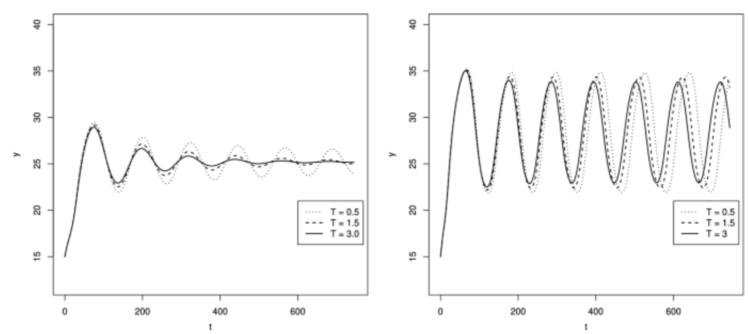

Figure 3.9: The plane of parameters (α, T) for system $m = 2$ and parameter $g = 0.016$ with investment function (3.6). Region I is the region of asymptotic stability, while region II and III are regions of parameters value for which a limit cycle solution exists.

3.3 Results

When we consider the dynamics of the model under the change of the growth rate parameter, we discover numerically two bifurcation values of the rate of growth parameter when the Hopf bifurcations occur.

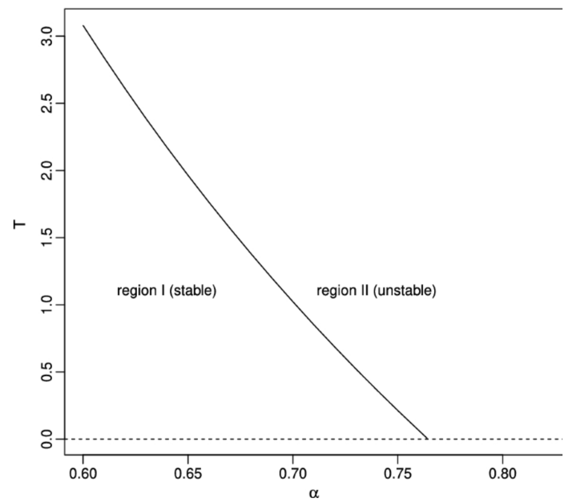

Figure 3.10: Trajectories of model $m = 2$ with investment function (3.6) for parameter $g = 0.016$ and $\alpha = 0.6$ (left panel) and $\alpha = 0.9$ (right panel).

Increasing the value of the rate of growth parameter, for a smaller value of this parameter the limit cycle emerges then for a larger value of this parameter the limit cycle is destroyed to a stable focus through the Hopf bifurcations. Therefore, the cyclic behavior takes place in some interval of the rate of growth parameter values. Outside of the interval the systems through damping oscillation goes to a stable stationary solution. The similar dynamic behavior with two Hopf bifurcations separating stable, unstable and stable regions was also found in the macroeconomic model extending the Calvo and Obstfeld framework [15].

Table 2: The Hopf bifurcation points $g_{bi,1}$ and $g_{bi,2}$ for subsequent approximations

model	$g_{bi,1} = 1$	$g_{bi,2} = 2$
m_1	0.01011989	0.02032586
m_2	0.01011919	0.02032671
m_3	0.01011909	0.02032693
m_4	0.01011906	0.02032703

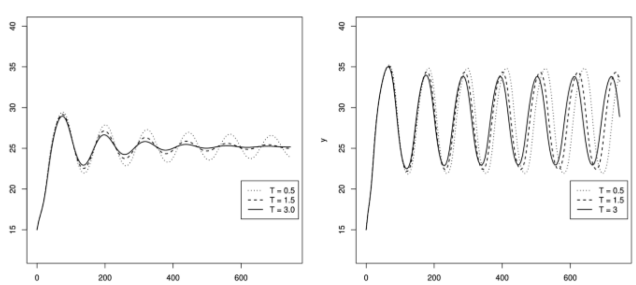

Figure 3.11: The plane of parameters (α, T) for system $(m = 1)$ and $(m = 2)$ with investment function (2.41). Here, it is assumed that $g = 0.016$ (left panel) and $g = 0.011$ (right panel). The dashed line is for model $m = 1$ and the dotted line is for model $m = 2$. These bifurcation curves separates Region I of asymptotic stability on the left side of curves and region II of limit cycle solution on the right side of curves.

There are two oscillating regimes. For lower and higher rates of growth the oscillations are damped and asymptotically stationary state is reached. For intermediate rates of growth the self-sustained oscillations of constant amplitude are present. For some model parameters this intermediate interval of rate of growth values is obtained to be $(0.01011989, 0.0238466)$ for $3-$dimensional and $4-$dimensional systems. The range of this intermediate interval depends on the model parameters: α, γ and δ. All numerical analyses have been done with Dana and Malgrange's investment function for the French macroeconomic data [6].

- We use the linear chain trick technique for reducing the Kaldor-Kalecki model with distributed delay to the ordinary differential system.

- Depending on the value of the parameter m of the Γ distribution function the reduced system is $(m = 2)-$dimensional ordinary differential equation system.

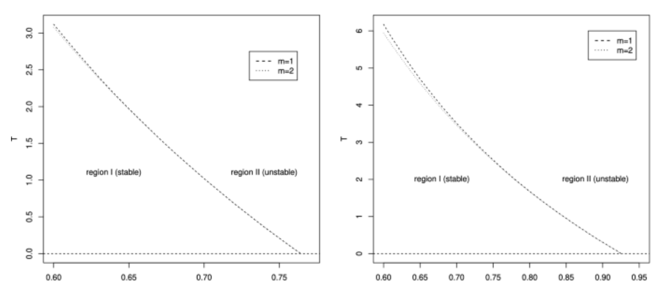

Figure 3.12: Trajectories of models $m = 1$ and $m = 2$ with investment function (2.41) for the same initial condition $(y = p = w = 15, k = 100)$ (left panel) for $\alpha = 0.6$ and $g = 0.016$. The left panel for $T = 0.5$ and the right panel for $T = 3$.

- When the time delay parameter increases, Then then super-critical Hopf bifurcation occurs .

- When the rate of growth parameter is increased, the limit cycle appears initially, then fades away. As a result, there are two super-critical Hopf bifurcations with two rate of growth parameter bifurcation values.

- For some values of parameters α and T, in the allowed range of the rate of growth parameter values, both for lower and higher values of the rate growth parameter the model has the stable stationary point while for the middle range of parameter values there is the limit cycle.

- The period of cycle increases and decreases as the rate of growth parameter increases in the range of unstable solution.
- Comparing the models with different m the stable region in the parameter space is slightly diminished as m is greater.

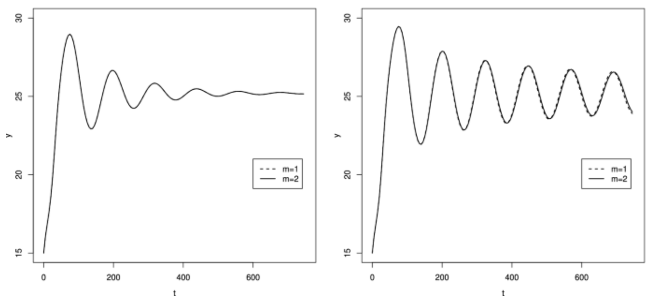

Figure 3.13: Trajectories of [1] models $m = 1$ and $m = 2$ with investment function (3.6) for the same initial condition ($y = p = w = 15, k = 100$) (left panel) for $\alpha = 0.9$ and $g = 0.016$. The left panel for $T = 0.5$ and right panel for $T = 3$.

CONCLUSION

The summary of this work presents a detailed study of the economic growth model with a time-delayed investment function. under the assumption that investment is time distributed. The Hopf bifurcation in these systems with respect to two parameters: the time delay parameter and the rate of growth parameter is studied. The main study is considered a generalization of the modified Kaldor model which is supposed to the investment is a linear function, and only the time-delay parameter plays a crucial role in Hopf bifurcation investigate. Finally, we discuss the numerical study in the previous papers in the last chapter.

BIBLIOGRAPHY

[1] W. W. Chang and D. J. Smyth, "The existence and persistence of cycles in a non-linear model: Kaldor's 1940 model re-examined," *The Review of Economic Studies*, vol. 38, no. 1, pp. 37–44, 1971.

[2] L. Guerrini, A. Krawiec, and M. Szydłowski, "Bifurcations in an economic growth model with a distributed time delay transformed to ode," *Nonlinear Dynamics*, vol. 101, no. 2, pp. 1263–1279, 2020.

[3] A. Krawiec and M. Szydlowski, "The kaldor-kalecki business cycle model," *Annals of Operations Research*, vol. 89, pp. 89–100, 1999.

[4] M. Kalecki, "A macrodynamic theory of business cycles," *Econometrica, Journal of the Econometric Society*, pp. 327–344, 1935.

[5] A. Krawiec and M. Szydłowski, "Economic growth cycles driven by investment delay," *Economic Modelling*, vol. 67, pp. 175–183, 2017.

[6] R.-A. Dana and P. Malgrange, "The dynamics of a discrete version of a growth cycle model," in *Analysing the structure of econometric models*, pp. 115–142, Springer, 1984.

[7] S. Lynch, *Dynamical Systems with Applications using Python*. Springer, 2018.

[8] J. K. Hale and H. Koçak, *Dynamics and bifurcations*, vol. 3. Springer Science & Business Media, 2012.

[9] R. Morgan, "Linearization and stability analysis of nonlinear problems," *Rose-Hulman Undergraduate Mathematics Journal*, vol. 16, no. 2, p. 5, 2015.

[10] E. A. N. A. Elnour *et al.*, *Some of the Limit Cycle Problems and Critical points for Planar Systems.* PhD thesis, Sudan University of Science and Technology, 2017.

[11] R. Seydel, *Practical bifurcation and stability analysis*, vol. 5. Springer Science & Business Media, 2009.

[12] T. A. Burton, *Stability & periodic solutions of ordinary & functional differential equations.* Courier Corporation, 2014.

[13] R. Mahardika, Y. Sumanto, *et al.*, "Routh-hurwitz criterion and bifurcation method for stability analysis of tuberculosis transmission model," in *Journal of Physics: Conference Series*, vol. 1217, p. 012056, IOP Publishing, 2019.

[14] J. Grasman and J. J. Wentzel, "Co-existence of a limit cycle and an equilibrium in kaldor's business cycle model and its consequences," *Journal of Economic Behavior & Organization*, vol. 24, no. 3, pp. 369–377, 1994.

[15] C. A. Mavi, "What can catastrophic events tell us about sustainability?," *Journal of Mathematical Economics*, vol. 83, pp. 70–83, 2019.