République Algérienne Démocratique et Populaire
Ministère de l'enseignement supérieur et de la recherche scientifique
Université Larbi Tébessi - Tébessa
Faculté des Sciences Exactes et des Sciences de la Nature et de la Vie
Département: Mathématiques et Informatique

Mémoire de fin d'études
Pour l'obtention du diplôme de MASTER
Domaine: Mathématiques et Informatique
Filière: Mathématiques
Option: Equations aux dérivées partielles et applications

Thème

Bifurcations of some Zeraoulia-Sprott mappings

Présenté Par:

Zedira Zineb & **Khaled Assia**

Devant le jury :

Mr, Belgacem REBIAI	PROF Univérsité Larbi Tébessi	Président
Mr, Zouhair DIAB	MCA Univérsité Larbi Tébessi	Examinateur
Mr, Elhadj ZERAOULIA	PROF Univérsité Larbi Tébessi	Encadreur

Date de soutenance :11 06 2022

بسم الله الرحمن الرحيم

شكر و عرفان

بسم الله الرحمن الرحيم

الحمد لله المعطاء، الذي سقانا من كأس العلم و العلماء، فرفعنا من بساط الأرض إلى منبر يضاهي عنان السماء، وأهدانا هوية العلم للبقاء، في عصر لا يرحم الجهل والجهلاء وبعد:

- ✓ على ضفة الامتنان ومن على منبر مصنوع من لوح الإقرار بالمعروف، أقف وزميلتي لنلقي كلمة شكر **لكل أساتذتنا** رموز الفخر وأصحاب سمو الفكر، من يذكرون فيطيب الذكر ولا يزرعون إلا ويثمر البذر، رفيعي المقام ومنابع العلم والإلهام، لن نوفيهم حقهم مهما خطت الأقلام، بكل أمانة نعترف وبكل إنصاف نقول بأنكم الأصالة وأنكم العلم، وأنكم فوق الرأي مميزون وفوق الاعتراف رائعون.
- ✓ بحق التوجيهات اللاتي أقمنا بهن على ذمة إشرافه وحق إرشاده الحميد، سقاه الله على إثره من بحر علمه المزيد، نقر بجميل المشرف النبيل الذي لا يضل الطالب تحت إشرافه ولا يغوى، على كل حرف نافع فضيل، وبما أن المجالس مدارس فقد كان لنا في مدرسته الخير الوفير وبقدر إخلاصه في توجيهنا وأكثر، نتقدم إليه بالشكر الجزيل إلى الأستاذ:

زرارواية الحاج

إهداء

بسم الله الرحمن الرحيم

الحمد لله المعطاء المرجو سخاؤه، خالق القضاء المضمون بقاؤه، رافع السماء المستحق فداؤه، مشرف العلم والعلماء الذي شمل العالمين إنعامه وعم جميع المخلوقات إكرامه وبعد:

فأهدي ثمرة سنيني وجهدها

إلى قصيدة القلوب المشهود فضلها، مفتاح الدروب البهي ظلها، بطلة الاحلام المستحيل مثلها، وعروس الأيام المكتوب عدلها، هدية الحياة المرجو نيلها، هي ربيع البيت وأغنية أركانه، ضحكة ليله وبهجة نهاره.

أمي الحبيـــبة

إلى من رصع فينا معنى الأدب، وجعل معدننا قح الذهب، وواصل المسيرة معنا من عز الشتاء إلى حر الصيهد، وضحى بسنينه فأعطى ووهب، و غاص لأجلنا في معنى التعب، مهما حييت لن أجد لمثله قلبا محب، فعجبا له من أب تقي وألف عجب.

أبي الحبيـــب

إلى أنصاف السعادة وبديلات الأمومة، مجروحات الشهادة وشريكات الغنيمة، بنات الحياء بعقول حكيمة وشقيقات الوفاء بقلوب رحيمة، أميرات الأخلاق بقيم سليمة ورفيعات الأذواق بهمم زعيمة، كريمات الموائد ورفيقات الشدائد.

أخـــواتي

إلى أصحاب المعروف ومعنى السند، نعم الموصوف ونعم العضد، أبطال الأخوة وملوك النخوة، رموز النجاح ومعاني الصلاح.

إخـــوتي

زينب

إهداء

بسم الله الرحمن الرحيم

إلى أعز الناس .. الذي لا يجزيه مني كلام .. إلى من تشققت يداه في سبيل رعايتي .. الذي فارقنا قبل تخرجي بأيام فله أهدي ما حصدت.

أبي رحمه الله

إلى أحن إنسانة .. التي لم تدخر نفسا في تربيتي .. إلى التي تستقبلني بابتسامة وتودعني بدعوة .. حماك الله وأدامك .. عصفورا مغردا يملأ حياتي بأعذب الألحان.

أمي الغالية

إلى الذين ظفرت بهم هدية من الأقدار .. إخوة فعرفوا معنى الأخوة .. أراكم بسمتي وأرى جمال الأيام أنتم.

إخواني مقداد، علاء، عنتر

إلى من أفتقد حرارة تصفيقه فرحا بإنجازي في هذه اللحظة.

أخي المرحوم كريم

إلى زهرات حياتي .. اللاتي يحملن في عيونهن ذكريات طفولتي.

أخواتي مبروكة، زكية، نجوى، فتيحة، ابتسام

إلى من تقاسمت معها ظلمة الرحم .. إلى روحي .. التي رسمت في حروف ..
و حروفي التي تمثلت في روح .. يا ميلادي الثاني .. وأنفاس عطري الباقي ..
يا هدية الرحمان في زمن ندر فيه الإخوان.

توأمي أسماء

إلى قطع الماس الصغيرة .. آدم عبد المعز, عبد الرحمان, أنس, ساجد,
عبد الباسط, عبد الصمد, براءة, ألاء, محمد إلياس, طه أمين, مودة, سند, بهاء.

أحفاد العائلة

إلى زوجات إخواني سميحة, شادلية .. وخطيبة أخي زينة .. إلى أزواج أخواتي
عبد الحفيظ, حسين.

إلى جدتاي وجداي وأخص بالذكر جدتي سعدية .. إلى أعمامي و عماتي.

إلى أخوالي عمر, عبد اللطيف, الطيب .. وزوجاتهم وأبنائهم.

إلى خالاتي رشيدة, العالية, عانس, نوة, فجرة .. وأزواجهم وأبنائهم.

أقارب العائلة

إلى رفيقات دربي حورية, مفيدة, زينب, نسيمة.

صديقاتي

إليكم جميعا أهدي ربيع حروفي و شذى كلماتي.

Abstract

The aim of this memory is to study the dynamical behaviors of some Zeraoulia-Sprott piecewise smooth mappings in one and two dimensional. These maps are characterized by a highly dispersed behavioral nature that reaches the extreme of robust chaos as a result of the border collision bifurcations that occurs especially in this type of maps.

Key words: *Border collision bifurcations, piecewise smooth map, chaotic behavior.*

Resumé

L'objectif de cette mémoire est d'étudier le comportement dynamique de certaines applications lisses par morceaux issues d'applications de Zeraoulia-Sprott prises à une et deux dimensions, qui se caractérisent par une nature comportementale très dispersé et atteint l'extreme du chaos robuste en raison de les bifurcations collision de frontière qui se produit surtout dans ce type d'applications.

Mots clés: *Bifurcations collision de la frontière*, *applications lisse par morceaux, comportement chaotique.*

ملخص

الهدف من هذه المذكرة هو دراسة السلوك الديناميكي لبعض التطبيقات الناعمة المعرفة بالأجزاء من تطبيقات زراولية-سبروت، المأخوذة في البعدين واحد وإثنان والتي تتميز بطبيعة سلوكية شديدة التشتت تصل حد الفوضى القوية نتيجة تشعبات تصادم الحدود، التي تحدث بشكل خاص في هذا النوع من التطبيقات.

الكلمات المفتاحية: تشعبات تصادم الحدود، التطبيقات الناعمة المعرفة بالأجزاء، سلوك فوضوي.

Contents

1 Border Collision Bifurcations **5**

1.1 Piecewise smooth maps 5

1.1.1 One-dimensional piecewise smooth maps 6

1.1.2 Two-dimensional piecewise smooth maps 12

2 Bifurcations of the 1-D Zeraoulia map **18**

2.1 One-dimensional piecewise smooth map 18

2.1.1 The normal form 18

2.1.2 The fixed points 22

2.2 The border collision bifurcations 22

3 Bifurcations of some 2-D Zeraoulia-Sprott mappings **27**

3.1 A unified piecewise smooth chaotic mapping that contains the Hénon and the Lozi systems 27

3.1.1 Two-dimensional piecewise smooth map 27

3.1.2 The border collision bifurcations 35

3.2 The discrete hyper-chaotic double scroll 37

3.2.1 Two-dimensional piecewise smooth map 37

3.2.2 Stability of fixed points 41

3.3 Conclusion 45

List of Figures

1.1 Partitioning of the parameter space into regions with the same qualitative phenomena. The labeling of regions refers to various bifurcation scenarios. $1)$ *Persistence of stable fixed points,* $2)$ *Persistence of unstable fixed points*, $3)$ *No fixed point to stable and unstable fixed points,* $4)$ *No fixed point to two unstable fixed points and chaotic attractor*, $5)$ *No fixed point to two unstable fixed points,* $6)$ *Supercritical border collision period doubling,* $7)$ *Subcritical border collision period doubling,* $8)$ *A stable fixed point to periodic or chaotic attractor*. The regions shown in primed numbers have the same bifurcation behavior as the unprimed ones when μ is varied in the opposite direction. 10

1.2 The parameter region $0 < a < 1$ and $b < -1$, showing the type of attractor for $\mu > 0$. Regions P_n correspond the existence of stable period n orbit, inside the shaded region there exists chaotic attractors. 11

1.3 The transformation of coordinates from the two-dimensional piecewise smooth map to the normal form. 15

1.4 The types of fixed points of the normal form map. 16

2.1 Bifurcation diagrams for the map $N_1(x,\mu) = \begin{cases} ax+\mu, & \text{for } x \leq 0 \\ bx+\mu, & \text{for } x \geq 0 \end{cases}$. (a) $a = 0.5$, $b = -3.5$: At $\mu = 0$, N_1 exhibits a border-collision bifurcation from a period-1 attractor to a period-3 attractor at $x = 0$. (b) $a = 0.5$, $b = -4.15$: At $\mu = 0$, N_1 exhibits a border-collision bifurcation from a fixed point attractor to a six-piece chaotic attractor at $x = 0$. (c) $a = 0.5$, $b = -4.4$: At $\mu = 0$, N_1 exhibits a border-collision bifurcation from a fixed point attractor to a three-piece chaotic attractor at $x = 0$. (d) $a = 0.5$, $b = -5.5$: At $\mu = 0$, N_1 exhibits a border-collision bifurcation from a fixed point attractor to a one-piece chaotic attractor at $x = 0$ [6]. 21

3.1 (a) The original Hénon chaotic attractor obtained from the H mapping with its basin of attraction (white) for $a = 1.4$ and $b = 0.3$. (b) The original Lozi chaotic attractor obtained from the L mapping with its basin of attraction (white) for $a = 1.4$ and $b = 0.3$. 28
3.2 (a) The transition Hénon-like chaotic attractor obtained for the unified chaotic map (3.1) with its basin of attraction (white) for $\alpha = 0.2$. (b) The graph of the function $f_{0.2}$. (c) The transition Lozi-like chaotic attractor obtained for the unified chaotic map (3.1) with its basin of attraction (white) for $\alpha = 0.8$. (d) The graph of the function $f_{0.8}$. 29
3.3 The classical double scroll attractor obtained for $\alpha = 9.35$, $\beta = 14.79$, $m_0 = \frac{-1}{7}$, $m_1 = \frac{2}{7}$. 38
3.4 The discrete hyperchaotic double scroll attractor obtained from the map (3.6) for $a = 3.36$, $b = 1.4$, $m_0 = -0.43$, and $m_1 = 0.41$ with initial conditions $x = y = 0.1$. . . 39

Introduction

The bifurcation theory is an essential part in the study of dynamical systems, it appeared as a term for the first time with the mathematician **Henri poincaré** at the beginning of the 20^{th} century during his work on differential systems, and since then it has been in continuous development to this day. It is defined as a quantitative or qualitative change in the solution of a dynamical system with a modification of the parameters on which it depends and there are two types of them: *local bifurcation* which can be analyzed entirely by changes in the stability of local equilibrium properties, periodic orbit or other invariant sets as the parameters cross critical thresholds, and *global bifurcation* which often occur when the larger invariant sets of the system collide with each other, or with the equilibriums of the system. They cannot be detected only with an analysis of the stability of the equilibria (fixed points).

In this work, we focus on a new type of bifurcations called the **border collision bifurcations** belongs to the *global bifurcations* and especially occurs in **piecewise smooth maps** when a fixed point (or periodic point) meets the *switching manifold* and is divided into two types namely *border collision pair bifurcation* and *border crossing bifurcation*.

In particular, we study the bifurcation theory for continuous piecewise smooth discrete-time systems in one and two dimensions. For more details we divide this thesis into 3 chapters as follows:

- **Chapter 1,** is devoted to presenting the essential results on the chaotic dynamics and bifurcations in one and two-dimensional piecewise smooth maps.
- **Chapter 2,** is limited to the study of the theory of border collision bifurcation in 1-D piecewise smooth **Zeraoulia mapping**.
- **Chapter 3**, is also concerned with the study the bifurcations mentioned previously in 2-D piecewise smooth **Zeraoulia-Sprott mappings.**

Chapter 1

Border Collision Bifurcations

In this chapter, we will talk about a new type of bifurcations, completely different from everything we studied upon previously such as saddle node, pitchfork, hopf..., called **the border collision bifurcations.** It appeared as a term for the first time in [5], although it was previously presented in the Russian literature under the name **C-bifurcation** attributed to the scientist Feigen in [4], that especially occurs in piecewise smooth maps and the reason for this is due to the fact that the latter is very effective in modeling the non-smoothness in the systems accurately and an example of this from physics (switching circuits), as this type of bifurcation is clearly manifested from a mathematical point of view at the border which namely *switching surface*, means that this bifurcation occurs when the nature of the fixed point is changed as it crosses the switching surface and it belongs to the category of global bifurcation that lead to the so-called **robust chaos**. But we are only concerned with studying some parts of these bifurcations.

1.1 Piecewise smooth maps

This section is based on the study of the piecewise smooth map in one and two dimensional through three main points represented in defining the map and presenting some of its properties, in addition to the normal form and its fixed point for both dimensions and finally addressing the border collision bifurcations. Consider a map $F : \mathbb{R}^m \to \mathbb{R}^m$ as follow:

$$x_{n+1} = F(x_n), \quad x_0 \in \mathbb{R} \tag{1.1}$$

Some properties

- The map (1.1) is a piecewise smooth, if the phase space $\mathbb{R}^m$ can be partitioned into a finite number J of disjoint non-empty open regions R_i, $i = 1, ..., J$, and a boundary Σ, so that

$$\mathbb{R}^m = \left(\bigcup_{i=1}^{J} R_i\right) \cup \Sigma.$$

- The boundary Σ made up of a union of continuously differentiable surfaces which separate the regions R_i.
- F is smooth in each regions R_i.
- Non-smoothness of F occurs on Σ, which is called **switching surface** or switching manifold.
- The map (1.1) is also known as **hybrid system**. For more details see [7].

The most important results about these maps are about the existence of relation between the chaotic behaviors and the border collision bifurcations. Note that the analysis of this relation is based on some ingredients. The first of which is the *affinity of the corresponding normal forms* for fixed points on the borders, and second is the *behavior of fixed points* (or periodic points) depending on the bifurcation parameter associated with the various cases, and this study is carried out in one and two-dimensions as follows which taken from [2] and [1] as follows:

1.1.1 One-dimensional piecewise smooth maps

Consider the following 1-D piecewise smooth system:

$$x_{n+1} = f\ (x_n, \mu) = \begin{cases} g\,(x, \mu)\,, & x < x_b \\ h\,(x, \mu)\,, & x > x_b \end{cases} \tag{1.2}$$

where μ is the bifurcation parameter, the smooth curve $x = x_b$, divided the state space into two regions R_L and R_R given by:

$$\begin{cases} R_L = \{x \in \mathbb{R} : x < x_b\} \\ R_R = \{x \in \mathbb{R} : x > x_b\} \end{cases}$$

and the boundary between them as is given by:

$$\Sigma = \{x \in \mathbb{R} : x = x_b\}$$

Some properties

- The map f is continuous, but its derivative is discontinuous at the borderline $x = x_b$.
- The functions g and h are both continuous and they have continuous derivatives in x everywhere except at x_b.
- $x_0\,(\mu)$ is a possible path of fixed points of f, this path depends continuously on μ.
- The fixed point possible hits the boundary at a critical parameter value $\mu_b : x_0\,(\mu_b) = x_b$.

The normal form

In order to facilitate and simplify the study of the border collision bifurcations in 1-D piecewise smooth map we need the following theorem:

Theorem 1.1 *The normal form of the piecewise smooth one-dimensional map (1.2) is given by* [2] *as:*

$$N_1(x,\mu) = \begin{cases} ax+\mu, & x<0 \\ bx+\mu, & x>0 \end{cases} \tag{1.3}$$

where μ *is a parameter, and* a, b *are the slopes of the graph at the two sides (*R_L *and* R_R*) of the border* $x = 0$.

Proof. The normal form (1.3) at a fixed point on the border is a piecewise affine approximation of the map in the neighborhood of the border point x_b.

- The method of derivation of such a form is as follows:

1. Let $\bar{x} = x - x_b$ and $\bar{\mu} = \mu - \mu_b$, then the equation (1.2) becomes:

$$\bar{f}\ (\bar{x},\bar{\mu}) = \begin{cases} g(\bar{x}+x_b, \bar{\mu}+\mu_b), & \bar{x}<0 \\ h(\bar{x}+x_b, \bar{\mu}+\mu_b), & \bar{x}>0 \end{cases} \tag{1.4}$$

   Hence, for map (1.4), we have the following properties:

   - The border is at $\bar{x} = 0$.
   - The state space is divided into two halves, $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0,\infty)$.
   - The fixed point of (1.4) is at the border for the parameter value $\bar{\mu} = 0$.

2. Expanding $\bar{f}$ to first order about $(0,0)$ gives:

$$\begin{cases} \bar{f}\ (\bar{x},\bar{\mu}) = \begin{cases} a\bar{x}+\bar{\mu}\upsilon + O(\bar{x},\bar{\mu}), & \bar{x}<0 \\ b\bar{x}+\bar{\mu}\upsilon + O(\bar{x},\bar{\mu}), & \bar{x}>0 \end{cases} \\ a = \lim_{x\to 0^-} \frac{\partial}{\partial x}\bar{f}\ \ (\bar{x},0) \\ b = \lim_{x\to 0^+} \frac{\partial}{\partial x}\bar{f}\ \ (\bar{x},0) \\ \upsilon = \lim_{x\to 0} \frac{\partial}{\partial \mu}\bar{f}\ \ (\bar{x},0) \end{cases} \tag{1.5}$$

   such that:

   - Due to the smoothness of f in μ, the last limit in (1.5) doesn't depend on the direction of approach of 0 by x.
   - Under the hypotheses $\upsilon \neq 0, |a| \neq 1$ and $|b| \neq 1$, the non-linear terms are negligible close to the border.

3. Finally, we define a new parameter $\mu'' = \bar{\mu}v$ and dropping the higher order terms as in [2], then the 1-D normal form is given by:

$$G_1(x, \bar{\mu}) = \begin{cases} a\bar{x} + \mu'', & \bar{x} < 0 \\ b\bar{x} + \mu'', & \bar{x} > 0 \end{cases}$$

which has the same form of (1.3).

■

The fixed points

- Let x_R^* and x_L^* be the possible fixed points of the system near the border to the right $(x > x_b)$ and left $(x < x_b)$ of the border, respectively. Then in the normal form (1.3) we have

$$\begin{cases} x_R^* = \frac{\mu}{1-b} > 0, \text{ if } b < 1 \wedge \mu > 0 \\ \qquad\qquad \text{and} \\ x_L^* = \frac{\mu}{1-a} < 0, \text{ if } a < 1 \wedge \mu < 0 \end{cases}$$

Border collision bifurcation scenarios

In the following section, we discuss some border collision bifurcation scenarios from x_b for μ near μ_b.

- Border collision bifurcation scenarios can be obtained by various combinations of the parameters $a \geq b$ as μ is varied. It is the same for $a < b$ which are summarized in Figure 1.1, because the normal form (1.3) is invariant under the transformation $x \to -x$, $\mu \to -\mu$, $a \rightleftarrows b$. See also [2]:

Scenario 1: *(Persistence of stable fixed point) or Period-1 → Period-1.*

If $-1 < b \leq a < 1$, then there is no bifurcation and a stable fixed point for $\mu < 0$ persists and remains stable for $\mu > 0$.

Scenario 2: *(Persistence of unstable fixed point) or No Attractor → No Attractor.*

If $1 < b \leq a$ or $b \leq a < -1$, then there is no bifurcation and an unstable fixed point for $\mu < 0$ persists and remains unstable for $\mu > 0$.

Scenario 3: *(Merging and annihilation of stable and unstable fixed points) or No Fixed Point → Period-1.*

If $-1 < b < 1 < a$, then there is a bifurcation from no fixed point for $\mu < 0$ to two fixed points x_L (unstable) and x_R (stable) for $\mu > 0$.

Scenario 4: *(Merging and annihilation of two unstable fixed points, plus chaos). No fixed point $\rightarrow$ chaos.*

If $a > 1$ and $\frac{-a}{a-1} < b < -1$, then there is a bifurcation from no fixed point to two unstable fixed points plus a growing chaotic attractor as μ is increased through zero.

Scenario 5: *(Merging and annihilation of two unstable fixed points) or No fixed point $\rightarrow$ No attractor.*

If $a > 1$ and $b < \frac{-a}{a-1}$, then there is a bifurcation from no fixed point to two unstable fixed points as μ is increased through zero and there is an unstable chaotic orbit for $\mu > 0$.

Scenario 6: *(Supercritical border collision period doubling) or Period-1 $\rightarrow$ Period-2.*

If $b < -1 < a < 0$ and $-1 < ab < 1$, then there is a bifurcation from a stable fixed point x_L to an unstable fixed point x_R plus a stable period-2 orbit as μ is increased through zero.

Scenario 7: *(Subcritical border collision period doubling) or Period-1 $\rightarrow$ No Attractor.*

If $b < -1 < a < 0$ and $ab > 1$, then there is a bifurcation from a stable fixed point x_L plus an unstable period-2 orbit to an unstable fixed point x_R as μ is increased though zero.

Scenario 8: *(Emergence of periodic or chaotic attractor from stable fixed point) or Period-1 $\rightarrow$ Periodic or Chaotic Attractor.*

If $0 < a < 1$, $b < -1$ and $ab < -1$, then there is a bifurcation from a stable fixed point x_L to an unstable fixed point x_R plus a period-n attractor, $n \geq 2$ or a chaotic attractor which is depends on the pair of parameters (a, b) as shown in Figure 1.2 as μ is increased through zero.

- Now we give the following definitions. For more details see [11]:

Definition 1.1 *The border collision pair bifurcation is a kind of border collision bifurcations and its similar to saddle node bifurcation (or tangent bifurcation) in smooth systems. In this bifurcation, the smooth map has two fixed points (one side of the border and the other fixed point is on the opposite side) for positive (respectively, negative) values of μ, and no fixed points for negative (respectively, positive) values of μ. Hence, the border collision pair bifurcation occurs if:*

$$b < 1 < a$$

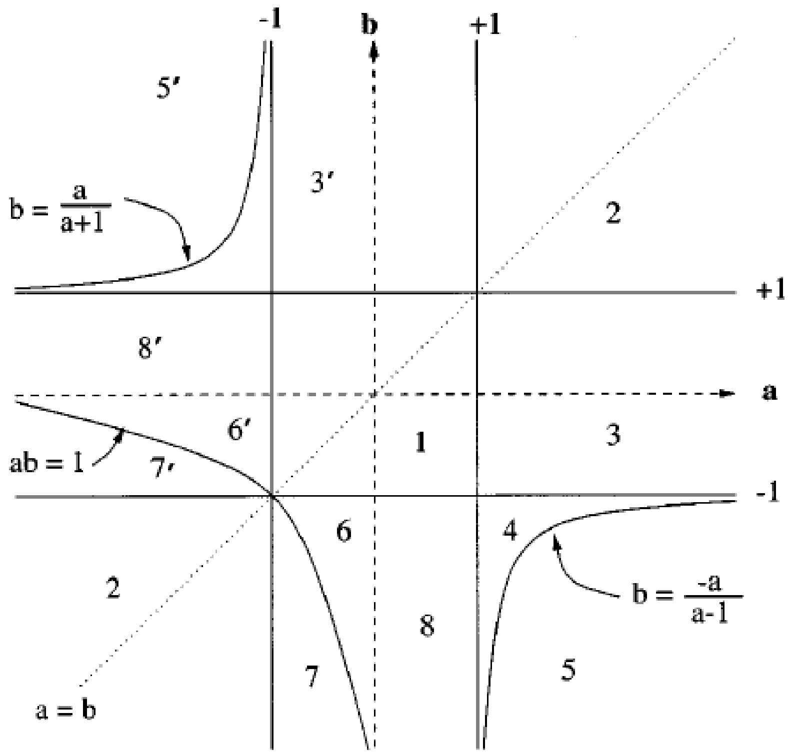

Figure 1.1: Partitioning of the parameter space into regions with the same qualitative phenomena. The labeling of regions refers to various bifurcation scenarios. 1) *Persistence of stable fixed points,* 2) *Persistence of unstable fixed points*, 3) *No fixed point to stable and unstable fixed points,* 4) *No fixed point to two unstable fixed points and chaotic attractor*, 5) *No fixed point to two unstable fixed points,* 6) *Supercritical border collision period doubling,* 7) *Subcritical border collision period doubling,* 8) *A stable fixed point to periodic or chaotic attractor*. The regions shown in primed numbers have the same bifurcation behavior as the unprimed ones when μ is varied in the opposite direction.

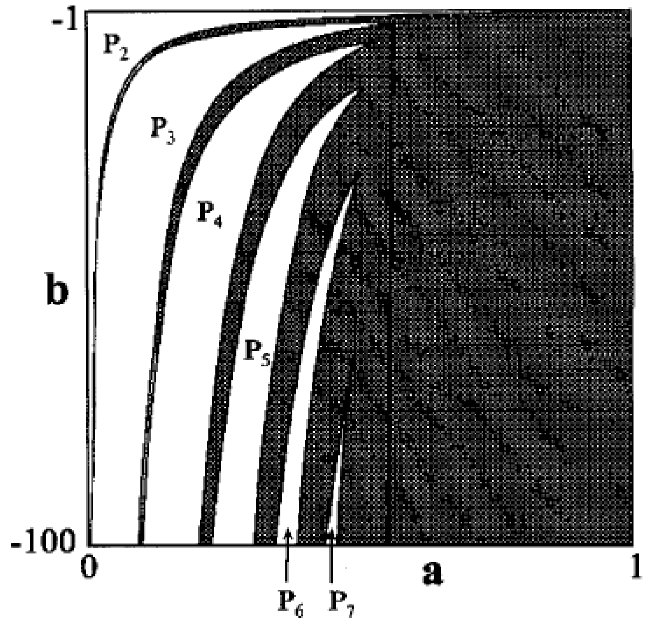

Figure 1.2: The parameter region $0 < a < 1$ and $b < -1$, showing the type of attractor for $\mu > 0$. Regions P_n correspond the existence of stable period n orbit, inside the shaded region there exists chaotic attractors.

Definition 1.2 *The border crossing bifurcation is a kind of border collision bifurcations, it has some similarities with period doubling bifurcation in smooth maps (supercritical period doubling bifurcation in smooth maps with one distinction). In this bifurcation, the fixed point persists and crosses the border as μ is varied through zero and other attractors or repellers appear or disappear as a result of the bifurcation. Indeed, border crossing bifurcation occurs if:*

$$a > -1 \text{ and } b < -1$$

Remark 1.1 *From the previous definitions, we can summarize the above scenarios as follows:*

- *The two scenarios 1 and 2 belongs to the **Scenario A** "Persistence of stable fixed point", at $\mu = 0$.*
- *The three scenarios 3, 4 and 5 belongs to the **Scenario B** "Border collision pair bifurcation".*
- *The last three scenarios 6, 7 and 8 belongs to the **Scenario C** "Border crossing bifurcation".*

1.1.2 Two-dimensional piecewise smooth maps

Let us consider the following 2-D piecewise smooth system given by:

$$g(\hat{x},\hat{y},\rho) = \begin{pmatrix} g_1 = \begin{pmatrix} f_1(\hat{x},\hat{y},\rho) \\ f_2(\hat{x},\hat{y},\rho) \end{pmatrix}, & \text{if } \hat{x} < S(\hat{y},\rho) \\ g_2 = \begin{pmatrix} f_3(\hat{x},\hat{y},\rho) \\ f_4(\hat{x},\hat{y},\rho) \end{pmatrix}, & \text{if } \hat{x} > S(\hat{y},\rho) \end{pmatrix} \tag{1.6}$$

where ρ is the bifurcation parameter, the smooth curve $\hat{x} = S(\hat{y},\rho)$ divided the phase plane into two regions R_L and R_R given by:

$$\begin{cases} R_L = \{(\hat{x},\hat{y}) \in \mathbb{R}^2, \ \ \hat{x} < S(\hat{y},\rho)\} \\ R_R = \{(\hat{x},\hat{y}) \in \mathbb{R}^2, \ \ \hat{x} > S(\hat{y},\rho)\} \end{cases}$$

and the boundary between them as:

$$\Sigma = \left\{(\hat{x},\hat{y}) \in \mathbb{R}^2, \ \ \hat{x} = S(\hat{y},\rho)\right\}$$

Some properties

- The map g is continuous, but its derivative is discontinuous at the borderline $\hat{x} = S(\hat{y},\rho)$.
- The functions g_1 and g_2 are both continuous and have continuous derivatives.
- The one-sided partial derivatives at the border are finite and in each subregion R_L and R_R.
- The map (1.6) has one fixed point in R_L and one fixed point in R_R for a value ρ_* of the parameter ρ.

The normal form

The results outlined above in 1-D normal form give a complete description of the bifurcations as μ is varied it has been shown in [2], for 2-D piecewise smooth maps, a normal form for border collision bifurcation can again be written as shown in [1] as follows:

Theorem 1.2 *The normal form of the piecewise smooth two-dimensional map (1.6) is given by:*

$$N_2(x,y) = \begin{cases} \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x < 0 \\ \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x > 0 \end{cases} \tag{1.7}$$

where μ is a parameter and $\tau_{L,R}$, $\delta_{L,R}$ are the traces and determinants of the corresponding matrices of the linearized map in the two subregions R_L and R_R.

Proof. The normal form (1.7) at a fixed point on the border is a piecewise affine approximation of the map in the neighborhood of the borderline $\hat{x} = S(\hat{y}, \rho)$.

- The method of derivation of such a form is as follows:

1. Let $\tilde{x} = \hat{x} - S(\hat{y}, \rho)$ and $\tilde{y} = \hat{y}$, this ρ-dependent change of variables moves the border to the $\tilde{y}$-axis, then the equation (1.6) becomes:

$$g(\tilde{x} + S(\hat{y}, \rho), \hat{y}, \rho) = f(\tilde{x}, \tilde{y}, \rho) \tag{1.8}$$

   Hence, for the map (1.8), we have the following properties:

   - The border is $\tilde{x} = 0$.
   - The phase space is divided into two halves L and R (for left and right), by the next transformation of coordinates.
   - The map (1.8) has a fixed point $P_* = (0, \tilde{y}_*(\rho_*))$ on the border when $\rho = \rho_*$.

2. The transformation of coordinates is summarized in these steps:

   - Let e_1 be a tangent vector in the $\tilde{y}$ direction and suppose that the vector e_1 maps to a vector e_2.
   - Assume e_2 is not parallel to e_1.
   - Define new coordinates again as shown in Figure 1.3.
   - Choose the point P_* as the new origin for e_1 in the $\bar{y}$ direction and e_2 in the $\bar{x}$ direction.
   - In $\bar{x} - \bar{y}$ coordinates, the fixed point P_* is now $(0,0)$ and the border is given by $\bar{x} = 0$.
   - Define the new parameter $\bar{\mu} = \rho - \rho_*$, so $\bar{\mu}_* = 0$.
   - Rescale $\bar{x}$ and $\bar{y}$ again such that at $\bar{\mu} = 0$ a unit vector along the $\bar{y}$-axis maps to a unit vector along the $\bar{x}$-axis. Then, the map $f(\tilde{x}, \tilde{y}, \rho)$ can be written as $F(\bar{x}, \bar{y}, \bar{\mu})$.

3. Now, write the map $F(\bar{x}, \bar{y}, \bar{\mu})$ in the side L in the matrix form as:

$$F_L(\bar{x}, \bar{y}, \bar{\mu}) = \begin{pmatrix} f_1(\bar{x}, \bar{y}, \bar{\mu}) \\ f_2(\bar{x}, \bar{y}, \bar{\mu}) \end{pmatrix}, \quad \text{and} \quad F_L(0,0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   and linearizing $F(\bar{x}, \bar{y}, \bar{\mu})$ in the neighbourhood of $(0,0,0)$, we have

$$F_L(\bar{x}, \bar{y}, \bar{\mu}) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \bar{\mu} \begin{pmatrix} v_{Lx} \\ v_{Ly} \end{pmatrix} + O(\bar{x}, \bar{y}, \bar{\mu}) \quad \text{for} \quad \bar{x} < 0 \tag{1.9}$$

where

$$\begin{cases} J_{11} = \lim_{\bar{x}\to 0^-,\bar{y}\to 0} \frac{\partial}{\partial \bar{x}} f_1(\bar{x},\bar{y},0) \\ J_{12} = \lim_{\bar{x}\to 0^-,\bar{y}\to 0} \frac{\partial}{\partial \bar{y}} f_1(\bar{x},\bar{y},0) \\ J_{21} = \lim_{\bar{x}\to 0^-,\bar{y}\to 0} \frac{\partial}{\partial \bar{x}} f_2(\bar{x},\bar{y},0) \\ J_{22} = \lim_{\bar{x}\to 0^-,\bar{y}\to 0} \frac{\partial}{\partial \bar{y}} f_2(\bar{x},\bar{y},0) \\ v_{Lx} = \lim_{\bar{x}\to 0^-,\bar{y}\to 0} \frac{\partial}{\partial \bar{\mu}} f_1(\bar{x},\bar{y},0) \\ v_{Ly} = \lim_{\bar{x}\to 0^-,\bar{y}\to 0} \frac{\partial}{\partial \bar{\mu}} f_2(\bar{x},\bar{y},0) \end{cases}$$

Then, the equation (1.9) becomes:

$$F_L(\bar{x},\bar{y},\bar{\mu}) = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \bar{\mu} \begin{pmatrix} v_{Lx} \\ v_{Ly} \end{pmatrix} + O(\bar{x},\bar{y},\bar{\mu}) \quad \text{for } \bar{x} < 0$$

such that

$$J_{11} = \tau_L \text{ (trace)} \quad \text{and} \quad J_{21} = -\delta_L \text{ (determinant)}$$

and since a unit vector along the $\bar{y}$ axis maps to a unit vector along the $\bar{x}$ axis at $\bar{\mu} = 0$, we have

$$J_{12} = 1 \quad \text{and} \quad J_{22} = 0$$

Similarly, for side R we obtain:

$$F_R(\bar{x},\bar{y},\bar{\mu}) = \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \bar{\mu} \begin{pmatrix} v_{Rx} \\ v_{Ry} \end{pmatrix} + O(\bar{x},\bar{y},\bar{\mu}) \quad \text{for } \bar{x} > 0$$

Continuity of the map implies:

$$\begin{pmatrix} v_{Lx} \\ v_{Ly} \end{pmatrix} = \begin{pmatrix} v_{Rx} \\ v_{Ry} \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

4. Make another change of variables as follow: Let $x = \bar{x}$, $y = \bar{y} - \bar{\mu} v_y$, and $\mu = \bar{\mu}(v_x + v_y)$ with $(v_x + v_y) \neq 0$. The choice of axis is independent of the parameter. Then, we have the normal form:

$$N(x,y) = \begin{cases} \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x < 0 \\ \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x > 0 \end{cases} \tag{1.10}$$

where μ is the parameter and $\tau_{L,R}$, $\delta_{L,R}$ are the traces and determinants of the corresponding matrices of the linearized map in the two subregions R_L and R_R given by:

$$\begin{cases} R_L = \{(x,y) \in \mathbb{R}^2\}, & x > 0 \\ R_R = \{(x,y) \in \mathbb{R}^2\}, & x > 0 \end{cases}$$

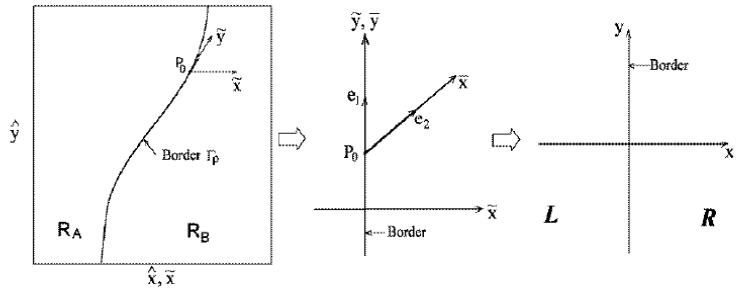

Figure 1.3: The transformation of coordinates from the two-dimensional piecewise smooth map to the normal form.

in the regions R_L and R_R, the map (1.10) is smooth and the boundary between them is given by:

$$\Sigma = \left\{(x,y) \in \mathbb{R}^2, \quad x = 0, \quad y \in \mathbb{R}\right\}$$

■

Remark 1.2 *There is a relation between the normal form of the piecewise smooth one-dimensional map and the normal form of the piecewise smooth two-dimensional map, where we can move from* (1.7) *to* (1.3) *when* δ_i *are zero for* $i = L, R$.

The fixed points

- Let P_L and P_R be the possible fixed points of the system near the border to the right: $x < S(\hat{y}, \rho)$ and left: $x > S(\hat{y}, \rho)$ of the border respectively. Then in the normal form (1.7) we have

$$\begin{cases} P_L = \left(\frac{\mu}{1-\tau_L+\delta_L}, \frac{-\delta_L\mu}{1-\tau_L+\delta_L}\right) \in R_L \\ P_R = \left(\frac{\mu}{1-\tau_R+\delta_R}, \frac{-\delta_R\mu}{1-\tau_R+\delta_R}\right) \in R_R \end{cases}$$

  with eigenvalues $\lambda_{L\,1.2}$ and $\lambda_{R\,1.2}$ respectively.

- The stability of the fixed points is determined by the eigenvalues of the corresponding Jacobian matrix, i.e.,

$$\lambda = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\delta}\right)$$

Border collision bifurcations

The border collision bifurcations can be obtained by various combinations of the values τ_L, τ_R, δ_L and δ_R as μ is varied through zero and because our study of this bifurcations in this dimension is

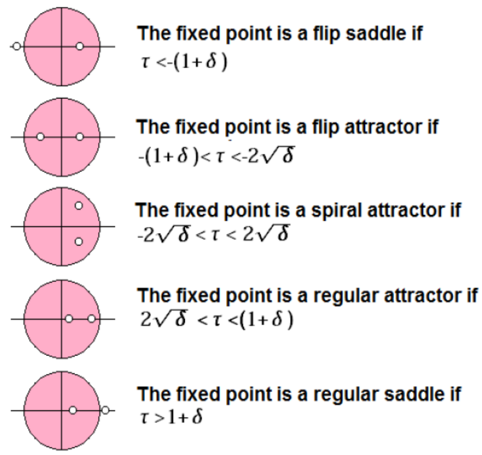

Figure 1.4: The types of fixed points of the normal form map.

limited only to a part that is the classification of fixed points under the both conditions $|\delta_L| < 1$ and $|\delta_R| < 1$. So the possible types of fixed points of the normal form map (1.7) shown in Figure 1.4 are given by:

(1) ***For positive determinant***

(1.a) For $2\sqrt{\delta} < \tau < (1+\delta)$, then the Jacobian matrix has two real eigenvalues $0 < \lambda_{1L}, \lambda_{2L} < 1$ and the fixed point is a **regular attractor**.

(1.b) For $\tau > 1+\delta$, then the Jacobian matrix has two real eigenvalues $0 < \lambda_{1L} < 1$, $\lambda_{2L} > 1$ and the fixed point is a **regular saddle**.

(1.c) For $-(1+\delta) < \tau < -2\sqrt{\delta}$, then the Jacobian matrix has two real eigenvalues $-1 < \lambda_{1L}, \lambda_{2L} < 0$ and the fixed point is a **flip attractor**.

(1.d) For $\tau < -(1+\delta)$, then the Jacobian matrix has two real eigenvalues $-1 < \lambda_{1L} < 0$, $\lambda_{2L} < -1$ and the fixed point is a **flip saddle**.

(1.e) For $0 < \tau < 2\sqrt{\delta}$, then the Jacobian matrix has two complex eigenvalues $|\lambda_{1L}|, |\lambda_{2L}| < 1$ and the fixed point is a **clockwise spiral**.

(1.g) For $-2\sqrt{\delta} < \tau < 0$, then the Jacobian matrix has two complex eigenvalues $|\lambda_{1L}|, |\lambda_{2L}| < 1$ and the fixed point is a **counter-clockwise spiral**.

(2) ***For negative determinant***

(2.a) For $-(1+\delta) < \tau < (1+\delta)$, then the Jacobian matrix has two real eigenvalues $-1 < \lambda_{1L} < 0, 0 < \lambda_{2L} < 1$ and the fixed point is a **flip attractor**.

(2.b) For $\tau > (1+\delta)$, then the Jacobian matrix has two real eigenvalues $\lambda_{1L} > 1, -1 < \lambda_{2L} < 0$ and the fixed point is a **flip saddle**.

(2.c) For $\tau < -(1+\delta)$, then the Jacobian matrix has two real eigenvalues $0 < \lambda_{1L} < 1, \lambda_{2L} < -1$ and the fixed point is a **flip saddle**. See also [11].

Chapter 2

Bifurcations of the 1-D Zeraoulia map

This chapter is concerned with the application of what was addressed in the first chapter to the one-dimensional Zeraoulia map and it is the piecewise linear logistic map. The study will be based on its definition and the normal form with its fixed points and finally the study of bifurcations in the neighborhood of the fixed point.

2.1 One-dimensional piecewise smooth map

Consider the piecewise logistic map given by [8] as:

$$x_{k+1} = f(x_k\ , \alpha) = \alpha\,|x|\,(1 - |x|) \tag{2.1}$$

where α is the bifurcation parameter, the smooth curve $x = 0$ divided the state space into two regions R_L and R_R given by:

$$\begin{cases} R_L = \{x \in \mathbb{R} : x < 0\} \\ R_R = \{x \in \mathbb{R} : x > 0\} \end{cases}$$

and the boundary between them as:

$$\Sigma = \{x \in \mathbb{R} : x = 0\}$$

So, the piecewise logistic map (2.1) can be written again as follow:

$$x_{k+1} = f(x_k\ , \alpha) = \begin{cases} -\alpha x\,(1 + x) & \text{if } \ x < 0 \\ \alpha x\,(1 - x) & \text{if } \ x > 0 \end{cases}$$

2.1.1 The normal form

In order to determine the associated normal form for the piecewise logistic map (2.1), we should do three main steps which are as follows:

1. We calculate the fixed points of the f mapping.

2. We derive the piecewise logistic map (2.1) on both side.

3. We choose an appropriate coordinate transformation because in our case the choice of axis is independent of the parameter.

Fixed points

The fixed points of the map (2.1) are the real solutions of the system:

$$f_\alpha(x) = x \Longleftrightarrow \alpha|x|(1-|x|) = x$$

therefore, we get the following equations:

$$\begin{cases} -\alpha x(1+x) = x, & x<0 \\ \quad \alpha x(1-x) = x, & x>0 \end{cases} \tag{2.2}$$

$$\Leftrightarrow \begin{cases} \alpha x^2 + \alpha x + x = 0, & x<0 \\ \alpha x^2 - \alpha x + x = 0, & x>0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x(\alpha(x+1)+1) = 0, & x<0 \\ x(\alpha(x-1)+1) = 0, & x>0 \end{cases}$$

- ***In the side L:*** The possible fixed points are:

$$\begin{cases} x_{1,L} = 0, \text{ (unacceptable)} \\ x_{2,L} = -\frac{\alpha+1}{\alpha} \end{cases}$$

such that

$$x_{2,L} = -\frac{\alpha+1}{\alpha} < 0 \Leftrightarrow \begin{cases} \text{if: } \begin{cases} \alpha>0 \ \wedge -(\alpha+1)<0 \\ \alpha<0 \ \wedge -(\alpha+1)>0 \end{cases} \Leftrightarrow \begin{cases} \alpha>0 \wedge \alpha>-1 \\ \alpha<0 \wedge \alpha<-1 \end{cases} \\ \Leftrightarrow \begin{cases} \alpha \in]0,+\infty[\cap]-1,+\infty[=]0,+\infty[\\ \alpha \in]-\infty,0[\cap]-\infty,-1[=]-\infty,-1[\end{cases} \\ \text{then, } \alpha \in]-\infty,-1[\cup]0,+\infty[\end{cases}$$

hence, the only negative solution of the first equation from (2.2) is:

$$x_L = -\frac{\alpha+1}{\alpha}, \quad \text{for } \alpha \in]-\infty,-1[\cup]0,+\infty[$$

- ***In the side R:*** The possible fixed points are:

$$\begin{cases} x_{1,R} = 0, \ \text{(unacceptable)} \\ x_{2,R} = \frac{\alpha-1}{\alpha} \end{cases}$$

such that

$$x_{2,R} = \frac{\alpha-1}{\alpha} > 0 \Leftrightarrow \begin{cases} \text{if: } \begin{cases} \alpha > 0 \ \wedge \alpha - 1 > 0 \\ \alpha < 0 \ \wedge \alpha - 1 < 0 \end{cases} \Leftrightarrow \begin{cases} \alpha > 0 \ \wedge \alpha > 1 \\ \alpha < 0 \ \wedge \alpha < 1 \end{cases} \\ \Leftrightarrow \begin{cases} \alpha \in \left]0, +\infty\right[\cap \left]1, +\infty\right[= \left]1, +\infty\right[\\ \alpha \in \left]-\infty, 0\right[\cap \left]-\infty, 1\right[= \left]-\infty, 0\right[\end{cases} \\ \text{then, } \alpha \in \left]-\infty, 0\right[\cup \left]1, +\infty\right[\end{cases}$$

hence, the only positive solution of the second equation from (2.2) is:

$$x_R = \frac{\alpha-1}{\alpha}, \ \text{ for } \ \alpha \in \left]-\infty, 0\right[\cup \left]1, +\infty\right[$$

Then, the piecewise logistic map (2.1) has two fixed points given by:

$$\begin{cases} x_L = -\frac{\alpha+1}{\alpha} \in R_L \\ x_R = \frac{\alpha-1}{\alpha} \in R_R \end{cases}, \ \text{ for } \ \alpha \in \left]-\infty, -1\right[\cup \left]1, +\infty\right[$$

Derivation

- The derivative of the map (2.1) evaluated at a point x in the both regions R_L and R_R is given by:

$$\begin{cases} Df_L(x) = \alpha(-2x - 1) \\ Df_R(x) = \alpha(-2x + 1) \end{cases}$$

- The derivative of the map (2.1) evaluated at a fixed points in the both regions R_L and R_R is given by:

$$\begin{cases} Df_L(x_L) = 2 + \alpha \\ Df_R(x_R) = 2 - \alpha \end{cases}$$

The coordinate transformation

The normal form of the map (2.1) is given by :

$$N_1(x, \mu) = \begin{cases} (2+\alpha)\, x + \mu, & x < 0 \\ (2-\alpha)\, x + \mu, & x > 0 \end{cases} \tag{2.3}$$

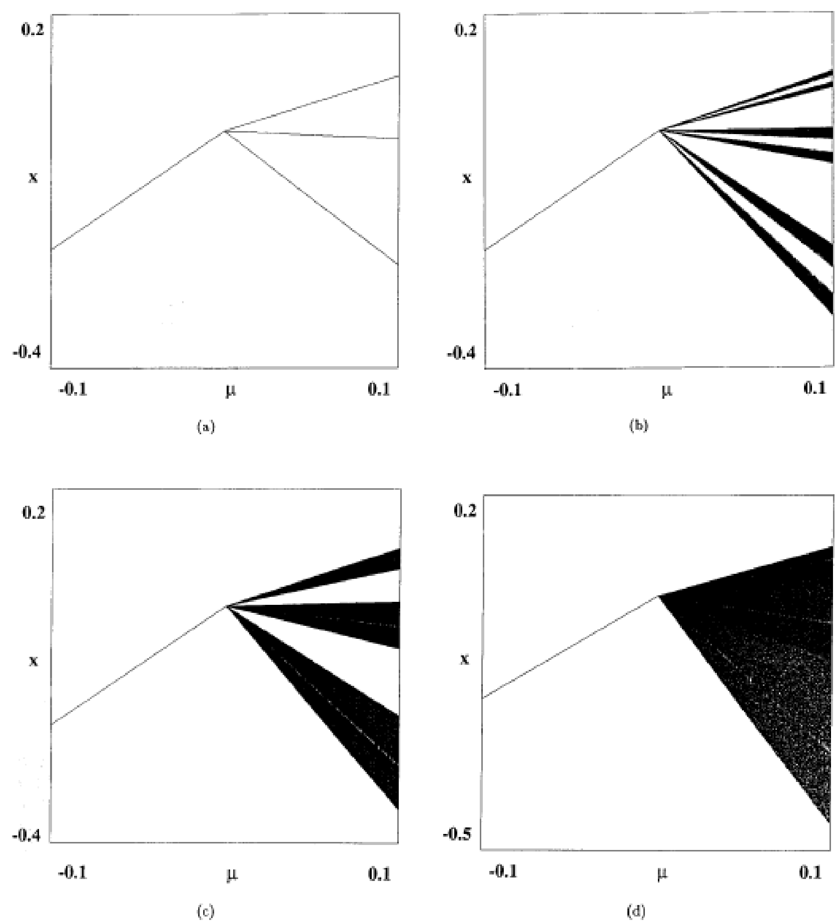

Figure 2.1: Bifurcation diagrams for the map $N_1(x,\mu) = \begin{cases} ax+\mu, & \text{for } x \leq 0 \\ bx+\mu, & \text{for } x \geq 0 \end{cases}$. (a) $a = 0.5$, $b = -3.5$: At $\mu = 0$, N_1 exhibits a border-collision bifurcation from a period-1 attractor to a period-3 attractor at $x = 0$. (b) $a = 0.5$, $b = -4.15$: At $\mu = 0$, N_1 exhibits a border-collision bifurcation from a fixed point attractor to a six-piece chaotic attractor at $x = 0$. (c) $a = 0.5$, $b = -4.4$: At $\mu = 0$, N_1 exhibits a border-collision bifurcation from a fixed point attractor to a three-piece chaotic attractor at $x = 0$. (d) $a = 0.5$, $b = -5.5$: At $\mu = 0$, N_1 exhibits a border-collision bifurcation from a fixed point attractor to a one-piece chaotic attractor at $x = 0$ [6].

2.1.2 The fixed points

- In the normal form (1.3) we have two fixed points x_{L^*} and x_{R^*} near the border to the right $(x<0)$ and the left $(x>0)$ of the border. So from (2.3) we have:

$$\begin{cases} x_{L^*} = -\frac{\mu}{1+\alpha} < 0, & \text{if } \alpha < -1 \wedge \mu < 0 \\ x_{R^*} = \frac{\mu}{\alpha-1} > 0, & \text{if } \alpha > 1 \wedge \mu > 0 \end{cases}$$

2.2 The border collision bifurcations

Since the border collision bifurcations of the original map f is the same as that of the normal form (2.3) as shown in Figure 2.1, the reason for this is due to the appearance of the bifurcation parameter α in the fixed points and because the study of this bifurcation is limited to one of the two cases ($a \geq b$ or $a < b$) plus $\alpha \in I =]-\infty, -1[\cup]1, +\infty[$ so:

1. For $\alpha \in I_1 =]-\infty, -1[\Leftrightarrow \alpha < -1 \Leftrightarrow \begin{cases} \alpha + 2 < 1 \\ 2 - \alpha > 3 \end{cases} \Leftrightarrow \begin{cases} a < 1 \\ b > 3 \end{cases} \Rightarrow a < b$

2. For $\alpha \in I_2 =]1, +\infty[\Leftrightarrow \alpha > 1 \Leftrightarrow \begin{cases} \alpha + 2 > 3 \\ 2 - \alpha < 1 \end{cases} \Leftrightarrow \begin{cases} a > 3 \\ b < 1 \end{cases} \Rightarrow a > b$

therefore, we study only on the field $I_2 =]1, +\infty[$.
<u>**Scenario 1:**</u> *(Persistence of stable fixed point) or Period-1 $\rightarrow$ Period-1.*

- If $-1 < b \leq a < 1 \Leftrightarrow -1 < 2 - \alpha \leq 2 + \alpha < 1$, such that $\alpha \in I_2 =]1, +\infty[$:

$$\Leftrightarrow \begin{cases} -1 < 2 - \alpha \\ 2 - \alpha \leq 2 + \alpha \\ 2 + \alpha < 1 \end{cases} \Leftrightarrow \begin{cases} \alpha < 3 \\ \alpha > 0 \\ \alpha < -1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha \in]-\infty, 3[\cap]1, +\infty[=]1, 3[\subset I_2 \\ \alpha \in]0, +\infty[\cap]1, +\infty[=]1, +\infty[= I_2 \\ \alpha \in]-\infty, -1[\cap]1, +\infty[= \varnothing \end{cases}$$

then, the **Scenario 1** is not hold.

<u>**Scenario 2:**</u> *(Persistence of unstable fixed point) or No Attractor $\rightarrow$ No Attractor.*

- If $1 < b \leq a \Leftrightarrow 1 < 2-\alpha \leq 2+\alpha$, such that $\alpha \in I_2 =]1,+\infty[$:

$$\Leftrightarrow \begin{cases} 1 < 2-\alpha \\ 2-\alpha \leq 2+\alpha \end{cases} \Leftrightarrow \begin{cases} \alpha < 1 \\ \alpha > 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha \in]-\infty,1[\cap]1,+\infty[= \varnothing \\ \alpha \in]0,+\infty[\cap]1,+\infty[=]1,+\infty[= I_2 \end{cases}$$

- If $b \leq a < -1 \Leftrightarrow 2-\alpha \leq 2+\alpha < -1$, such that $\alpha \in I_2 =]1,+\infty[$:

$$\Leftrightarrow \begin{cases} 2-\alpha \leq 2+\alpha \\ 2+\alpha < -1 \end{cases} \Leftrightarrow \begin{cases} \alpha > 0 \\ \alpha < -3 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha \in]0,+\infty[\cap]1,+\infty[=]1,+\infty[= I_2 \\ \alpha \in]-\infty,-3[\cap]1,+\infty[= \varnothing \end{cases}$$

  then, the **Scenario 2** is not hold.

Scenario 3: *(Merging and annihilation of stable and unstable fixed points) or No Fixed Point → Period-1*.

- If $-1 < b < 1 < a \Leftrightarrow -1 < 2-\alpha < 1 < 2+\alpha$, such that $\alpha \in I_2 =]1,+\infty[$:

$$\Leftrightarrow \begin{cases} -1 < 2-\alpha \\ 2-\alpha < 1 \\ 1 < 2+\alpha \end{cases} \Leftrightarrow \begin{cases} \alpha < 3 \\ \alpha > 1 \\ \alpha > -1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha \in]-\infty,3[\cap]1,+\infty[=]1,3[\subset I_2 \\ \alpha \in]1,+\infty[= I_2 \\ \alpha \in]-1,+\infty[\cap]1,+\infty[=]1,+\infty[= I_2 \end{cases}$$

  then, the **Scenario 3** is hold for $\alpha \in]1,3[$ which implies that there is a bifurcation from no fixed point for $\mu < 0$ to two fixed points x_L (unstable) and x_R (stable) for $\mu > 0$.

Scenario 4: *(Merging and annihilation of two unstable fixed points, plus chaos). No fixed point → chaos*.

- If

$$\begin{cases} a > 1 \\ \text{and} \\ \frac{-a}{a-1} < b < -1 \end{cases} \Leftrightarrow \begin{cases} 2+\alpha > 1 \\ \text{and} \\ \frac{-2-\alpha}{1+\alpha} < 2-\alpha < -1 \end{cases} \Leftrightarrow \begin{cases} \alpha > -1 \\ \text{and} \\ -2-\alpha < (2-\alpha)(1+\alpha) < -(1+\alpha) \end{cases}$$

such that $\alpha \in I_2 =]1, +\infty[$:

$$\Leftrightarrow \begin{cases} \alpha > -1 \\ \text{and} \\ \begin{cases} -(2+\alpha) < (2-\alpha)(1+\alpha) \\ (2-\alpha)(1+\alpha) < -(1+\alpha) \end{cases} \end{cases} \Leftrightarrow \begin{cases} \alpha > -1 \\ \text{and} \\ \begin{cases} \alpha^2 - 2\alpha - 4 < 0 \\ \alpha^2 - 2\alpha - 3 > 0 \end{cases} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha \in]-1, +\infty[\cap]1, +\infty[=]1, +\infty[= I_2 \\ \text{and} \\ \begin{cases} \alpha \in \left]-\sqrt{5}+1, \sqrt{5}+1\right[\cap]1, +\infty[= \left]1, \sqrt{5}+1\right[\subset I_2 \\ \alpha \in]-\infty, -1[\cup]3, +\infty[\cap]1, +\infty[=]3, +\infty[\subset I_2 \end{cases} \end{cases}$$

then, the **Scenario 4** is hold for $\alpha \in \left]3, \sqrt{5}+1\right[$ which implies that there is a bifurcation from no fixed point to two unstable fixed points plus a chaotic attractor as μ is increased through zero.

Scenario 5: *(Merging and annihilation of two unstable fixed points) or No fixed point $\rightarrow$ No attractor.*

- If:

$$\begin{cases} a > 1 \\ \text{and} \\ b < \frac{-a}{a-1} \end{cases} \Leftrightarrow \begin{cases} 2+\alpha > 1 \\ \text{and} \\ 2-\alpha < -\frac{2+\alpha}{1+\alpha} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha > -1 \\ \text{and} \\ \alpha^2 - 2\alpha - 4 > 0 \end{cases}$$

such that $\alpha \in I_2 =]1, +\infty[$:

$$\Leftrightarrow \begin{cases} \alpha \in]-1, +\infty[\cap]1, +\infty[=]1, +\infty[= I_2 \\ \text{and} \\ \alpha \in \left]-\infty, -\sqrt{5}+1\right[\cup \left]\sqrt{5}+1, +\infty\right[\cap]1, +\infty[= \left]\sqrt{5}+1, +\infty\right[\subset I_2 \end{cases}$$

then, the **Scenario 5** is hold for $\alpha \in \left]\sqrt{5}+1, +\infty\right[$ which implies that there is a bifurcation from no fixed point to two unstable fixed points as is increased through zero, and there is an unstable chaotic orbit for $\mu > 0$.

Scenario 6: *(Supercritical border collision period doubling) or Period-1 → Period-2.*

- If:

$$\begin{cases} b < -1 < a < 0 \\ \text{and} \\ -1 < ab < 1 \end{cases} \Leftrightarrow \begin{cases} 2-\alpha < -1 < 2+\alpha < 0 \\ \text{and} \\ -1 < (2+\alpha)(2-\alpha) < 1 \end{cases}$$

such that $\alpha \in I_2 =]1, +\infty[$:

$$\Leftrightarrow \begin{cases} 2-\alpha < -1 \\ -1 < 2+\alpha \\ 2+\alpha < 0 \end{cases} \Leftrightarrow \begin{cases} \alpha > 3 \\ \alpha > -3 \\ \alpha < -2 \end{cases} \Leftrightarrow \begin{cases} \alpha \in]3, +\infty[\cap]1, +\infty[=]3, +\infty[\subset I_2 \\ \alpha \in]-3, +\infty[\cap]1, +\infty[=]1, +\infty[= I_2 \\ \alpha \in]-\infty, -2[\cap]1, +\infty[= \varnothing \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{cases} -1 < (2+\alpha)(2-\alpha) \\ (2+\alpha)(2-\alpha) < 1 \end{cases} \Leftrightarrow \begin{cases} -1 < 4-\alpha^2 \\ 4-\alpha^2 < 1 \end{cases} \Leftrightarrow \begin{cases} \alpha^2 < 5 \\ \alpha^2 > 3 \end{cases} \\ \begin{cases} \alpha \in \left]-\sqrt{5}, \sqrt{5}\right[\cap]1, +\infty[= \left]1, \sqrt{5}\right[\subset I_2 \\ \alpha \in \left]-\infty, -\sqrt{3}\right] \cup \left[\sqrt{3}, \infty\right[\cap]1, +\infty[= \left]\sqrt{3}, +\infty\right[\subset I_2 \end{cases} \end{cases}$$

then, the **Scenario 6** is not hold.

Scenario 7: *(Subcritical border collision period doubling) or Period-1 → No Attractor.*

- If:

$$\begin{cases} b < -1 < a < 0 \\ \text{and} \\ ab > 1 \end{cases} \Leftrightarrow \begin{cases} 2-\alpha < -1 < 2+\alpha < 0 \\ \text{and} \\ (2-\alpha)(2+\alpha) > 1 \end{cases}$$

such that $\alpha \in I_2 =]1, +\infty[$:

$$\Leftrightarrow \begin{cases} \begin{cases} 2-\alpha<-1 \\ -1<2+\alpha \\ 2+\alpha<0 \end{cases} \Leftrightarrow \begin{cases} \alpha>3 \\ \alpha>-3 \\ \alpha<-2 \end{cases} \\ \Leftrightarrow \begin{cases} \alpha \in \left]3,+\infty\right[\cap \left]1,+\infty\right[= \left]3,+\infty\right[\subset I_2 \\ \alpha \in \left]-3,+\infty\right[\cap \left]1,+\infty\right[= \left]1,+\infty\right[= I_2 \\ \alpha \in \left]-\infty,-2\right[\cap \left]1,+\infty\right[= \varnothing \end{cases} \\ \text{and} \\ 4-\alpha^2>1 \Leftrightarrow \alpha^2<3 \Leftrightarrow \alpha \in \left]-\sqrt{3},\sqrt{3}\right[\cap \left]1,+\infty\right[= \left]1,\sqrt{3}\right[\subset I_2 \end{cases}$$

then, the **Scenario 7** is not hold.

Scenario 8: *(Emergence of periodic or chaotic attractor from stable fixed point) or Period-1 $\rightarrow$ Periodic or Chaotic Attractor.*

- If :

$$\begin{cases} 0<a<1 \wedge b<-1 \\ \text{and} \\ ab<-1 \end{cases} \Leftrightarrow \begin{cases} 0<2+\alpha<1 \wedge\ 2-\alpha<-1 \\ \text{and} \\ (2+\alpha)(2-\alpha)<-1 \end{cases}$$
$$\Leftrightarrow \begin{cases} \begin{cases} 0<2+\alpha \wedge \alpha>3 \\ 2+\alpha<1 \wedge \alpha>3 \end{cases} \\ \text{and} \\ \alpha^2>5 \end{cases}$$

such that $\alpha \in I_2 = \left]1,+\infty\right[$:

$$\Leftrightarrow \begin{cases} \begin{cases} \alpha>-2 \wedge \alpha>3 \\ \alpha<-1 \wedge \alpha>3 \end{cases} \Leftrightarrow \begin{cases} \alpha \in \left]-2,+\infty\right[\cap \left]3,+\infty\right[\cap I_2 = \left]3,+\infty\right[\subset I_2 \\ \alpha \in \left]-\infty,-1\right[\cap \left]3,+\infty\right[= \varnothing \end{cases} \\ \text{and} \\ \alpha \in \left]-\infty,-\sqrt{5}\right[\cup \left]\sqrt{5},+\infty\right[\cap I_2 = \left]\sqrt{5},+\infty\right[\subset I_2 \end{cases}$$

then, the **Scenario 8** is not hold.

Chapter 3

Bifurcations of some 2-D Zeraoulia-Sprott mappings

This chapter is also devoted for the application to the theoretical part, but to two 2-D Zeraoulia-Sprott mappings. As one of them represents a chaotic model resulting from the unification of two maps that have a chaotic behavior, and the other is a modified version of the Lozi map and has the same non-linearity used in the Chua circuit. However, the latter applies to the border collision bifurcation which relates to three regions while the study in the first chapter is only related to two regions, and for this we only studying stability which is at the heart of the study of bifurcations and an integral part of it.

3.1 A unified piecewise smooth chaotic mapping that contains the Hénon and the Lozi systems

In this section, we will study the **unified chaotic model,** starting from its definition to its normal form and fixed points and finally is the bifurcations in the neighborhood of the fixed points.

3.1.1 Two-dimensional piecewise smooth map

Let us consider the unified chaotic map given by [10] as:

$$U\ (x,y)=\begin{pmatrix} 1-1.4f_{\alpha}\left(x\right)+y \\ 0.3x \end{pmatrix} \tag{3.1}$$

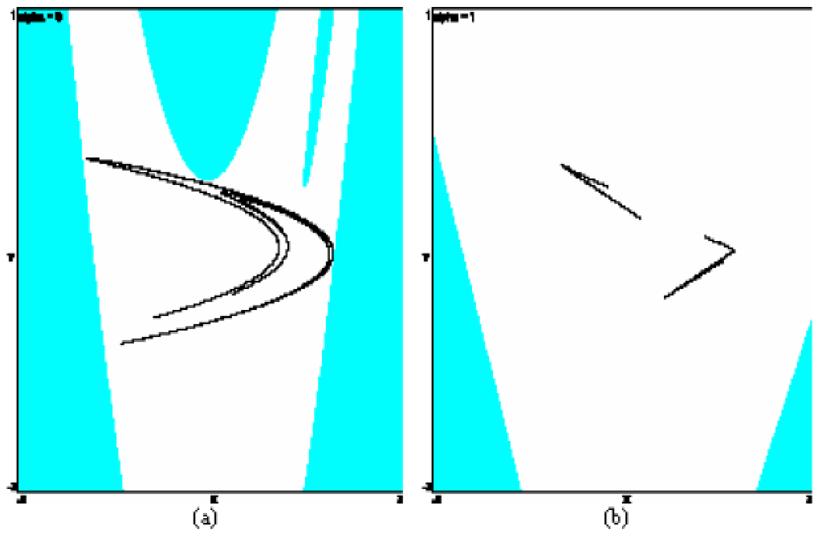

Figure 3.1: (a) The original Hénon chaotic attractor obtained from the H mapping with its basin of attraction (white) for $a = 1.4$ and $b = 0.3$. (b) The original Lozi chaotic attractor obtained from the L mapping with its basin of attraction (white) for $a = 1.4$ and $b = 0.3$.

which defined by two discrete mathematical models, the **Hénon** and the **Lozi** maps shown in Figure 3.1 (a) and (b) given as follows:

$$H(x,y) = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix} \quad \text{and} \quad L(x,y) = \begin{pmatrix} 1 - a\ |x| + y \\ bx \end{pmatrix}$$

such that $0 \leq \alpha \leq 1$ is the bifurcation parameter and the function f_α shown in Figure 3.2 is given by:

$$f_\alpha(x) = \alpha\ |x| + (1-\alpha)\, x^2$$

So, the unified chaotic map (3.1) can be written as follow:

$$U(x,y) = \begin{pmatrix} \begin{cases} 1.4(\alpha-1)\,x^2 + 1.4\alpha x + y + 1, & \text{if } (x,y) \in R_L \\ 1.4(\alpha-1)\,x^2 - 1.4\alpha x + y + 1, & \text{if } (x,y) \in R_R \end{cases} \\ 0.3x \end{pmatrix}$$

where the smooth curve $x = 0$, divided the phase plan into two regions R_L and R_R, given by:

$$\begin{cases} R_L = \{(x,y) \in \mathbb{R}^2,\ \ x < 0\} \\ R_R = \{(x,y) \in \mathbb{R}^2,\ \ x > 0\} \end{cases}$$

and the boundary between them as:

$$\Sigma = \left\{(x,y) \in \mathbb{R}^2,\ \ x = 0\right\}$$

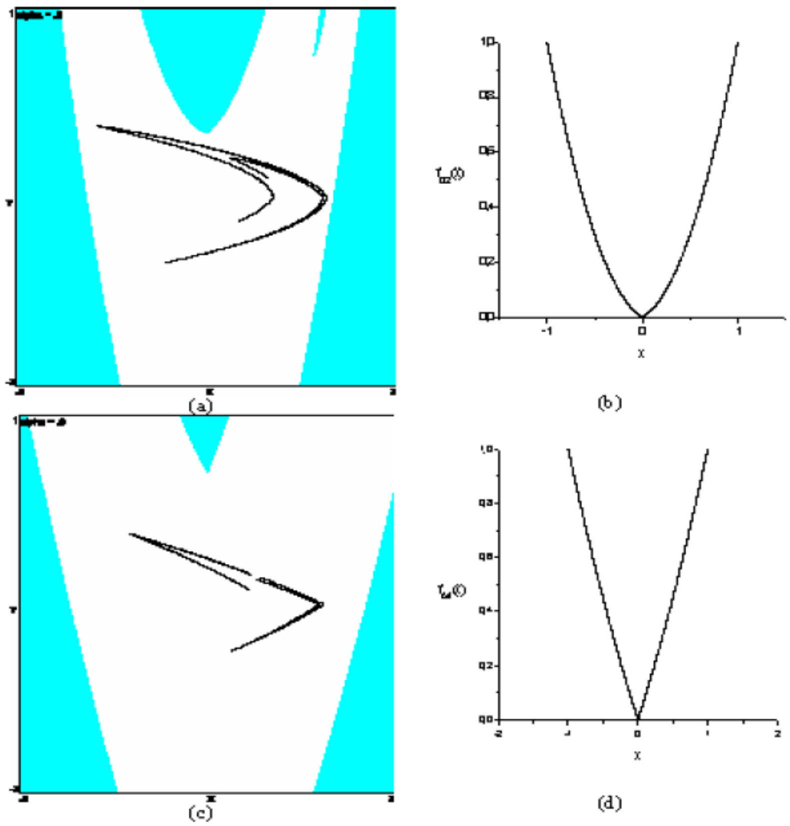

Figure 3.2: (a) The transition Hénon-like chaotic attractor obtained for the unified chaotic map (3.1) with its basin of attraction (white) for $\alpha = 0.2$. (b) The graph of the function $f_{0.2}$. (c) The transition Lozi-like chaotic attractor obtained for the unified chaotic map (3.1) with its basin of attraction (white) for $\alpha = 0.8$. (d) The graph of the function $f_{0.8}$.

The normal form

In order to determine the normal form of the unified chaotic map (3.1). Its sufficient to just follow the steps mentioned previously when we study the first map.

Fixed points The fixed points of the unified chaotic map (3.1) are the real solutions of the system

$$U(x,y)=(x,y)\Leftrightarrow\begin{pmatrix}1-1.4f_\alpha(x)+y\\0.3x\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$$

So, we get two equations:

$$\begin{aligned}&\begin{cases}1.4(\alpha-1)x^2+1.4\alpha x+y+1=x, & \text{for } x<0 \text{ and } 0.3x=y\\1.4(\alpha-1)x^2-1.4\alpha x+y+1=x, & \text{for } x>0 \text{ and } 0.3x=y\end{cases}\\ \Longleftrightarrow &\begin{cases}1.4(\alpha-1)x^2+(1.4\alpha-0.7)x+1=0, & \text{for } x<0 \text{ and } 0.3x=y\\1.4(\alpha-1)x^2-(1.4\alpha+0.7)x+1=0, & \text{for } x>0 \text{ and } 0.3x=y\end{cases}\end{aligned}\tag{3.2}$$

- ***In the side L:*** The discriminant of the first equation from (3.2) is:

  $$\Delta=1.96\alpha^2-7.56\alpha+6.09$$

  to find the sign of discriminant, we set $1.96\alpha^2-7.56\alpha+6.09=0$, and solve a quadratic equation:

  $$\Delta_*=b^2-4ac=9.408>0$$

  so, we get two solutions are:

  $$\begin{cases}\alpha_1=\frac{-b-\sqrt{\Delta}}{2a}=1.1461\\\alpha_2=\frac{-b+\sqrt{\Delta}}{2a}=2.711\end{cases}$$

  therefore, the discriminant Δ is only positive on $]-\infty,1.1461[\cup]2.711,+\infty[$ which means that is also positive for $\alpha\in[0,1[$. Thus, we conclude two different solutions of this equation are:

  $$\begin{cases}x_{1.L}=\frac{-0.7\alpha+0.35+\frac{\sqrt{1.96\alpha^2-7.56\alpha+6.09}}{2}}{1.4(\alpha-1)}<0\\\text{and}\\x_{2.L}=\frac{-0.7\alpha+0.35-\frac{\sqrt{1.96\alpha^2-7.56\alpha+6.09}}{2}}{1.4(\alpha-1)}>0\end{cases}$$

Because the denominator of $x_{1.L}$ and $x_{2.L}$ is always negative for $\alpha \in [0,1[$ which means that:

$$\begin{cases} -0.7\alpha + 0.35 + \frac{\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09}}{2} > 0 \\ \qquad\qquad \wedge \\ -0.7\alpha + 0.35 - \frac{\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09}}{2} < 0 \end{cases}$$

to ensure this we do the following:

1. For the first solution $x_{1.L}$ we have:

   – **If:** $-0.7\alpha + 0.35 + \frac{\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09}}{2} < 0$, for $\alpha \in J = [0,1[$:

$$\Leftrightarrow \begin{cases} \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} < 1.4\alpha - 0.7, & \text{if } \alpha \in \left[0, \frac{1}{2}\right[\\ \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} < 1.4\alpha - 0.7, & \text{if } \alpha \in \left[\frac{1}{2}, 1\right[\end{cases}$$
$$\Leftrightarrow \begin{cases} \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} - 1.4\alpha + 0.7 < 0, & \text{if } \alpha \in \left[0, \frac{1}{2}\right[\\ 1.96\alpha^2 - 7.56\alpha + 6.09 < (1.4\alpha - 0.7)^2, & \text{if } \alpha \in \left[\frac{1}{2}, 1\right[\end{cases}$$
$$\Leftrightarrow \begin{cases} \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} - 1.4\alpha + 0.7 < 0, & \text{if } \alpha \in \left[0, \frac{1}{2}\right[\\ -5.6\alpha < -5.6, & \text{if } \alpha \in \left[\frac{1}{2}, 1\right[\end{cases}$$
$$\Leftrightarrow \begin{cases} \text{no solution found} \\ \text{solution is: } \alpha \in \ [1, +\infty[\cap \left[\frac{1}{2}, 1\right[= \varnothing \end{cases}$$

   then, the numerator of $x_{1.L}$ is not negative for $\alpha \in [0,1[$.

2. For the second solution $x_{2.L}$ we have:

   – **If:** $-0.7\alpha + 0.35 - \frac{\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09}}{2} > 0$, for $\alpha \in J = [0,1[$:

$$\Leftrightarrow \begin{cases} \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} < -1.4\alpha + 0.7, & \text{if } \alpha \in \left[0, \frac{1}{2}\right[\\ \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} < -1.4\alpha + 0.7, & \text{if } \alpha \in \left[\frac{1}{2}, 1\right[\end{cases}$$
$$\Leftrightarrow \begin{cases} 1.96\alpha^2 - 7.56\alpha + 6.09 < (-1.4\alpha + 0.7)^2, & \text{if } \alpha \in \left[0, \frac{1}{2}\right[\\ \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} + 1.4\alpha - 0.7 < 0, & \text{if } \alpha \in \left[\frac{1}{2}, 1\right[\end{cases}$$
$$\Leftrightarrow \begin{cases} -5.6\alpha < -5.6, & \text{if } \alpha \in \left[0, \frac{1}{2}\right[\\ \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} + 1.4\alpha - 0.7 < 0, & \text{if } \alpha \in \left[\frac{1}{2}, 1\right[\end{cases}$$
$$\Leftrightarrow \begin{cases} \text{solution is: } \alpha \in \ [1, +\infty[\cap \left[0, \frac{1}{2}\right[= \varnothing \\ \text{no solution found} \end{cases}$$

   then, the numerator of $x_{2.L}$ is not positive for $\alpha \in [0,1[$. Hence, the only negative solution of the first equation from (3.2) is:

$$x_L = \frac{-0.7\alpha + 0.35 + \frac{\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09}}{2}}{1.4(\alpha - 1)}$$

- ***In the side R:*** The discriminant of the second equation from (3.2) is:

$$\Delta = 1.96\alpha^2 - 3.64\alpha + 6.09$$

to find the sign of discriminant, we set $1.96\alpha^2 - 3.64\alpha + 6.09 = 0$, and solve a quadratic equation:

$$\Delta_{**} = b^2 - 4ac = -34.496 < 0$$

since, $\Delta_{**} < 0$, the sign of the polynomial is the sign of $1.96 > 0$ and $\alpha \in [0, 1[$, that means $1.96\alpha^2 - 3.64\alpha + 6.09 > 0$. Thus, we conclude two different solutions of this equation are:

$$\begin{cases} x_{1.R} = \frac{0.7\alpha+0.35-\frac{\sqrt{1.96\alpha^2-3.64\alpha+6.09}}{2}}{1.4(\alpha-1)} > 0 \\ \text{and} \\ x_{2.R} = \frac{0.7\alpha+0.35+\frac{\sqrt{1.96\alpha^2-3.64\alpha+6.09}}{2}}{1.4(\alpha-1)} < 0 \end{cases}$$

Because the denominator of $x_{1.R}$ and $x_{2.R}$ is always negative for $\alpha \in [0, 1[$ which means that:

$$\begin{cases} 0.7\alpha + 0.35 - \frac{\sqrt{1.96\alpha^2-3.64\alpha+6.09}}{2} < 0 \\ \wedge \\ 0.7\alpha + 0.35 + \frac{\sqrt{1.96\alpha^2-3.64\alpha+6.09}}{2} > 0 \end{cases}$$

to ensure this we do the following:

1. For the first solution $x_{1.R}$:
   - **If:** $0.7\alpha + 0.35 - \frac{\sqrt{1.96\alpha^2-3.64\alpha+6.09}}{2} > 0$, for $\alpha \in J = [0, 1[$:

$$\begin{aligned} &\Leftrightarrow \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} < 1.4\alpha + 0.7, && \text{for } \alpha \in [0, 1[\\ &\Leftrightarrow 1.96\alpha^2 - 3.64\alpha + 6.09 < (1.4\alpha + 0.7)^2, && \text{for } \alpha \in [0, 1[\\ &\Leftrightarrow -5.6\alpha < -5.6, && \text{for } \alpha \in [0, 1[\\ &\Leftrightarrow \text{solution is: } \alpha \in]1, +\infty[\cap [0, 1[= \varnothing \end{aligned}$$

   then, the numerator of $x_{1.R}$ is not positive for $\alpha \in [0, 1[$.

2. For the second solution $x_{2.R}$:
   - It is obvious that the numerator of $x_{1.R}$ is already positive for $\alpha \in J = [0, 1[$ so:

$$\alpha \in \mathbb{R} \cap [0, 1[= [0, 1[\subset J$$

Therefore, the only positive solution of the second equation of (3.2) is:

$$x_R = \frac{0.7\alpha + 0.35 - \frac{\sqrt{1.96\alpha^2-3.64\alpha+6.09}}{2}}{1.4\,(\alpha - 1)}$$

Then, the unified chaotic map (3.1) has two **fixed points** given by:

$$P_L = (x_L, 0.3x_L) \in R_L \text{ and } P_R = (x_R, 0.3x_R) \in R_R$$

such that:

$$\begin{cases} x_L = \frac{-0.7\alpha+0.35+\frac{\sqrt{1.96\alpha^2-7.56\alpha+6.09}}{2}}{1.4(\alpha-1)} \\ x_R = \frac{0.7\alpha+0.35-\frac{\sqrt{1.96\alpha^2-3.64\alpha+6.09}}{2}}{1.4(\alpha-1)} \end{cases}$$

Remark 3.1 *We note that in our study we have excluded the case $\alpha = 1$, because it is a forbidden value in the denominator of both solutions x_L and x_R. For this reason we study only for $\alpha \in J = [0, 1[$.*

The Jacobian matrix Obviously, we get the Jacobian matrix of any map, when we derive this map so:

1. The Jacobian matrix of the unified chaotic map (3.1) evaluated at a point (x, y) in the both regions R_L and R_R is given by:

$$\begin{cases} J_L(x,y) = \begin{pmatrix} 1.4\alpha - 2.8x + 2.8x\alpha & 1 \\ 0.3 & 0 \end{pmatrix} \\ J_R(x,y) = \begin{pmatrix} 2.8x\alpha - 1.4\alpha - 2.8x & 1 \\ 0.3 & 0 \end{pmatrix} \end{cases}$$

2. The Jacobian matrix of the unified chaotic map (3.1) evaluated at a fixed points in the both regions R_L and R_R is given by:

$$\begin{cases} J_L(P_L) = \begin{pmatrix} 0.7 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} & 1 \\ 0.3 & 0 \end{pmatrix} \\ J_R(P_R) = \begin{pmatrix} 0.7 - \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} & 1 \\ 0.3 & 0 \end{pmatrix} \end{cases}$$

3. The **eigenvalues** of $J_L(P_L)$ and $J_R(P_R)$ are the solutions of the characteristic polynomials:

$$\begin{cases} \lambda^2 - \tau_L\lambda + \delta_L = 0 \\ \lambda^2 - \tau_R\lambda + \delta_R = 0 \end{cases}$$

- ***In the side L:*** The characteristic polynomial of $J_L(P_L)$ can be written as:

$$\lambda^2 - \left(0.7 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09}\right)\lambda - 0.3 = 0$$

and the discriminant of this equation is:

$$\left(0.7+\sqrt{1.96\alpha^2-7.56\alpha+6.09}\right)^2+1.2>0,\quad \text{for all } \alpha\in\mathbb{R}\cap[0.1[=[0.1[=J$$

So, we get two different solutions are:

$$\begin{cases}\lambda_{1,L}=\frac{\sqrt{1.96\alpha^2-7.56\alpha+6.09}+\sqrt{1.96\alpha^2-7.56\alpha+1.4\sqrt{1.96\alpha^2-7.56\alpha+6.09}+7.78}}{2}+0.35\\[2ex] \lambda_{2,L}=\frac{\sqrt{1.96\alpha^2-7.56\alpha+6.09}-\sqrt{1.96\alpha^2-7.56\alpha+1.4\sqrt{1.96\alpha^2-7.56\alpha+6.09}+7.78}}{2}+0.35\end{cases}$$

- ***In the side R:*** The characteristic polynomial of $J_R(P_R)$ can be written also as:

$$\lambda^2-\left(0.7-\sqrt{1.96\alpha^2-3.64\alpha+6.09}\right)\lambda-0.3=0$$

and the discriminant of this equation is:

$$\left(0.7-\sqrt{1.96\alpha^2-3.64\alpha+6.09}\right)^2+1.2>0,\quad \text{for all } \alpha\in\mathbb{R}\cap[0.1[=[0.1[=J$$

So, we get two solutions are:

$$\begin{cases}\lambda_{1,R}=\frac{-\sqrt{1.96\alpha^2-3.64\alpha+6.09}+\sqrt{1.96\alpha^2-3.64\alpha+1.4\sqrt{1.96\alpha^2-3.64\alpha+6.09}+7.78}}{2}+0.35\\[2ex] \lambda_{2,R}=\frac{-\sqrt{1.96\alpha^2-3.64\alpha+6.09}-\sqrt{1.96\alpha^2-3.64\alpha+1.4\sqrt{1.96\alpha^2-3.64\alpha+6.09}+7.78}}{2}+0.35\end{cases}$$

Hence, The **eigenvalues** of $J_L(P_L)$ and $J_R(P_R)$ are:

$$\begin{cases}\lambda_{1,L}=\frac{\sqrt{1.96\alpha^2-7.56\alpha+6.09}+\sqrt{1.96\alpha^2-7.56\alpha+1.4\sqrt{1.96\alpha^2-7.56\alpha+6.09}+7.78}}{2}+0.35\\[2ex] \lambda_{2,L}=\frac{\sqrt{1.96\alpha^2-7.56\alpha+6.09}-\sqrt{1.96\alpha^2-7.56\alpha+1.4\sqrt{1.96\alpha^2-7.56\alpha+6.09}+7.78}}{2}+0.35\end{cases}$$

$$\begin{cases}\lambda_{1,R}=\frac{-\sqrt{1.96\alpha^2-3.64\alpha+6.09}+\sqrt{1.96\alpha^2-3.64\alpha+1.4\sqrt{1.96\alpha^2-3.64\alpha+6.09}+7.78}}{2}+0.35\\[2ex] \lambda_{2,R}=\frac{-\sqrt{1.96\alpha^2-3.64\alpha+6.09}-\sqrt{1.96\alpha^2-3.64\alpha+1.4\sqrt{1.96\alpha^2-3.64\alpha+6.09}+7.78}}{2}+0.35\end{cases}$$

The coordinate transformations The normal form of the map (3.1) is given by :

$$N_2(x,y)=\begin{cases}\begin{pmatrix}0.7+\sqrt{1.96\alpha^2-7.56\alpha+6.09} & 1\\ 0.3 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}\mu, & \text{if } x<0\\ \begin{pmatrix}0.7-\sqrt{1.96\alpha^2-3.64\alpha+6.09} & 1\\ 0.3 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}\mu, & \text{if } x>0\end{cases}$$

The fixed points

In the normal form (1.8) we have two fixed points P_L and P_R near the border to the right ($x < 0$) and left ($x > 0$) of the border. So from (3.3) we have:

$$\begin{cases} P_L = \left(\frac{\mu}{-\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09}}, \frac{0.3\mu}{-\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09}} \right) \in R_L \\ P_R = \left(\frac{\mu}{\sqrt{1.96\alpha^2 - 3.64\alpha + 6.09}}, \frac{0.3\mu}{\sqrt{1.96\alpha^2 - 3.64\alpha + 6.09}} \right) \in R_R \end{cases}$$

3.1.2 The border collision bifurcations

The dynamics of the system (3.1) is governed by five parameters $\tau_L, \tau_R, \delta_L, \delta_R$ and μ, so according to the text of this bifurcation in the theoretical part we conclude the possible types of fixed points of the normal form map (3.3):

1. ***For positive determinant:*** Because the determinant of the corresponding matrices of the linearized map in the both side is not positive we study only the case for negative determinant.

2. ***For negative determinant:***

- ***In the side L:***

2.a For

$$-(1 + \delta_L) < \tau_L < 1 + \delta_L \Leftrightarrow -0.7 < 0.7 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} < 0.7$$

such that $\alpha \in J = [0, 1[$ and $J_1, J_2 \subset J$:

$$\Leftrightarrow \begin{cases} -0.7 < 0.7 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09}, & \text{for } \alpha \in J_1 \\ 0.7 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} < 0.7, & \text{for } \alpha \in J_2 \end{cases}$$
$$\Leftrightarrow \begin{cases} 1.4 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} > 0, & \text{for } \alpha \in J_1 \\ \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} < 0, & \text{for } \alpha \in J_2 \end{cases}$$
$$\Leftrightarrow \begin{cases} \text{solution is: } \alpha \in \mathbb{R} \cap J_1 = J_1 \\ \text{no solution found } \forall \alpha \in \mathbb{R} \end{cases}$$

Then **(2.a)** is not hold, which implies that the fixed point is not a flip attractor.

2.b For

$$\tau_L > 1 + \delta_L \Leftrightarrow 0.7 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} > 0.7$$

such that $\alpha \in J = [0, 1[$ and $J_3 \subset J$:

$$\begin{aligned} &\Leftrightarrow \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} > 0, \quad \text{for } \alpha \in J_3 \\ &\Leftrightarrow \quad 1.96\alpha^2 - 7.56\alpha + 6.09 > 0, \quad \text{for } \alpha \in J_3 \end{aligned}$$

we previously checked that it is greater than zero if $\alpha \in]-\infty, 1.1461[\cup]2.711, +\infty[$, solution is:

$$J_3 =]-\infty, 1.146[\cup]2.711, +\infty[\cap J = J$$

Then **(2.b)** is hold, which implies that the fixed point is a flip saddle.

2.c For

$$\tau_L < -(1 + \delta_L) \Leftrightarrow 0.7 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} < -0.7$$

such that $\alpha \in J = [0, 1[$

$$\Leftrightarrow 1.4 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} < 0, \text{ no solution found } \forall\alpha \in \mathbb{R}$$

Then **(2.c)** is not hold, which implies that the fixed point is not a flip saddle.

- ***In the side R:***

2.a For

$$-(1 + \delta_R) < \tau_R < 1 + \delta_R \Leftrightarrow -0.7 < 0.7 - \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} < 0.7$$

such that $\alpha \in J = [0, 1[$ and $J_4, J_5 \subset J$:

$$\Leftrightarrow \begin{cases} -0.7 < 0.7 - \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09}, & \text{for } \alpha \in J_4 \\ 0.7 - \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} < 0.7, & \text{for } \alpha \in J_5 \end{cases} \tag{3.3}$$

We have the first inequality from (3.4):

$$\begin{aligned} -0.7 &< 0.7 - \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09}, \quad \text{for } \alpha \in J_4 \\ &\Leftrightarrow \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} < 1.4, \quad \text{for } \alpha \in J_4 \\ &\Leftrightarrow 1.96\alpha^2 - 3.64\alpha + 6.09 < 1.96, \quad \text{for } \alpha \in J_4 \\ &\Leftrightarrow 1.96\alpha^2 - 3.64\alpha + 4.13 < 0, \quad \text{for } \alpha \in J_4 \end{aligned} \tag{3.4}$$

we set $1.96\alpha^2 - 3.64\alpha + 4.13 = 0$, and solve a quadratic equation:

$$\Delta = b^2 - 4ac = -19.1296 < 0$$

since $\Delta < 0$, the sign of the polynomial is from the sign of $1.96 > 0$, then $1.96\alpha^2 - 3.64\alpha + 4.13 > 0$, and this contradicts the previous result in (3.5), since the first inequality of (3.4) is not hold. Then **(2.a)** is not hold, which implies that the fixed point is not a flip attractor.

2.b For

$$\tau_R > 1 + \delta_R \Leftrightarrow 0.7 - \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} > 0.7$$

such that $\alpha \in J = [0, 1[$:

$$\Leftrightarrow \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} < 0, \text{ no solution found } \forall \alpha \in \mathbb{R}$$

Then **(2.b)** is not hold, which implies that the fixed point is not a flip saddle.

2.c For

$$\tau_R < -(1 + \delta_R) \Leftrightarrow 0.7 - \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} < -0.7$$

such that $\alpha \in J = [0, 1[$ and $J_6 \subset J$:

$$\begin{aligned} &\Leftrightarrow 1.4 < \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09}, && \text{for } \alpha \in J_6 \\ &\Leftrightarrow 1.96 < 1.96\alpha^2 - 3.64\alpha + 6.09, && \text{for } \alpha \in J_6 \\ &\Leftrightarrow 1.96\alpha^2 - 3.64\alpha + 4.13 > 0, && \text{for } \alpha \in J_6 \end{aligned}$$

we set $1.96\alpha^2 - 3.64\alpha + 4.13 = 0$ and solve a quadratic equation:

$$\Delta = -19.1296 < 0$$

since $\Delta < 0$, the sign of the polynomial is from the sign of $1.96 > 0$, then $1.96\alpha^2 - 3.64\alpha + 4.13 > 0$, and the solution is $J_6 = \mathbb{R} \cap]0, 1[= [0, 1[= J$. Then **(2.c)** is hold, which implies that the fixed point is a flip saddle.

3.2 The discrete hyper-chaotic double scroll

In this section, our study of the **discrete hyper-chaotic double scroll map,** will be based on some of its basic properties which are the map definition, its fixed points and finally the Jacobian matrix which has a great role in the study of stability near fixed points.

3.2.1 Two-dimensional piecewise smooth map

Consider the discrete hyper-chaotic double scroll map given in [9] as follows:

$$f(x, y) = \begin{pmatrix} x - ah(y) \\ bx \end{pmatrix} \tag{3.5}$$

where a and b are the bifurcation parameters, and the characteristic function h called *double scroll attractor* shown in Figure 3.3 is given by:

$$h(x) = \frac{2m_1 x + (m_0 - m_1)(|x + 1| - |x - 1|)}{2}$$

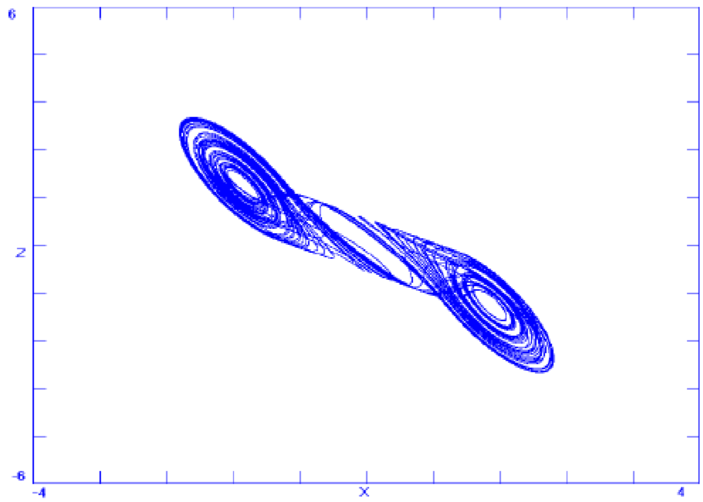

Figure 3.3: The classical double scroll attractor obtained for $\alpha = 9.35$, $\beta = 14.79$, $m_0 = \frac{-1}{7}$, $m_1 = \frac{2}{7}$.

such that $m_0 < 0$ and $m_1 > 0$, are respectively the slopes of the inner and outer sets of the original **Chua circuit** which proposed as follows:

$$\begin{cases} x' = \alpha\left(y - h\left(x\right)\right) \\ y' = x - y + z \\ z' = -\beta y \end{cases}$$

So, the discrete hyper-chaotic double scroll map shown in Figure 3.4 and can be given by:

$$f\left(x,y\right) = \left(\begin{array}{c} \begin{cases} x - a\left(m_1 y + \left(m_0 - m_1\right)\right), & \text{if } (x,y) \in R_1 \\ x - a m_0 y, & \text{if } (x,y) \in R_2 \\ x - a\left(m_1 y - \left(m_0 - m_1\right)\right), & \text{if } (x,y) \in R_3 \end{cases} \\ bx \end{array}\right)$$

Due to the shape of the vector field f of the map, the plane can be divided into three linear regions denoted by:

$$\begin{cases} R_1 = \{(x,y) \in \mathbb{R}^2, \ \ y \geq 1\} \\ R_2 = \{(x,y) \in \mathbb{R}^2, \ \ |y| \leq 1\} \\ R_3 = \{(x,y) \in \mathbb{R}^2, \ \ y \leq -1\} \end{cases}$$

where in each of these regions the map (3.6) is linear.

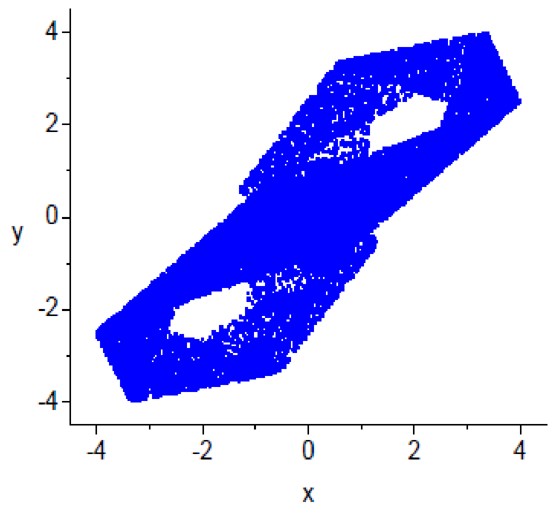

Figure 3.4: The discrete hyperchaotic double scroll attractor obtained from the map (3.6) for $a = 3.36$, $b = 1.4$, $m_0 = -0.43$, and $m_1 = 0.41$ with initial conditions $x = y = 0.1$.

Fixed points

The fixed points of the discrete hyper-chaotic double scroll map (3.6) are the real solutions of the system:

$$f\ (x, y) = (x, y) \Longleftrightarrow \begin{pmatrix} x - ah(y) \\ bx \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

therefore, we get the following equations:

$$\begin{cases} x - a\,(m_1 y + (m_0 - m_1)) = x, & \text{and } \ bx = y, & \text{for } y \geq 1 \\ x - am_0 y = x, & \text{and } \ bx = y, & \text{for } |y| \ \leq 1 \\ x - a\,(m_1 y - (m_0 - m_1)) = x, & \text{and } \ bx = y, & \text{for } y \leq -1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} a\,(m_1 bx + (m_0 - m_1)) = 0, & \text{and } \ bx = y, & \text{for } y \geq 1 \\ am_0 bx = 0, & \text{and } \ bx = y, & \text{for } |y| \ \leq 1 \\ a\,(m_1 bx - (m_0 - m_1)) = \ 0, & \text{and } \ bx = y, & \text{for } y \leq -1 \end{cases}$$

Now, we discuss the cases from the existence of the fixed points:

Case 1: For $y \geq 1$ we have:

$$x_1 = \frac{m_1 - m_0}{bm_1} \Rightarrow y_1 = \frac{m_1 - m_0}{m_1}, \quad abm_1 \neq 0$$

such that:

$$y_1 = \frac{m_1 - m_0}{m_1} \geq 1 \Leftrightarrow m_1 - m_0 \geq m_1 \Leftrightarrow m_0 < 0, \ \text{ and } \ m_1 > 0$$

So, the fixed point (x_1, y_1) exist in R_1 if $m_1 m_0 < 0$.

Case 2: For $|\ y| \leq 1$ we have:

$$x_2 = 0 \Rightarrow y_2 = 0, \text{ and } abm_0 \neq 0$$

Hence, the fixed point (x_2, y_2) exist in R_2 if $m_0 \neq 0$.

Case 3: For $y \leq -1$ we have:

$$x_3 = \frac{m_0 - m_1}{bm_1} \Rightarrow y_3 = \frac{m_0 - m_1}{m_1}, \text{ and } abm_1 \neq 0$$

such that:

$$y_3 = \frac{m_0 - m_1}{m_1} \leq -1 \Leftrightarrow m_0 - m_1 \leq -m_1 \Leftrightarrow m_0 < 0, \text{ and } m_1 > 0$$

Hence, the fixed point (x_3, y_3) exist in R_3 if $m_1 m_0 < 0$. Then, the double scroll map (3.6) has the fixed points given by:

$$\begin{cases} P_2 = (0,0)\,, & \text{if } m_0 m_1 > 0 \\ \begin{cases} P_1 = \left(\frac{m_1 - m_0}{bm_1}, \frac{m_1 - m_0}{m_1}\right) \\ P_2 = (0,0) \\ P_3 = \left(\frac{m_0 - m_1}{bm_1}, \frac{m_0 - m_1}{m_1}\right) \end{cases}, & \text{if } m_0 m_1 < 0 \end{cases}$$

Jacobian matrix

By deriving the map (3.6) at each area defined by it, we get the following:

- The Jacobian matrix of the double scroll map (3.6) evaluated at the fixed points P_1, P_2 and P_3 are given by:

$$\begin{cases} J_1(x,y) = J_1(P_1) = \begin{pmatrix} 1 & -abm_1 \\ 1 & 0 \end{pmatrix} \\ J_2(x,y) = J_2(P_2) = \begin{pmatrix} 1 & -abm_0 \\ 1 & 0 \end{pmatrix} \\ J_3(x,y) = J_3(P_3) = \begin{pmatrix} 1 & -abm_1 \\ 1 & 0 \end{pmatrix} \end{cases}$$

we note here that $J_1(P_1) = J_3(P_3)$, so we can again write the Jacobian matrix as follows:

$$\begin{cases} J_{1,3}(x,y) = J_{1,3}(P_{1,3}) = \begin{pmatrix} 1 & -abm_1 \\ 1 & 0 \end{pmatrix} \\ J_2(x,y) = J_2(P_2) = \begin{pmatrix} 1 & -abm_0 \\ 1 & 0 \end{pmatrix} \end{cases} \tag{3.6}$$

- The eigenvalues of the corresponding Jacobian matrices (3.7) is given by the solutions of their characteristic polynomials. Which are given also respectively by:

$$\begin{cases} \lambda^2 - \lambda + abm_1 = 0 \\ \lambda^2 - \lambda + abm_0 = 0 \end{cases} \tag{3.7}$$

3.2.2 Stability of fixed points

The stability is a part of the border bifurcations, we resort to studying it in this case near the fixed point, because it is not possible to study the border collision bifurcation through the normal form, and on it we rely on the following theory:

Theorem 3.1 *Let (x^*, y^*) be a fixed point of f and assume that $f \in C^1$.*

- *If $|\lambda| < 1,$ for every eigenvalues λ of $Df\ (x^*, y^*)$, then (x^*, y^*) is an asymptotically stable fixed point of f.*

- *If $|\lambda| > 1,$ for some eigenvalues λ of $Df\ (x^*, y^*)$, then (x^*, y^*) is not a Lyapunov stable fixed point of f. See* [11].

We conclude from the previous theorem, that to study the stability of the fixed point, we perform three main steps:

1. We evaluate the Jacobian matrix at the fixed point.

2. We calculate the eigenvalues from the solution of the characteristic polynomial.

3. We compare the resulting eigenvalues with the unit disk.

Since, we did the first step previously its enough that we only start from the second step, specifically from the statement (3.7) therefore:

- We have from the first quadratic equation of (3.8):

$$\Delta = b^2 - 4ac = 1 - 4abm_1, \text{ and } m_1 > 0$$

So, we distinguish three cases of the delta discriminant:

Case 1: Null discriminant:

$$\Delta = 1 - 4abm_1 = 0, \text{ if } abm_1 = \frac{1}{4}$$

then, we have one double eigenvalue:

$$\lambda = \frac{1}{2} < 1$$

Hence, the fixed points P_1 and P_3 are asymptotically stable fixed points of f if $abm_1 = \frac{1}{4}$.

Case 2: Positive discriminant:

$$\Delta = 1 - 4abm_1 > 0, \ \text{ if } \ abm_1 \in \left]-\infty, 0\right[\cup \left]0, \frac{1}{4}\right] = I_1$$

then, we get two real eigenvalues:

$$\begin{cases} \lambda_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{1-\sqrt{1-4abm_1}}{2} \\ \\ \lambda_2 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{1+\sqrt{1-4abm_1}}{2} \end{cases}$$

Now, we apply the theorem (3.1) as follow:

1. For the first eigenvalues λ_1: The case

$$|\lambda_1| < 1 \Leftrightarrow -1 < \frac{1-\sqrt{1-4abm_1}}{2} < 1 \Leftrightarrow -2 < 1 - \sqrt{1-4abm_1} < 2$$

$$\Leftrightarrow \begin{cases} \begin{cases} -2 < 1 - \sqrt{1-4abm_1} \\ 1 - \sqrt{1-4abm_1} < 2 \end{cases} \Leftrightarrow \begin{cases} \sqrt{1-4abm_1} < 3 \\ 1 + \sqrt{1-4abm_1} > 0 \end{cases} \\ \text{solution is: } \begin{cases} abm_1 \in \left]-2, \frac{1}{4}\right] \cap I_1 = \left]-2, 0\right[\cup \left]0, \frac{1}{4}\right] \subset I_1 \\ abm_1 \in \mathbb{R} \cap I_1 = I_1 \end{cases} \end{cases}$$

   then: $|\lambda_1| < 1$ is hold if $abm_1 \in \left]-2, 0\right[\cup \left]0, \frac{1}{4}\right]$.

2. For the second eigenvalues λ_2: The case

$$|\lambda_2| < 1 \Leftrightarrow -1 < \frac{1+\sqrt{1-4abm_1}}{2} < 1 \Leftrightarrow -2 < 1 + \sqrt{1-4abm_1} < 2$$

$$\Leftrightarrow \begin{cases} \begin{cases} -2 < 1 + \sqrt{1-4abm_1} \\ 1 + \sqrt{1-4abm_1} < 2 \end{cases} \Leftrightarrow \begin{cases} 3 + \sqrt{1-4abm_1} > 0 \\ \sqrt{1-4abm_1} < 1 \end{cases} \\ \text{solutions is: } \begin{cases} abm_1 \in \mathbb{R} \cap I_1 = I_1 \\ abm_1 \in \left]0, \frac{1}{4}\right] \cap I_1 = \left]0, \frac{1}{4}\right] \subset I_1 \end{cases} \end{cases}$$

   then $|\lambda_2| < 1$ is hold if $abm_1 \in \left]0, \frac{1}{4}\right]$. Hence, the fixed points P_1 and P_3 are asymptotically stable fixed points of f if $abm_1 \in \left]0, \frac{1}{4}\right]$.

Case 3: Negative discriminant:

$$\Delta = 1 - 4abm_1 < 0, \ \text{ if } \ abm_1 \in \left]\frac{1}{4}, +\infty\right[= I_2$$

then, we get two complex eigenvalues:

$$\begin{cases} \lambda_1 = \frac{-b-i\sqrt{\Delta}}{2a} = \frac{1-i\sqrt{1-4abm_1}}{2} \\ \\ \lambda_2 = \frac{-b+i\sqrt{\Delta}}{2a} = \frac{1+i\sqrt{1-4abm_1}}{2} \end{cases}$$

Now, we apply the theorem (3.1) as follow:

1. For the first eigenvalues λ_1: The case $|\lambda_1| < 1 \Leftrightarrow \sqrt{(Re.\lambda_1)^2 + (Im.\lambda_1)^2} < 1,\ \lambda_1 \in \mathbb{C}$

$$\Leftrightarrow \begin{cases} \left|\frac{1-i\sqrt{1-4abm_1}}{2}\right| < 1 \Leftrightarrow \sqrt{\frac{1}{4} + \frac{1-4abm_1}{4}} < 1 \Leftrightarrow \sqrt{\frac{2-4abm_1}{4}} < 1 \\ \text{solutions is: } abm_1 \in \left]\frac{-1}{2}, \frac{1}{2}\right] \cap I_2 = \left]\frac{1}{4}, \frac{1}{2}\right] \subset I_2 \end{cases}$$

   then: $|\lambda_1| < 1$ is hold if $abm_1 \in I_2$.

2. For the second eigenvalues λ_2: The case $|\lambda_2| < 1 \Leftrightarrow \sqrt{(Re.\lambda_2)^2 + (Im.\lambda_2)^2} < 1,\ \lambda_2 \in \mathbb{C}$

$$\Leftrightarrow \begin{cases} \left|\frac{1+i\sqrt{1-4abm_1}}{2}\right| < 1 \Leftrightarrow \sqrt{\frac{1}{4} + \frac{1-4abm_1}{4}} < 1 \Leftrightarrow \sqrt{\frac{2-4abm_1}{4}} < 1 \\ \text{solutions is: } abm_1 \in \left]\frac{-1}{2}, \frac{1}{2}\right] \cap I_2 = \left]\frac{1}{4}, \frac{1}{2}\right] \subset I_2 \end{cases}$$

   then: $|\lambda_2| < 1$ is hold if $abm_1 \in I_2$. Hence, the fixed points $P_1$and P_3 are asymptotically stable fixed points of f if $abm_1 \in \left]\frac{1}{4}, \frac{1}{2}\right]$.

- We have from the second quadratic equation of (3.8):

$$\Delta = b^2 - 4ac = 1 - 4abm_0, \quad \text{and} \quad m_0 < 0$$

So, we distinguish three cases of the delta discriminant:

Case 1: Null discriminant:

$$\Delta = 1 - 4abm_0 = 0, \quad \text{if} \quad abm_0 = \frac{1}{4}$$

then, we have one double eigenvalue:

$$\lambda = \frac{1}{2} < 1$$

Hence, P_2 are an asymptotically stable fixed point of f if $abm_0 = \frac{1}{4}$.

Case 2: Positive discriminant:

$$\Delta = 1 - 4abm_0 > 0, \quad \text{if} \quad abm_0 \in]-\infty, 0[\cup \left]0, \frac{1}{4}\right[= I_1$$

then, we get two real eigenvalues:

$$\begin{cases} \lambda_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{1-\sqrt{1-4abm_0}}{2} \\ \\ \lambda_2 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{1+\sqrt{1-4abm_0}}{2} \end{cases}$$

Now, we apply the theorem (3.1) as follow:

1. For the first eigenvalues λ_1: The case

$$|\lambda_1| < 1 \Leftrightarrow -1 < \frac{1-\sqrt{1-4abm_0}}{2} < 1 \Leftrightarrow -2 < 1-\sqrt{1-4abm_0} < 2$$

$$\Leftrightarrow \begin{cases} \begin{cases} -2 < 1-\sqrt{1-4abm_0} \\ 1-\sqrt{1-4abm_0} < 2 \end{cases} \Leftrightarrow \begin{cases} \sqrt{1-4abm_0} < 3 \\ 1+\sqrt{1-4abm_0} > 0 \end{cases} \\ \text{solution is: } \begin{cases} abm_0 \in \left]-2, \frac{1}{4}\right] \cap I_1 = \left]-2, 0\right[\cup \left]0, \frac{1}{4}\right] \subset I_1 \\ abm_0 \in \mathbb{R} \cap I_1 = I_1 \end{cases} \end{cases}$$

then $|\lambda_1| < 1$ is hold if $abm_0 \in \left]-2, 0\right[\cup \left]0, \frac{1}{4}\right]$.

2. For the second eigenvalues λ_2: The case

$$|\lambda_2| < 1 \Leftrightarrow -1 < \frac{1+\sqrt{1-4abm_0}}{2} < 1 \Leftrightarrow -2 < 1+\sqrt{1-4abm_0} < 2$$

$$\Leftrightarrow \begin{cases} \begin{cases} -2 < 1+\sqrt{1-4abm_0} \\ 1+\sqrt{1-4abm_0} < 2 \end{cases} \Leftrightarrow \begin{cases} 3+\sqrt{1-4abm_0} > 0 \\ \sqrt{1-4abm_0} < 1 \end{cases} \\ \text{solution is: } \begin{cases} abm_0 \in \mathbb{R} \cap I_1 = I_1 \\ abm_0 \in \left]0, \frac{1}{4}\right] \cap I_1 = \left]0, \frac{1}{4}\right] \subset I_1 \end{cases} \end{cases}$$

then $|\lambda_2| < 1$ is hold if $abm_0 \in \left]0, \frac{1}{4}\right]$. Hence, the fixed points P_2 are asymptotically stable fixed point of f if $abm_0 \in \left]0, \frac{1}{4}\right]$.

Case 3: Negative discriminant:

$$\Delta = 1 - 4abm_0 < 0, \text{ if } abm_0 \in \left]\frac{1}{4}, +\infty\right[$$

then, we get two complex eigenvalues:

$$\begin{cases} \lambda_1 = \frac{-b-i\sqrt{\Delta}}{2a} = \frac{1-i\sqrt{1-4abm_0}}{2} \\ \\ \lambda_2 = \frac{-b+i\sqrt{\Delta}}{2a} = \frac{1+i\sqrt{1-4abm_0}}{2} \end{cases}$$

Now, we apply the theorem (3.1) as follow:

1. For the first eigenvalues λ_1: The case $|\lambda_1| < 1 \Leftrightarrow \sqrt{(Re.\lambda_1)^2 + (Im.\lambda_1)^2} < 1$, $\lambda_1 \in \mathbb{C}$

$$\Leftrightarrow \begin{cases} \left|\frac{1-i\sqrt{1-4abm_0}}{2}\right| < 1 \Leftrightarrow \sqrt{\frac{1}{4} + \frac{1-4abm_0}{4}} < 1 \Leftrightarrow \sqrt{\frac{2-4abm_0}{4}} < 1 \\ \text{solution is: } abm_0 \in \left]\frac{-1}{2}, \frac{1}{2}\right] \cap I_2 = \left]\frac{1}{4}, \frac{1}{2}\right] \subset I_2 \end{cases}$$

then $|\lambda_1| < 1$ is hold if $abm_0 \in I_2$.

2. For the second eigenvalues λ_2: If $|\lambda_2| < 1 \Leftrightarrow \sqrt{(Re.\lambda_2)^2 + (Im.\lambda_2)^2} < 1$, $\lambda_2 \in \mathbb{C}$

$$\Leftrightarrow \begin{cases} \left|\frac{1+i\sqrt{1-4abm_0}}{2}\right| < 1 \Leftrightarrow \sqrt{\frac{1}{4} + \frac{1-4abm_0}{4}} < 1 \Leftrightarrow \sqrt{\frac{2-4abm_0}{4}} < 1 \\ \text{solution is: } abm_0 \in \left]\frac{-1}{2}, \frac{1}{2}\right] \cap I_2 = \left]\frac{1}{4}, \frac{1}{2}\right] \subset I_2 \end{cases}$$

then $|\lambda_2| < 1$ is hold if $abm_0 \in I_2$. Hence, the fixed points P_1 and P_3 are asymptotically stable fixed points of f if $abm_0 \in \left]\frac{1}{4}, \frac{1}{2}\right]$.

3.3 Conclusion

The *border collision bifurcation* is one of the most studied bifurcations for dynamical systems in recent years that occurs to *piecewise smooth maps* and the reason for this is due to the fact that the latter is very effective in modeling the non-smoothness in the systems accurately, i.e., characterized by a complex dynamical behavior. In view of the importance of the subject from a scientific and practical point of view, we choose it to be our topic and purpose in this thesis by presenting the theoretical part of this bifurcation and its importance in some piecewise smooth maps in one and two dimensional *Zeraoulia-Sprott mappings*. One of the benefits of this work is that the study of the dynamics of multi-dimensional systems can be reduced to the study of systems of lower dimension and even though there have been a lot of research on these systems and their bifurcation phenomena. Many aspects of the dynamics of such maps still remain unexplored and to be understood. Accordingly to remove the mystery of this type of maps, it has become necessary to pay a great attention to it in terms of scientific research.

Bibliography

[1] S. Banerjee and C. Grebogi, *Border collision bifurcations in two-dimensional piecewise smooth maps*, Phys, Rev, E, **59** (4), 4052-4061, 1999.

[2] S. Banerjee, M. S. Karthik, G. Yuan, and J. A. Yorke, *Bifurcations in one-dimensional piecewise smooth maps — theory and applications in switching circuits*, IEEE Trans, Circuits Systems I Fund. Theory Appl, **47** (3), 389-394, 2000.

[3] S. Banerjee, *Border collision bifurcations: theory and applications*, Department of Electrical Engineering, Indian Institute of Technology, Kharagpur — 721302, India.

[4] M. I. Feigin, *Doubling of the oscillation period with C-bifurcations in piecewise continuous systems*, Prikl, Mat, Meh, **34**, 861-869, 1970.

[5] H. E. Nusse and J. A. Yorke, *Border-collision bifurcations including "period two to period three" for piecewise smooth systems*, Phys, D, **57** (1-2), 39-57, 1992.

[6] H. E. Nusse and J. A. Yorke, *Border-collision bifurcations for piecewise smooth one-dimensional maps*, Internat, J, Bifur, Chaos Appl, Sci, Engrg, **5** (1), 189-207, 1995.

[7] C. H. Wong, *Border collision bifurcations in piecewise smooth systems*, A thesis for the degree of Doctor of Philosophy, University of Manchester, 2011.

[8] E. Zeraoulia, Preprint, 2009.

[9] E. Zeraoulia and J. C. Sprott, *The discrete hyperchaotic double scroll*. International Journal of Bifurcations and Chaos, **19** (3), 1023-1027, 2009(a).

[10] E. Zeraoulia and J. C. Sprott, *A unified chaotic mapping that contains the Hénon and the Lozi systems*, Annual Review of Chaos Theory, Bifurcations and Dynamical Systems, **1**, 50-60, 2012(a).

[11] E. Zeraoulia, *Dynamical systems, Theory and Applications*, Science Publishers, 2018.

[12] E. Zeraoulia, *Dynamical systems*, First year master's course, Tébessa University, 2020.