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Dynamic Properties of a Nonlinear Viscoelastic Kirchhoff Problem

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بسم الله الرحمن الرحيم

شكر وعرفان

نحمد الله و نشكره شكرا جزيلا إذ هو خالقنا، و معيننا

فهو الأولى بالشكر في كل الأوقات و الظروف.

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و على كل النصائح و التوجيهات.

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بسم الله الرحمان الرحيم و الصلاة و السلام على اشرف المرسلين و على اله
وصحبه اجمعين
الهم وعوني الى من جعل نفسه شمعة تحترق من اجل ان يرفع قدري و يزيل افتخاربكل
عني الى من تعب و شقى من اجل ان ينير دربي الى نور حياتي و اعز انسان في هذا الوجود
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الى بسمة* حياتي الى من افنت حياتها لاسعادنا و هزل جسدها لخدمتنا الى من كان دعائها
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حمزة
الى كل افراد عائلة زغلامي كبيرا و صغيرا *
الى اصدقائي و صديقاتي و رفيقات دربي الذين لا تحلو الحياة الا بهم*
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وصال

Abstract

In this thesis , we study the dynamic proprieties of a nonlinear viscoelastic Kirchhoff-type equation with initial conditions and acoustic boundary conditions (see [29]). We show that , the energy of the solution decays exponentially or polynomially. Our approach is based on integral inequalities and multiplier techniques. Instead of using a Lyapunov-type technique for some perturbed energy .

Keywords : Kirchhoff-type equation, Acoustic boudary condition , Original energy , Energy decay.

Resumé

Dans ce mémoire , nous étudions les propriétés dynamique de l’équation viscoélastique non linéaire de type Kirchhoff définit avec conditions initiales et conditions aux limites acoustiques (voir [29]). Nous montrons que, l’énergie de la solution décroit de manière exponentielle ou polynomiale. Notre approche est basée sur les inégalités intégrales et les téchniques de multiplicateur. Au lieu d’utiliser une téchnique de Lyapunov pour une certaine énergie perturbée.

Mots-clés : équation de type Kirchhoff , Condition aux limites acostique , énergie originelle , Décroissance d'énergie.

الملخص

في هذا العمل ، ندرس الخصائص الديناميكية لمعادلة اللزوجة غير الخطية من نوع كيرشوف مع الشروط الابتدائية والشروط الحدية الصوتية (أنظر[29]). نبرهن أن طاقة الحل تضمحل بشكل اسي أو كثير حدود. نعتمد على تقنية المتراجحات التكاملية ونظرية المؤثر. عوضا عن إستخدام تقنية ليابونوف لبعض الطاقة المضطربة.

الكلمات المفتاحية: معادلة كيرشوف ، شروط حدية صوتية ، الطاقة الاصلية ، إضمحلال الطاقة.

Contents

General Introduction

Our understanding of real-world phenomena and our technology today are largely based on Partial Differential Equations (PDEs). It is indeed thanks to the modeling of these phenomena through partial differential equations, which allow us to understand the role of such and such a parameter, and above all to obtain sometimes extremely precise forecasts. In particular the wave equations model several natural phenomena in: Physics, Chemistry, Biology...

In general, The physical model giving rise to the acoustic boundary condition is that of gas undergoing small irrotational perturbations from rest in a domain Ω with a smooth compact boundary.We assume that each point of the surface S acts like a spring in response to the excess pressure in the gas, and that there is no transverse tension between neighboring points of S, i.e., the "springs" are independent of each other. (Such a surface is called locally reacting.This type of boundary condition has been introduced by several authors beginning with Morse and Ingard in $[4]$, in this context, we can refer $[3, 8, 18]$.

In this work, we study the following initial boundary-value problem for the nonlinear viscoelastic Kirchhoff-type equation (see $[30]$):

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \displaystyle\int_0^t h(t-s)\,\Delta u(s)\,ds \\ \quad + a\,|u_t|^{m-2}u_t = |u|^{p-2}u & \text{in } \Omega\times(0,\infty), \\ u = 0 & \text{on } \Gamma_1\times(0,\infty), \\ M\left(\|\nabla u\|_2^2\right)\frac{\partial u}{\partial v} - \displaystyle\int_0^t h(t-s)\,\frac{\partial u(s)}{\partial v}ds = y_t & \text{on } \Gamma_0\times(0,\infty), \\ u_t + \alpha(x)\,y_t + \beta(x)\,y = 0 & \text{on } \Gamma_0\times(0,\infty), \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 & \text{in } \Omega, \end{cases} \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with C^2-boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, Γ_0 and Γ_1 are closed and disjoint, meas $(\Gamma_0) \geq 0$ and meas $(\Gamma_1) > 0$, $a \geq 0$, $m \geq 2$, and $P > 2$ are constants, v is the unit outward normal to Γ, $u_t = \frac{\partial u}{\partial t}$, $y_t = \frac{\partial y}{\partial t}$, $\Delta u = \sum_{i=1}^n \left(\partial^2 u / \partial x_i^2\right)$, M is a positive C^1 -function and h represents the kernel of the memory term, y is the normal displacement to the boundary at time t at the boundary point x, and α and β will be specified later.

When $h = 0$ and $M \equiv 1$, The first Eq. in (P) becomes a nonlinear wave equation, this equation has been extensively studied, and several results concerning existence and nonexistence have been established. When M is not a constant, the first Eq. in (P) is a Kirchhoff-type wave equation. This type of models was introduced by Kirchhoff in order to study nonlinear vibrations of an elastic string. Kirchhoff was the first to study the oscillations of stretched strings and plates. The existence and nonexistence of solutions in this case have been discussed by many authors.

For the first Eq. in (P) with $h \neq 0$ and $M \equiv 1$, Cavalcanti et al. [9] studied the case of $m = 2$ and the localized damping $a(x) u_t$. They obtained an exponential decay rate under the assumption that the kernel h decays exponentially. They studied the case of $m \geq 2$ in [8]. The results of this work were later improved by Cavalcanti et al. [11] and by Berrimi and Messaoudi [7].
The homogeneous Dirichlet boundary-value problems for Kirchhoff-type equations have been considered by many mathematicians. Nishihara and Yamada [30] considered the global solvability of the homogeneous Dirichlet boundary value problem for

$$\frac{\partial^2 u}{\partial t^2} - a\left(\int_\Omega |\nabla u|^2 \, dx\right) \Delta u + 2\gamma \frac{\partial u}{\partial t} = 0 \quad \text{in} \quad \Omega \times [0, \infty)$$

and showed the global existence, uniqueness, and asymptotic decay of solutions provided that the initial data $u_0 \, (u_0 \neq 0)$, u_1 are small and u_1 is much smaller than u_0 in some sense. Aassila and Benaissa [1] extended the global existence part of [12] to the case where $M(s) > 0$, $M\left(\|\nabla u_0\|^2\right) \neq 0$, and the equation contains the nonlinear dissipative term $|u_t|^{\alpha-2} u_t$. Ono [32], [37] proved the global existence of a solution to the homogeneous Dirichlet boundary value problem for

$$u_{tt} - M\left(\|\nabla u\|_2^2\right) \Delta u - a u_t = b\,|u|^{\beta-2} u \quad \text{in} \quad \Omega \times (0, \infty),$$

where a, $b > 0$ and $\beta > 2$ are constants and $M(s)$ is a C^1 function on $[0, \infty)$ satisfying

$$M(s) \geq m_0, \qquad sM(s) \geq \int_0^s M(\tau)\, d\tau \quad \text{for all} \quad s \in [0, \infty)$$

with $m_0 \geq 1$. Wu et al. [36] solved the general Kirchhoff-type equation

$$u_{tt} - M\left(\|\nabla u(t)\|_2^2\right) \Delta u + |u_t|^{m-2} u_t = |u|^{p-2} u$$

with homogeneous Dirichlet boundary condition and positive upper-bounded initial energy blow-ups. Applying the Banach contraction mapping principle, Gao et al. [17] proved the local existence of a solution to the homogeneous Dirichlet boundary-value problem for the higher-order nonlinear Kirchhoff-type equation

$$u_{tt} + M\left(\|D^m u(t)\|_2^2\right) (-\Delta)^m u + |u_t|^{q-2} u_t = |u|^{p-2} u,$$

where $p > q \geq 2$ and $m \geq 1$. Using Galerkin's method, Ono and Nishihara [31] proved the global existence and decay structure of a solutions to the homogeneous Dirichlet boundary-value problem for

$$u_{tt} - M\left(\|\nabla u\|_2^2\right) \Delta u - a \Delta u_t = b\,|u|^{\beta-2} u \quad \text{in} \quad \Omega \times (0, \infty)$$

without smallness conditions on the data. Wu [35] considered the strong damping integro-differential equation

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t h\left(t-s\right)\Delta u\left(s\right)ds - \Delta u_t = |u|^{p-2}\,u$$

with homogeneous Dirichlet boundary and showed that, under certain conditions on h, the solution is global in time and energy decays exponentially.

The mixed Dirichlet and Neumann homogenous boundary-value problems for Kirchhoff-type equations were considered in [16] by Gorain, who studied the uniform stability of two mixed Dirichlet and Neumann homogenous boundary-value problems for

$$u_{tt} + 2\delta u_t = \left(a^2 + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u \qquad \text{in} \quad \Omega\times(0,\infty)\,,$$

$$u_{tt} = \left(a^2 + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u \; + 2\lambda\Delta u_t \quad \text{in} \quad \Omega\times(0,\infty)\,.$$

Beale and Rosencrans [4] introduced acoustic boundary conditions of the general form

$$\frac{\partial u}{\partial v} = y_t \quad \text{on} \quad \Gamma_0\times(0,\infty)\,, \tag{1}$$

$$\gamma u_t + m\left(x\right)y_{tt} + \alpha\left(x\right)y_t + \beta\left(x\right)y = 0 \qquad \text{on} \quad \Gamma_0\times(0,\infty)\,, \tag{2}$$

and then Beale [5], [6] investigated the global existence and regularity of solutions for the wave equation

$$u_{tt} - \Delta u = 0$$

with conditions (1), (2) by semigroup methods.

These acoustic boundary conditions have great intuitive appeal. It is easy to imagine a music hall designed with these conditions in mind but with an absorbing portion of the boundary (for example, the floor). In recent years, wave equations with acoustic boundary conditions have been treated by many authors. Frota and Goldstein [15] studied the nonlinear Carrier wave equation

$$u_{tt} - M\left(\int_\Omega u^2 dx\right)\Delta u + |u_t|^\alpha\,u_t = 0$$

with the $u = 0$ on Γ_1, (1) and (2). They proved the existence of solutions, but gave no decay rate for solutions. Park and Park [34] considered a wave equation of memory type with acoustic boundary conditions

$$\begin{cases} u_{tt} - \Delta u + \int_0^t h(t-s)\,\Delta u(s)\,ds = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \dfrac{\partial u}{\partial v} - \int_0^t h(t-s)\,\frac{\partial u(s)}{\partial v}ds = y_t & \text{on } \Gamma_0 \times (0,\infty), \\ u_t + p(x)\,y_t + q(x)\,y = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ u(x,0) = u_0, \qquad u_t(x,0) = u_1 & \text{in } \Omega, \end{cases}$$

investigated the influence of the kernel function h, and obtained the general decay rates of solutions when h does not necessarily decay exponentially.

In $[19] - [21]$, Li et al. proved, respectively, the existence and uniqueness, the uniform energy decay rates, and the limit behavior of the solution to the nonlinear viscoelastic Marguerre–von Kàrmàn shallow shells system. Li et al. $[22] - [25]$ proved the global existence and uniqueness of a solution and decay estimates for the nonlinear viscoelastic equation with boundary dissipation. The same authors studied the blow-up phenomenon for some evolution equations in $[26] - [28]$.

Motivated by the above work, we intend to study the energy decay for the problem (P). By using multipliers techniques, we prove that, under certain conditions on M, h, α, β, and on the initial data, the solution to the problem exists globally, and we obtain the uniform decay rate. The main author's contributions in our study is:

(A) The non linearity of viscolastic kirchhoff equation;

(B)They introduce the Lyapunov -type technique for some perturbed energy, they concentrate on the original energy;

(C) The assumptions on the initial data and the relaxation function h are weaker, and the estimates are precise.

The thesis is organized as follows. In chaptre 1, we introduced and stated without proofs some important materials must be need in the proof. In chaptre 2, we study a nonlinear viscoelastic Kirchhoff-type equation with initial conditions and acoustic boundary conditions, we showed that the energy of the solution decays exponentially or polynomially as $t \to +\infty$.

Chapter 1

Preliminaries

In this chapter, we recall the main notions that we will need, after present the normed spaces, Banach and Hilbert, L^p spaces, and sobolev spaces, we will introduce some necessary inequalities and importants Lemmas and theorems.

1.1 Functional Spaces

1.1.1 Normed spaces and Banach spaces

Definition 1.1 *The linear space V is endowed by a binary operation $(v_1, v_2) \to v_1 + v_2 : V \times V \to V$ which makes it a commutative group and furthermore it is equipped with a multiplication $(a, x) \to ax : R \times V \to V$ satisfying*

$$\begin{aligned} (a_1 + a_2)\, v &= av_1 + av_2, \\ a\,(v_1 + v_2) &= av_1 + av_2, (a_1 a_2)\, v \\ &= a_1\,(a_2 v) \end{aligned}$$

and

$$1.v = v.$$

Definition 1.2 *Let V be linear space. A non-negative, degree-1 homogeneous, subadditive functional $\|\cdot\|_V : V \to \mathbb{R}$ called a norm if it vanishes only at 0, often, we will write briefly $\|\cdot\|$ instead of $\|\cdot\|_V$, if the following properties are satisfing respectively:*

$$\begin{aligned} \|v\| &\geq 0 \\ \|av\| &= |a|\,\|v\| \\ \|u + v\| &\leq \|u\|\,\|v\| \end{aligned}$$

for any $v \in V$ *and* $a \in \mathbb{R}$ *and* $\|v\| = 0 \Rightarrow v = 0$*. A linear space equipped with* a *norm is called a normed linear space . If the last property (i.e.*$\|v\|_V = 0 \Rightarrow v = 0$ *) is missing, we call such a functional a seminorm.*

Definition 1.3 *A Banach space is a complete normed linear space* V*. Its dual space* V' *is the linear space of all continuous linear functional* $u : V \to \mathbb{R}$*.*

Notation 1.1 *We can consider the linear space* $\ell(V, \mathbb{R})$*, being also denoted by* V' *and called the dual space to* V*. The original space* V *is then called predual to* V'*.*

Proposition 1.1 V' *equipped with the norm* $\|\cdot\|_{V'}$ *defined by*

$$\|u\|_{V'} = \sup \{|u(x)| : \|x\| \leq 1\},$$

is also a Banach space. If V is a Banach space such that, for any

$$v \in V, \ V \to \mathbb{R} : u \Rightarrow \|u+v\|^2 - \|u-v\|^2$$

is linear, then V is called a Hilbert space. In this case, we define the inner product (also called scalar product) by

$$(u, v) = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2.$$

Definition 1.4 *Since* u *is linear we see that*

$$u : X \to X'',$$

is a linear isometry of V *onto a closed subspace of* V''*, we denote this by*

$$V \to V''.$$

Let V be a Banach space and $u \in V'$. Denote by

$$\begin{array}{rcl} \varphi_u & : & V \to \mathbb{R} \\ x & \to & \varphi_u(V) \end{array}$$

when u cover V', we obtain a family $(\varphi_u)_{u \subset V'}$ of applications to V in $\mathbb{R}$

Proposition 1.2 *The weak star topology on V' is the weakest topology on V' for which every $(\varphi_x)_{x\in v}$ is continuous.*

Theorem 1.1 *Let V be Banach space. Then, V is reflexive, if and only if,*

$$B_V = \{x \in V : \|x\| \leq 1\},$$

is compact with the weak topology $\sigma(V, V')$.

Corollary 1.1 *Every weakly convergent sequence in V' must be bounded if V is a Banach space. In particular, every weakly convergent sequence in a reflexive Banach V must be bounded.*

Definition 1.5 *Let V be a Banach space and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in V. Then u_n converges strongly to u in V if and only if*

$$\lim_{t\to\infty} \|u_n - u\|_V = 0,$$

and this is denoted by $u_n \to u$, or $\lim\limits_{t\to\infty} u_n = u$.

1.1.2 The $L^p(\Omega)$ spaces

Definition 1.6 *Let $1 \leq p \leq \infty$ and let Ω be an open domain in $\mathbb{R}^n$; $n \in \mathbb{N}$ Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

Notation 1.2 *If $p = \infty$; we have*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \textit{ is measurable and there exists a constant } C \textit{ suchthat } |f(x)| \leq C \textit{ i.e} \in \Omega\}.$$

Also, we denote by

$$\|f\|_\infty = \inf\{C, |f(x)| \leq C \text{ a.e} \in \Omega\}.$$

Notation 1.3 *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$; we denote by q the conjugate of p i.e. $\frac{1}{P} + \frac{1}{q} = 1$.*

Theorem 1.2 *$L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$.*

Remark 1.1 *In particularly, when $P = 2$; $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)\, g(x)\, dx,$$

is a Hilbert space.

Theorem 1.3 *For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space*

The $L^p(0,T;X)$ spaces

Let X be a Banach space, denote by $L^p(0,T;X)$ the space of measurable functions such that

$$\left(\int_0^T \|f(t)\|_X^p\, dt\right)^{\frac{1}{p}} = \|f\|_{L^p(0,T;X)} < \infty \text{ for } 1 \leq p \leq \infty$$

If $p = \infty$

$$\|f\|_{L^p(0,T;X)} = \sup_{t\in]0,T[} ess \|f(t)\|_X$$

Theorem 1.4 *The space $L^p(0,T;X)$ is complete.We denote by $D'(0,T;X)$ the space of distributions in $]0,T[$ which take its values in X and us define*

$$D'(0,T;X) = \pounds(D]0,T[,X)$$

where $\pounds(\phi,\varphi)$ is the space of the linear continuous applications of ϕ to φ Since $u \in D'(0,T;X)$; we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right), \forall\varphi \in D(]0,T[)$$

and since, we have $u \in L^p(0,T;X)$

$$u(\varphi) = \int_0^T u(t)\,\varphi(t)\, dt, \forall\varphi \in D(]0,T[)$$

We will introduce some basic results on the $L^p(0,T;X)$ space.

Lemma 1.1 *Let*

$$f \in L^p(0,T;X) \quad \text{and} \quad \frac{\partial f}{\partial t} \in L^p(0,T;X), (1 \leq p \leq \infty)$$

then the function f is continuous from $[0,T]$ to X: i.e $f \in C^1(0,T;X)$

Lemma 1.2 *Let $\varphi =]0,T[\times\Omega$ an open bounded domain in $\mathbb{R}\times\mathbb{R}^n$; and g_μ, g ; are two functions in $L^q\left(]0,T[, L^q(\Omega)\right), 1<q<\infty$ such that*

$$\|g_\mu\|_{L^q(]0,T[,L^q(\Omega))} \leq c, \forall\mu\in\mathbb{N}$$

and $g_\mu \to g$ in φ ; then $g_\mu \to g$ in $L^q(\varphi)$

Theorem 1.5 *$L^p(0,T;X)$ equipped with the norm $\|\cdot\|_{L^q(]0,T[,X)}$, $1\leq p\leq\infty$ is a Banach space.*

Proposition 1.3 *Let X be a reflexive Banach space, X' it's dual, and $1\leq p\ q\leq\infty\ \frac{1}{P}+\frac{1}{q}=1$. Then the dual of $L^P(0,T;X)$ is identify algebraically and topologically with $L^q(0,T;X')$.*

Definition 1.7 *Let X, Y be Banach space, $X\subset Y$ with continuous embedding, then we have*

$$L^P(0,T;X)\subset L^P(0,T;Y),$$

with continuous embedding. The following compactness criterion will be useful for nonlinear evolution problem, especially in the limit of the nonlinear terms

Definition 1.8 ***(Local L^P spaces)*** *Let G be an open set in $\mathbb{R}^N$. The local L^P space on G consists of all L-measurable functions f defined a.e on G such that for every compact set $K\subset G$, the characteristic function $f\times k$ has a finite L^P norm ; that is*

$$\int_K |f(x)|^p\,dx \ \leq\infty \quad \text{if } 1\leq p\leq\infty$$

f is essentially bounded on K if $p=\infty$
This set is denoted $L^p_{loc}(G)$ .from our result on finite measur spaces ,
we have at once for $1\leq p\leq q\leq\infty$

$$L^\infty_{loc}(G)\subset L^q_{loc}(G)\subset L^p_{loc}(G)\subset L^1_{loc}(G).$$

The spaces $\mathbf{C}^k(\Omega)$ et $\mathbf{C}^\infty(\Omega)$, $\mathbf{0}\leq\mathbf{k}\leq\infty$

Definition 1.9 *We denote by $C(\Omega)$ where $C^0(\Omega)\,(resp. C^1(\Omega))$,the space of continuous functions (resp. continuously differentiable) on Ω with numerical values (i.e real or complex). For $k\in\mathbb{N}, k\geq 2$, we pose*

$$C^k(\Omega)=\left\{u\in C^{k-1}(\Omega) : \frac{\partial u}{\partial x_i}\in C^{k-1}(\Omega); i=1,...,n\right\}$$

it is the space of k times continuously differentiable functions on Ω. Finally we note

$$C^\infty(\Omega)=\cap_{k\in\mathbb{N}}C^k(\Omega),$$

1.1.3 Hilbert space

Definition 1.10 *A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

Theorem 1.6 ***(Riesz)*** *If $(H; \langle ., .\rangle)$ is a Hilbert space, $\langle ., .\rangle$ being a scalar product on H ; then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$ and $\|f\|'_H = \|x\|_H$.*

Remark 1.2 *From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.*

Theorem 1.7 *Let $(u_n)_{n\in N}$ is a bounded sequence in the Hilbert space H;it posses a subsequence which converges in the weak topology of H*

Theorem 1.8 *In the Hilbert space, all sequence which converges in the weak topology is bounded.*

Theorem 1.9 *Let $(u_n)_{n\in N}$ be a sequence which converges to u, in the weak topology and $(v_n)_{n\in N}$ is an other sequence which converge weakly to v ; then*

$$\lim_{n\to\infty} \langle v_n, u_n\rangle$$

Theorem 1.10 *Let X be a normed space, then the unit ball*

$$B' \equiv \{x \in X : \|x\| \leq 1\},$$

of X' is compact in $\sigma(X', X)$.

1.1.4 Sobolev spaces

Modern theory of differential equations is based on spaces of function whose derivatives exist in a generalized sense and enjoy a suitable integrability.

Proposition 1.4 *Let Ω be an open domain in $\mathbb{R}^n$, Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi\rangle = \int_\Omega f(x)\varphi(x)\,dx, \text{forall } \varphi \in D(\Omega)$$

where $1 \leq p \leq \infty$, and it's well-known that f is unique.

Let $m \in \mathbb{N}$ and $p \in [0, \infty]$. The$W^{m,p}(\Omega)$,p(W) is the space of all $f \in L^p(\Omega)$, defined as $W^{m,p}(\Omega)$, such that $\partial^\alpha f \in L^p(\Omega)$for all $\alpha \in \mathbb{N}^m$ such that

$$|\alpha| = \sum_{j=1}^{n} \alpha_j \leq m, \text{ where, } \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} .. \partial_n^{\alpha_n}.$$

Theorem 1.11 *$W^{m,p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p}, 1 \leq p \leq \infty, \text{ for all } f \in W^{m,p}(\Omega)$$

Definition 1.11 *Denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.*

Definition 1.12 *When $p = 2$, we prefer to denote by $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ supplied with the norm*

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}}$$

which do at $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx$$

Theorem 1.12

1. *$H^m(\Omega)$ supplied with inner product $\langle ., . \rangle_{H^m(\Omega)}$ is a Hilbert space.*
2. *If $m \geq m'$, $H^m(\Omega) \to H^{m'}(\Omega)$, with continuous imbedding .*

Lemma 1.3 *Since $D(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω, and we have*

$$D(\Omega) \to H_0^m(\Omega) \to L^2(\Omega) \to H^{-m}(\Omega) \to D'(\Omega)$$

The next results are fundamental in the study of partial differential equations

Theorem 1.13 *Assume that Ω is an open domain in $\mathbb{R}^n$ $(N \geq 1)$, with smooth boundary $\partial\Omega$. Then,*

1. If $1 \leq p \leq \infty$, we have $W^{1,p}$, for every $q \in [p, p^*]$, where $p^* = \frac{np}{n-p}$.
2. If $p = n$ we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, \infty)$.
3. If $p > n$ we have $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, where $\alpha = \frac{p-n}{p}$.

The $W^{m,p}(\Omega)$ spaces

Proposition 1.5 *Let Ω be an open domain in $\mathbb{R}^n$. Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi \rangle = \int_\Omega f(x)\,\varphi(x)\,dx, \text{ for all } \varphi \in D(\Omega)$$

where $1 \leq p \leq \infty$ and it's well-known that f is unique. Now, we will introduce the Sobolev spaces: The Sobolev space $W^{K,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order K have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p; D^\alpha f \in L^p(\Omega).\forall\alpha\ ;\ |\alpha| \leq k\}.$$

With this definition, the Sobolev spaces admit a natural norm:

$$f \to \|f\|_{w^{k,p}}(\Omega) = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, \text{ for } p < +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\|.\|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 \leq p \leq \infty$ and a separable space for $1 \leq p \leq \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f,g)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)}$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $C^\infty(\bar{\Omega})$ and $C^m(\bar{\Omega})$. The closure of $D(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H^m_0(\Omega)$ (respectively $W^{K,P}_0(\Omega)$). Now, we introduce a space of functions with values in a space X (a separable Hilbert space).The space $L^2(a,b\ ;\ X)$ is a Hilbert space for the inner product

$$(f,g)_{L^2(a,b\,;\,X)} = \int_a^b (f(t), g(t))_X\,dt$$

we note that $L^\infty(a,b\ ;\ X) = (L^1(a,b\ ;\ X))'$. Now, we define the Sobolev spaces with values in a Hilbert space X. For $k \in N$, $p \in [1,\infty]$, we set:

$$W^{k,p}(a,b;X) = \left\{ v \in L^p(a,b\,;\,X)\ \frac{\partial v}{\partial x_i} \in L^p(a,b\,;\,X).\forall i \leq k \right\}$$

The Sobolev space $W^{k,p}(a,b\,;\,X)$ is a Banach space with the norm

$$\|f\|_{W^{k,p}(a,b\,;\,X)} = \left(\sum_{i=0}^{k} \left\| \frac{\partial f}{\partial x_i} \right\|_{L^P(a,b;X)}^p \right)^{\frac{1}{p}} .\text{for } p < +\infty$$

and

$$\|f\|_{W^{k,\infty}(a,b\,;\,X)} = \left(\sum_{i=0}^{k} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^\infty(a,b;X)} \right)^{\frac{1}{p}} .\text{for } p = +\infty$$

The spaces $W^{k,2}(a,b\,;\,X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b\,;\,X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^i} \right)_X dt.$$

Theorem 1.14 *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ (where $p = n$; $p = 1$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in W^{1,P}(\mathbb{R}^n).$$

Corollary 1.2 *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [P, P^*]$$

with continuous imbedding. For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [n, +\infty[$$

Theorem 1.15 *Let $p > n$, then*

$$W^{1,p}\left(\mathbb{R}^n\right) \subset L^{\infty}\left(\mathbb{R}^n\right)$$

with continuous imbedding.

Corollary 1.3 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have if $1 \leq p \leq \infty$, then $W^{1,p}\left(\Omega\right) \subset L^{p^*}\left(\Omega\right)$ where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$*
if $p = n$; then $W^{1,p}\left(\Omega\right) \subset L^q\left(\Omega\right), \forall q \in [p, +\infty[$
if $p > n$; then $W^{1,p}\left(\Omega\right) \subset L^{\infty}\left(\Omega\right)$ with continuous imbedding. Moreover, if $p > n$ we have:

$$\forall u \in W^{1,p}\left(\Omega\right), \left|u\left(x\right) - u\left(y\right)\right| \leq C\left|x - y\right|^{\alpha}\left\|u\right\|_{W^{1,p}(\Omega)} \text{ a.e } x, y \in \Omega$$

with $\alpha = 1 - \frac{n}{p} > 0$ and C is a constant which depend on p, n and Ω In particular $W^{1,p}\left(\Omega\right) \subset C\left(\bar{\Omega}\right)$

Lemma 1.4 *(Sobolev-Poincarés inequality)*

$$\text{If } 2 \leq q \leq \frac{2n}{n-2},\ n \geq 3\ ;\ \text{and},\ n = 1, 2;$$

then

$$\left\|u\right\|_q \leq C\left(q, \Omega\right)\left\|\nabla u\right\|_2,\ \forall u \in H_0^1\left(\Omega\right)$$

Remark 1.3 *For all $\varphi \in H^2\left(\Omega\right), \Delta\varphi \in L^2\left(\Omega\right)$ and for Γ sufficiently smooth, we have*

$$\left\|\varphi\left(t\right)\right\|_{H^2(\Omega)} \leq C\left\|\Delta\varphi\left(t\right)\right\|_{L^2(\Omega)}$$

Proposition 1.6 *(Green's formula). For all $u \in H^1\left(\Omega\right)$ we have*

$$-\int_{\Omega} \Delta u v dx = \int_{\Omega} \nabla u \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma$$

where $\frac{\partial u}{\partial \eta}$ is a normal derivation of u at Γ.

1.2 Some integral inequalities

Young, Holder's inequalities

Notation 1.4 *Let $1 \leq p \leq \infty$, we denote by q the conjugate exponent,*

$$\frac{1}{P}+\frac{1}{q}=1$$

We define the convolution product of a function $f\in L^{1}\left(\mathbb{R}^{n}\right)$ with a function $g\in L^{p}\left(\mathbb{R}^{n}\right)$.

Theorem 1.16 *Let $f\in L^{1}\left(\mathbb{R}^{n}\right)$ and $g\in L^{p}\left(\mathbb{R}^{n}\right)$ with $1\leq p\leq\infty$. Then for a.e. $x\in\mathbb{R}^{n}$ the function $y\rightarrow f\left(x-y\right)g\left(y\right)$ is integrable on $\mathbb{R}^{n}$ and we define*

$$\left(f*g\right)\left(x\right)=\int_{R^{n}}f\left(x-y\right)g\left(y\right)dy$$

In addition $(f*g)\in L^{p}(\mathbb{R}^{n}\ )$ and

$$\left\|f*g\right\|_{p}\leq\left\|f\right\|_{1}\left\|g\right\|_{p}$$

Theorem 1.17 *Assume $f\in L^{p}\left(\mathbb{R}^{n}\right)$ and $g\in L^{q}\left(\mathbb{R}^{n}\right)$ with $1\leq p\leq\infty$, $1\leq q\leq\infty$ and $\frac{1}{r}=\frac{1}{P}+\frac{1}{q}-1\geq 0$. Then $(f*g)\in L^{r}(\mathbb{R}^{n}\ )$ and*

$$\left\|f*g\right\|_{r}\leq\left\|f\right\|_{p}\left\|g\right\|_{q}$$

Theorem 1.18 *Assume that $f\in L^{p}$ and $g\in L^{q}$ with*

$1\leq p\leq\infty$. Then $(f\ g)\in L^{1}$ and

$$\left\|f\ g\right\|\leq\left\|f\right\|_{p}\left\|g\right\|_{q}$$

Corollary 1.4 *(general form) Let $f_{1},f_{2},...f_{k}$ be k functions such that, $f_{i}\in L^{p}\left(\Omega\right)$,$1\leq i\leq k$, and*

$$\frac{1}{p}=\frac{1}{p_{1}}+\frac{1}{p_{2}}+...+\frac{1}{p_{k}}\leq 1$$

Then, the produc $f_{1}f_{2}...f_{k}\in L^{p}\left(\Omega\right)$ and $\left\|f_{1}f_{2}...f_{k}\right\|_{p}\leq\left\|f_{1}\right\|_{p_{1}}...\left\|f_{k}\right\|_{p_{k}}$.

1.2.1 Minkowski inequality

Lemma 1.5 *For $1\leq p\leq\infty$, we have*

$$\left\|u+v\right\|_{L^{p}}\leq\left\|u\right\|_{L^{p}}+\left\|v\right\|_{L^{p}}.$$

1.2.2 Cauchy-Schwarz inequality

Lemma 1.6 *Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \, \|x_2\| \, .$$

The equality sign holds if and only if x_1 and x_2 are dependent. We will give here some integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Lemma 1.7 *Let* $1 \leq p \leq r \leq q$*,* $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ *, and* $1 \leq \alpha \leq 1$*. Then*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\alpha} \, \|u\|_{L^q}^{1-\alpha} \, .$$

Lemma 1.8 *If* $\mu(\Omega) < \infty$*,* $1 \leq p \leq q \leq \infty$*, then* $L^q \to L^p$*, and Since our study based on some known algebraic inequalities, we want to recall few of them here.*

Lemma 1.9 *For all* $a, b \in \mathbb{R}^+$*, we have*

$$ab \leq \delta a^2 + \frac{b^2}{4\delta'},$$

where δ is any positive constant.

Lemma 1.10 *For all* $a, b \geq 0$*, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q'},$$

where, $\frac{1}{P} + \frac{1}{q} = 1$.

Lemma 1.11 *If* $h \in C^1(R)$ *and* u *is the solution to* $(2.1) - (2.5)$*, then*

$$\int_0^t h(t-s)\,(u(s), u_t(t))\,ds = -\frac{1}{2}\frac{d}{dt}\left(h \circ u - \int_0^t h(s)\,ds\,\|u\|_2^2\right) + \frac{1}{2}h' \circ u - \frac{1}{2}h(t)\,\|u\|_2^2 \, .$$

Proof. Indeed

$$\begin{aligned}\int_0^t h(t-s)\,(u_t(t), u(s))\,ds &= \int_0^t h(t-s)\,(u_t(t), (u(s) - u(t)))\,ds + \int_0^t h(t-s)\,(u_t(t), u(t))\,ds \\ &= -\frac{1}{2}\int_0^t h(t-s)\,\frac{d}{dt}\,\|u(t) - u(s)\|_2^2\,ds + \frac{1}{2}\int_0^t h(s)\,ds\frac{d}{dt}\,\|u\|_2^2\end{aligned}$$

$$= -\frac{1}{2}\frac{d}{dt}\left(h \circ u - \int_0^t h(s)\,ds \,\|u\|_2^2\right) + \frac{1}{2}h' \circ u - \frac{1}{2}h(t)\,\|u\|_2^2 .$$

■

Chapter 2

Energy Decay of nonlinear viscoelastic Kirchhoff type equation with Acoustic Control Boundary Conditions

This chapter is devoted to provide the general decay of solution by using the multiplier techniques and some integral inequalities for the problem of a Nonlinear Viscoelastic Kirchhoff-Type Equation with Acoustic Control Boundary Conditions

2.1 Statement of problem

We study the following Problem

$$\begin{aligned}
&u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t h(t-s)\,\Delta u(s)\,ds \\
&\quad + a\,|u_t|^{m-2}u_t = |u|^{p-2}u \qquad \text{in } \Omega\times(0,\infty), \qquad (2.1)\\
&u = 0 \qquad \text{on } \Gamma_1\times(0,\infty), \qquad (2.2)\\
&M\left(\|\nabla u\|_2^2\right)\frac{\partial u}{\partial v} - \int_0^t h(t-s)\,\frac{\partial u(s)}{\partial v}ds = y_t \qquad \text{on } \Gamma_0\times(0,\infty), \qquad (2.3)\\
&u_t + \alpha(x)\,y_t + \beta(x)\,y = 0 \qquad \text{on } \Gamma_0\times(0,\infty), \qquad (2.4)\\
&u(x,0) = u_0, \quad u_t(x,0) = u_1 \qquad \text{in } \Omega, \qquad (2.5)
\end{aligned}$$

2.2 Preliminaries and Assumptions

Throughout this works, we use the space

$$V=\left\{u\in H^{1}\left(\Omega\right)\mid u=0 \text{ on } \Gamma_{1}\right\},$$

the scalar products

$$\left(u,v\right)=\int_{\Omega}u\left(x\right)v\left(x\right)dx,\qquad \left(u,v\right)_{\Gamma_{0}}=u\left(x\right)v\left(x\right)dS,$$

and the norms

$$\left\|u\right\|_{L^{p}(\Omega)}=\left(\int_{\Omega}\left|u\right|^{p}dx\right)^{\frac{1}{P}},\ \ \left\|u\right\|_{L^{p}(\Gamma_{0})}=\left(\int_{\Gamma_{0}}\left|u\right|^{p}dS\right)^{\frac{1}{p}}.$$

To simplify notation, we denote $\left\|u\right\|_{L^{p}(\Omega)}$ and $\left\|u\right\|_{L^{p}(\Gamma_{0})}$ by $\left\|u\right\|_{P}$ and $\left\|u\right\|_{P,\Gamma_{0}}$, respectively. The symbol $h*u$ stands for convolution, that is,

$$h*u=\int_{0}^{t}h\left(t-s\right)u\left(s\right)ds,$$

and by $\circ$ we denote

$$h\circ\nabla u\left(t\right)=\int_{0}^{t}h\left(t-s\right)\int_{\Omega}\left|\nabla u\left(s\right)-\nabla u\left(t\right)\right|^{2}dxds=\int_{0}^{t}h\left(t-s\right)\left\|\nabla u\left(s\right)-\nabla u\left(t\right)\right\|^{2}ds.$$

We make the following general assumptions on M, α, β, and h.

(A$_1$) The notation $M\left(s\right)$, $s\geq 0$, is used for a positive C^{1} function satisfying

$$M\left(s\right)\geq m_{0}>0\ \text{ and }\ \overline{M}\left(s\right)\leq sM\left(s\right),$$

where

$$\overline{M}\left(s\right)=\int_{0}^{s}M\left(t\right)dt.$$

(A$_2$) The notation $h:[0,\infty)\longrightarrow[0,\infty)$ is used for a nonincreasing C^{1} function satisfying

$$m_{0}-\int_{0}^{\infty}h\left(s\right)ds=l>0.$$

Furthermore, there exists a $\rho\in(2,\infty]$ and a $\xi>0$ such that

$$h^{'}\left(t\right)\leq-\xi h^{1+1/\rho}\quad \text{for all}\quad t\geq 0,$$

(for $\rho=\infty$, $1/\rho=0$ is set to 0).

(A$_3$) Functions $\alpha\left(x\right)$ and $\beta\left(x\right)$ are assumed to satisfy the conditions $\alpha\left(x\right)$, $\beta\left(x\right)\in C\left(\Gamma_{0}\right)$ and $\alpha\left(x\right)$, $\beta\left(x\right)>0$ for all $x\in\Gamma_{0}$. This assumption implies that there exist positive constants α_{i} and β_{i} $i=0,1$, such that,

$$\alpha_0 \leq \alpha(x) \leq \alpha_1, \quad \beta_0 \leq \beta(x) \leq \beta_1 \text{ for all } \quad x \in \Gamma_0.$$

(A$_4$) Either $2 < p \leq 2n/(n-2)$, $2 < m \leq 2n/(n-2)$, and $n \geq 3$ or $p > 2$, $m \geq 2$, and $n = 1, 2$.

Remark 2.1 *Assumption* (A_2) *implies that, for some* $t_0 > 0$*,*

$$\int_0^t h(s)\, ds \geq \int_0^{t_0} h(s)\, ds := h_0 \text{ for all } \quad t \geq t_0.$$

Remark 2.2 *Assumption* (A_2) *implies that, for* $\rho \in (2, \infty)$*,*

$$h(t) \leq \frac{k}{(1+t)^\rho} \text{ for all } \quad t \geq 0.$$

Therefore,

$$h^\sigma(t) \in L^1(0, \infty) \text{ for all } \quad t \geq 0 \text{ for any } \sigma > \frac{1}{\rho}.$$

Our result is based on the following existence and uniqueness theorem for a solution to problem $(2.1) - (2.5)$.

Theorem 2.1 *Suppose that Assumptions* (A_1)–(A_4)*, hold and* $(u_0, u_1) \in (V \cap H^2(\Omega)) \times V$ *. Then*

there exists a unique solution u of $(2.1) - (2.5)$ satisfying

$$u \in L^\infty_{loc}\left(0, \infty\, ;\, V \cap H^2(\Omega)\right),$$

$$u_t \in L^\infty_{loc}(0, \infty\, ;\, V), \quad u_{tt} \in L^\infty_{loc}\left(0, \infty\, ;\, V \cap L^2(\Omega)\right), \quad y, y_t \in L^2\left(0, \infty\, ;\, L^2(\Gamma_0)\right).$$

Moreover,

$$u \in C([0, \infty)\, ;\, V), \quad u_t \in C([0, \infty)\, ;\, L^2(\Omega)),$$

$$u(x,t) \longrightarrow u_0(x) \quad \text{in } V, \qquad u_t(x,t) \longrightarrow u_1(x) \quad \text{in } L^2(\Omega) \quad \text{as } t \longrightarrow 0.$$

Proof. This theorem is proved by using Galerkin's method and a calculus theorem in an abstract space [19], [33], [13]. In what follows, we shall use the following lemmas. ■

Lemma 2.1 ***(Poincare´ inequality,*** [2]***).***

Let q satisfy

$$\begin{cases} 2 \leq & q & \leq \frac{2n}{n-2}, & n \geq 3, \\ 2 \leq & q & & n = 1, 2. \end{cases}$$

Then there exists a positive constant c_* such that

$$\|u\|_q \leq c_*(q) \|\nabla u\|_2 \quad \text{for all} \quad u \in V.$$

Moreover, the trace theorem implies

$$\|u\|_{2,\Gamma_0}^2 \leq \lambda \|\nabla u\|_2^2 \quad \text{for all} \quad u \in V\ .$$

2.3 General Decay results

In order to define the energy function $E(t)$ of problem $(2.1) - (2.5)$, we give the following computation. Multiplying u_t by both sides of Eq. (2.1), integrating the resulting equation over Ω, we obtain

$$\begin{aligned} &\int_\Omega u_{tt} u_t dx - \int_\Omega M\left(\|\nabla u\|_2^2\right) \Delta u u_t dx + \int_\Omega \int_0^t h(t-s) \Delta u(s) u_t ds dx + a \int_\Omega |u_t|^{m-2} |u_t|^2 dx \\ = &\int_\Omega |u|^{p-2} u u_t dx. \end{aligned} \tag{2.6}$$

using Green's formula and (A_1) on the second and third terms in the left hand sid of (2.6)

$$\begin{aligned} -\int_\Omega M\left(\|\nabla u\|_2^2\right) \Delta u u_t dx &= \int_\Omega M\left(\|\nabla u\|_2^2\right) \nabla u \nabla u_t dx - \int_{\Gamma_0} M\left(\|\nabla u\|_2^2\right) \frac{\partial u}{\partial v} u_t dS \\ &= \frac{1}{2} \int_\Omega M\left(\|\nabla u\|_2^2\right) \frac{d}{dt} |\nabla u|^2 dx - \int_{\Gamma_0} M\left(\|\nabla u\|_2^2\right) \frac{\partial u}{\partial v} u_t dS \\ &= \frac{1}{2} \frac{d}{dt} (\overline{M}(\|\nabla u\|_2^2) - \int_{\Gamma_0} M\left(\|\nabla u\|_2^2\right) \frac{\partial u}{\partial v} u_t dS \end{aligned} \tag{2.7}$$

$$\int_\Omega \int_0^t h(t-s) \Delta u(s) u_t ds dx = -\int_\Omega \int_0^t h(t-s) \nabla u(s) \nabla u_t ds dx + \int_{\Gamma_0} \int_0^t h(t-s) \frac{\partial u(s)}{\partial v} u_t dS \tag{2.8}$$

we add and subtract the terms $-\int_\Omega \int_0^t h(t-s) \nabla u(s) \nabla u_t ds dx$ to (2.8) we get

$$\begin{aligned}
&-\int_{\Omega}\int_{0}^{t} h(t-s)\nabla u_t\left|\nabla u(s)-\nabla u(t)\right|^2 dsdx-\int_{\Omega}\int_{0}^{t} h(t-s)\nabla u_t\nabla u(t)\,dsdx\\
&= -\frac{1}{2}\int_{\Omega}\int_{0}^{t} h(t-s)\frac{d}{dt}\left|\nabla u(s)-\nabla u(t)\right|^2 dsdx-\frac{1}{2}\int_{\Omega}\int_{0}^{t} h(t-s)\frac{d}{dt}\left|\nabla u\right|^2 dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\frac{d}{dt}\int_{\Omega}\int_{0}^{t} h(t-s)\left|\nabla u(s)-\nabla u(t)\right|^2 dsdx-\frac{1}{2}\int_{\Omega}\int_{0}^{t} h'(t-s)\left|\nabla u(s)-\nabla u(t)\right|^2 dsdx\\
&\quad+\frac{1}{2}\frac{d}{dt}\int_{\Omega}\int_{0}^{t} h(t-s)\left|\nabla u\right|^2 dsdx-\frac{1}{2}\int_{\Omega}\int_{0}^{t} h'(t-s)\left|\nabla u\right|^2 dsdx\\
&= \frac{1}{2}\frac{d}{dt}\int_{0}^{t} h(t-s)\int_{\Omega}\left|\nabla u(s)-\nabla u(t)\right|^2 dxds-\frac{1}{2}\int_{0}^{t} h'(t-s)\int_{\Omega}\left|\nabla u(s)-\nabla u(t)\right|^2 dxds\\
&\quad+\frac{1}{2}\frac{d}{dt}\int_{\Omega}\int_{0}^{t} h(t-s)\left|\nabla u\right|^2 dsdx-\frac{1}{2}\int_{\Omega}\int_{0}^{t} h'(t-s)\left|\nabla u\right|^2 dsdx\\
&= \frac{1}{2}\frac{d}{dt}\int_{0}^{t} h(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|_2^2 ds-\frac{1}{2}\int_{0}^{t} h'(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|_2^2 ds\\
&\quad+\frac{1}{2}\frac{d}{dt}\int_{\Omega}\int_{0}^{t} h(t-s)\left|\nabla u\right|^2 dsdx-\frac{1}{2}\int_{\Omega} h(t-s)\left|\nabla u\right|^2 dsdx\\
&= \frac{1}{2}\frac{d}{dt}h\circ\nabla u-\frac{1}{2}h'\circ\nabla u+\frac{1}{2}\frac{d}{dt}\int_{0}^{t} h(t-s)\,ds\left\|\nabla u\right\|_2^2-\frac{1}{2}h(t-s)\left\|\nabla u\right\|_2^2 ds \qquad (2.9)
\end{aligned}$$

Nou we treat the source terme

$$\begin{aligned}
\int_{\Omega}|u|^{p-2} u u_t dx &= \int_{\Omega}|u|^{p-2}\left[\frac{1}{2}\frac{d}{dt}|u|^2\right] dx\\
&= \frac{1}{p}\frac{d}{dt}\int_{\Omega}|u|^p dx\\
&= \frac{1}{p}\frac{d}{dt}\|u\|_p^p \qquad (2.10)
\end{aligned}$$

using (2.2) (2.3) we obtain

$$-\int_{\Gamma_0} u_t\left(M\left(\|\nabla u\|_2^2\right)\frac{\partial u}{\partial v}-\int_0^t h(t-s)\frac{\partial u(s)}{\partial v}\right) ds=-\left(u_t,y_t\right)_{\Gamma_0}=\left(\alpha(x),y_t^2\right)_{\Gamma_0}+\frac{1}{2}\frac{d}{dt}\left(\beta(x),y^2\right)_{\Gamma_0} \qquad (2.11)$$

substituting(2.7),(2.8), (2.9), (2.10), (2.11)and using Lemma 2-11 into (2.6)

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|_2^2+\overline{M}(\|\nabla u\|_2^2)-\frac{2}{p}\|u\|_p^p\right)-\frac{1}{2}\frac{d}{dt}\left(\int_0^t h(s)\|\nabla u\|_2^2\,ds-h\circ\nabla u\right)+\frac{1}{2}\frac{d}{dt}\int_{\Gamma_0}(\beta(x),y^2)_{\Gamma_0}$$
$$=\quad -a\|u_t\|_m^m-\frac{1}{2}h(t)\|\nabla u\|_2^2+\frac{1}{2}h'\circ\nabla u-.\int_{\Gamma_0}\alpha(x)\,y_t^2dS$$

The above computation inspires us to define energy functional as

$$\begin{aligned}E(t) &= \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\left(\overline{M}\left(\|\nabla u\|_2^2\right)-\int_0^t h(s)\,ds\,\|\nabla u\|_2^2\right)+\frac{1}{2}h\circ\nabla u\\ &\quad+\frac{1}{2}\int_{\Gamma_0}\beta(x)\,y^2dS-\frac{1}{p}\|u\|_p^p \end{aligned}\tag{2.12}$$

Lemma 2.2 *The energy functional $E(t)$ satisfies*

$$E'(t)=-a\|u_t\|_m^m-\frac{1}{2}h(t)\|\nabla u\|_2^2+\frac{1}{2}h'\circ\nabla u-\int_{\Gamma_0}\alpha(x)\,y_t^2dS\le 0.$$

Proof. It is easy to see from the above computation and Assumptions (A_2) and (A_3) that

$$E'(t)=-a\|u_t\|_m^m-\frac{1}{2}h(t)\|\nabla u\|_2^2+\frac{1}{2}h'\circ\nabla u-\int_{\Gamma_0}\alpha(x)\,y_t^2dS\le 0.$$

In what follows, we prove that the energy functional $E(t)$ is nonnegative under appropriate conditions on the initial energy and data. By the definition of $E(t)$, using Assumptions (A_1) and (A_2) and Lemma(2.1) we obtaine

$$\begin{aligned}E(t) &\ge \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\left(m_0\|\nabla u\|_2^2-\int_0^t h(s)\|\nabla u\|_2^2\,ds\right)+\frac{1}{2}h\circ\nabla u+\frac{1}{2}\int_{\Gamma_0}\beta(x)\,y^2dS-\frac{1}{P}\|u\|_p^p\\ &\ge \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}l\|\nabla u\|_2^2+\frac{1}{2}h\circ\nabla u-\frac{c_*^p}{p}\|\nabla u\|_2^P\end{aligned}$$

$$\begin{aligned}E(t) &\ge \frac{1}{2}l\|\nabla u\|_2^2+\frac{1}{2}h\circ\nabla u-\frac{c_*^p}{P}\|\nabla u\|_p^p\\ &\ge \frac{1}{2}\left(l\|\nabla u\|_2^2+h\circ\nabla u\right)-\frac{c_*^p}{p}\|\nabla u\|_2^p\\ &\ge G\left(\left(l\|\nabla u\|_2^2+h\circ\nabla u\right)\right)^{\frac{1}{2}}\qquad\text{for all}\qquad t\ge 0,\end{aligned}\tag{2.13}$$

where

$$G(\lambda)=\frac{1}{2}\lambda^2-\frac{B^p}{p}\lambda^p\qquad\text{and}\qquad B=\frac{c_*}{\sqrt{l}}$$

It is easy to verify that $G(\lambda)$ has a maximum at $\lambda_1=B^{-p/(p-2)}$, and its maximum value is

$$\begin{aligned}G'(\lambda) &= \lambda-B^p\lambda^{p-1}=0\\ \frac{\lambda}{\lambda^{p-1}} &= B^p\\ \lambda^{2-p} &= B^p\end{aligned}$$

replace $\lambda_1 = B^{-p/(p-2)}$ in $G(\lambda)$

$$\begin{aligned} E_1 &= G(\lambda_1) = \frac{1}{2}\lambda_1^2 - \frac{B^p}{p}\lambda_1^p \\ E_1 &= \frac{1}{2}\lambda_1^2 - \frac{B^p}{p}\lambda_1^p = \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_1^2 > 0. \end{aligned}$$

■

Lemma 2.3 *Let* u *be the solution of* $(2.1)-(2.5)$, *and let the initial data satisfy* $E(0) < E_1$ *and* $l^{\frac{1}{2}}\|\nabla u_0\|_2 < \lambda_1$. *Then*

1.
$$\left(l\|\nabla u\|_2^2 + h \circ \nabla u\right)^{\frac{1}{2}} < \lambda_1 \quad for \quad t > 0;$$

2.
$$\|u\|_q^q \leq l^{(q-2)/(p-2)} c_*^{2(p-q)/(P-2)}(q)\|\nabla u\|_2^2 := c(q)\|\nabla u\|_2^2 \text{ for } q \text{ such that}$$
$$\begin{cases} 2 \leq q \leq \frac{2n}{n-2}, & n \geq 3, \\ 2 \leq q & n = 1, 2; \end{cases}$$

in particular,

$$\|u\|_p^p \leq l\|\nabla u\|_2^2;$$

3. the following inequality holds:

$$E(t) \geq \frac{1}{2}\|u_t\|_2^2 + \left(\frac{1}{2} - \frac{1}{p}\right) l\|\nabla u\|_2^2 + \frac{1}{2}\int_{\Gamma_0} \beta(x) y^2 dS + \frac{1}{2} h \circ \nabla u \geq 0;$$

4.
$$\|u\|_p^p \leq (2pk/(P-2))\, E(t) \quad \text{for some} \quad k < 1.$$

Proof. (1) To obtain the desired conclusion (1), we argue by contradiction. Indeed, if (1) does not hold,then it follows from the continuity of $u(t)$ that there exists a $t_0 > 0$ such that

$$\left(l\|\nabla u(t_0)\|_2^2 + (h \circ \nabla u)(t_0)\right)^{1/2} = \lambda_1,$$

which. together with (2.13), implies

$$E(t_0) \geq G\left(\left(l\|\nabla u(t_0)\|_2^2 + (h \circ \nabla u)(t_0)\right)^{1/2}\right) = G(\lambda_1) = E_1. \tag{2.14}$$

Using Lemma 2.2 and the assumption $E(0) < E_1$, we see that

$$E(t) \leq E(0) < E_1 \qquad \text{for all} \qquad t \geq 0. \tag{2.15}$$

Inequalities (3.12) and (3.13) contradict each other.
(2) Noting that $\lambda_1 = B^{\frac{-p}{p-2}} = \left(\frac{c_*}{\sqrt{l}}\right)^{\frac{-p}{p-2}} = l^{\frac{p}{2(P-2)}} c_*^{-\frac{p}{p-2}}$ we see from conclusion (1) that

$$l \left\|\nabla u\right\|_2^2 \leq l \left\|\nabla u\right\|_2^2 + h \circ \nabla u < \lambda_1^2 = \left(B^{\frac{-p}{p-2}}\right)^2 = \left(l^{\frac{p}{2(P-2)}} c^{-\frac{p}{p-2}}\right)^2$$

that is,

$$\left\|\nabla u\right\|_2^2 \leq l^{\frac{2}{p-2}} c^{-\frac{2p}{p-2}},$$

which, together with Lemma 2.1, implies

$$\begin{aligned} \|u\|_q^q &\leq c_*^q \left\|\nabla u\right\|_2^q \leq c_*^q \left\|\nabla u\right\|_2^2 \left(\left\|\nabla u\right\|_2^2\right) \left(\left\|\nabla u\right\|_2^2\right)^{\frac{q-2}{2}} \leq c_*^q \left(l^{\frac{2}{p-2}} c_*^{-\frac{2p}{p-2}}\right)^{\frac{q-2}{2}} \left\|\nabla u\right\|_2^2 \\ \|u\|_q^q &\leq l^{\frac{q-2}{P-2}} c_*^{\frac{2(p-q)}{p-2}} \left\|\nabla u\right\|_2^2 \end{aligned} \tag{2.16}$$

(3) From (2.12) and (2.16), we obtain

$$\begin{aligned} E(t) &= \frac{1}{2}\left\|u_t\right\|_2^2 + \frac{1}{2}\left(\overline{M}(\left\|\nabla u\right\|_2^2) - \int_0^t h(s)\,ds \left\|\nabla u\right\|_2^2\right) + \frac{1}{2} h \circ \nabla u \\ &+ \frac{1}{2}\int_{\Gamma_0} \beta(x)\, y^2 dS - \frac{1}{P}\|u\|_P^P \end{aligned}$$

$$\begin{aligned} E(t) &\geq \frac{1}{2}\left\|u_t\right\|_2^2 + \frac{1}{2}\left(m_0 - \int_0^t h(s)\,ds\right)\left\|\nabla u\right\|_2^2 + \frac{1}{2} h \circ \nabla u + \int_{\Gamma_0} \beta(x)\, y^2 dS - \frac{l}{p}\left\|\nabla u\right\|_2^2 \\ &\geq \frac{1}{2}\left\|u_t\right\|_2^2 + \frac{1}{2} l \left\|\nabla u\right\|_2^2 + \frac{1}{2} h \circ \nabla u + \int_{\Gamma_0} \beta(x)\, y^2 dS - \frac{l}{p}\left\|\nabla u\right\|_2^2 \end{aligned}$$

$$E(t) \geq \frac{1}{2}\left\|u_t\right\|_2^2 + \left(\frac{1}{2} - \frac{1}{p}\right) l \left\|\nabla u\right\|_2^2 + \frac{1}{2}\int_{\Gamma_0} \beta(x)\, y^2 dS + \frac{1}{2} h \circ \nabla u \geq 0. \tag{2.17}$$

(4) By Lemma 2.1 and (2.15), we have

$$\|u\|_p^p \leq c_*^p \left\|\nabla u\right\|_2^p \leq c_*^p \left\|\nabla u\right\|_2^2 \left(\left\|\nabla u\right\|_2^2\right)^{\frac{P-2}{2}}$$

using(2.17)

$$E(t) \geq \frac{1}{2}\left\|u_t\right\|_2^2 + \left(\frac{l(p-2)}{2p}\right)\left\|\nabla u\right\|_2^2 + \frac{1}{2}\int_{\Gamma_0} \beta(x)\, y^2 dS + \frac{1}{2} h \circ \nabla u \geq 0$$

$$E(t) \geq \left(\frac{l(p-2)}{2p}\right)\left\|\nabla u\right\|_2^2$$

implies that

$$\left(\|\nabla u\|_2^2\right)^{\frac{p-2}{2}} \leq \left[\frac{2p}{l(p-2)}E(t)\right]^{\frac{p-2}{2}}$$

so

$$\begin{aligned}
\|u\|_p^p &\leq c_*^p \|\nabla u\|_2^2 \left[\frac{2p}{l(p-2)}E(t)\right]^{\frac{P-2}{2}} \leq \frac{c_*^p}{l}\left[\frac{2p}{l(p-2)}E(0)\right]^{\frac{p-2}{2}} l\|\nabla u\|_2^2 \\
\|u\|_p^p &\leq kl\|\nabla u\|_2^2 \leq \frac{2pk}{p-2}E(t) \qquad \text{for all} \quad t\geq 0, \qquad (2.18)
\end{aligned}$$

where

$$k = \frac{c_*^p}{l}\left[\frac{2p}{l(p-2)}E(0)\right]^{\frac{p-2}{2}}$$

and

$$k = \frac{c_*^p}{l}\left[\frac{2p}{l(p-2)}E(0)\right]^{\frac{p-2}{2}} \leq \frac{c_*^p}{l}\left[\frac{2p}{l(p-2)}E_1\right]^{\frac{p-2}{2}}$$

replace $\lambda_1 = B^{\frac{-p}{p-2}}$ whith E_1

$$E_1 = \left(\frac{1}{2}-\frac{1}{p}\right)\lambda_1^2 = \left(\frac{1}{2}-\frac{1}{p}\right)B^{\frac{-2p}{p-2}} = \left(\frac{p-2}{2p}\right)B^{\frac{-2p}{p-2}}$$

replace E_1 whith k

$$k = \frac{c_*^p}{l}\left[\frac{2p}{l(p-2)}E(0)\right]^{\frac{p-2}{2}} \leq \frac{c_*^p}{l}\left[\frac{2p}{l(p-2)}E_1\right]^{\frac{p-2}{2}} = \frac{c_*^p}{l}\left(B^{\frac{-2p}{p-2}}\right)^{\frac{p-2}{2}} = 1$$

where

$$\begin{aligned}
\left(\frac{c_*}{\sqrt{l}}\right)^p &= B^p \\
\left(B^{\frac{-2p}{p-2}}\right)^{\frac{p-2}{2}} &= B^{-p}
\end{aligned}$$

The proof of Lemma 3.2 is complet. ■

Remark 2.3 *We conclude from the above inequality that $E(0) < E_1$ if and only if $\kappa < 1$.*

Below we state our main result and then make use of the above assumptions and preliminaries to prove it. By c_i and C_i we denote different positive constants

Lemma 2.4 $(see[14])$.

Let $E(t)$ be a nonnegative decreasing function defined on $[0, \infty)$. If

$$\int_s^{+\infty} E(t)dt \leq CE(s) \quad \text{for all} \quad s \geq s_0$$

for some constants S_0 and $C > 0$, then

$$E(t) \leq E(0)e^{1-t/(s_0+C)} \quad \text{for all} \quad t \geq 0.$$

Lemma 2.5 $(See[14])$.

Let $E(t)$ be a nonnegative decreasing function defined on $[0, \infty)$. If

$$\int_s^{+\infty} E^{1+\theta}(t)dt \leq CE^{\theta}(0)\,E(s) \quad \text{for all} \quad s \geq s_0$$

for some constants s_0 and $C > 0$, then

$$E(t) \leq E(0)\left[\frac{(s_0+C)(1+\theta)}{\theta t + s_0 + C}\right]^{1/\theta} \quad forall \quad t \geq 0.$$

Theorem 2.2 *Let u be the global solution of problem $(2.1)-(2.5)$ with conditions $(A_1)-(A_4)$, $E(0) < E_1$, and $l^{\frac{1}{2}}\|\nabla u_0\|_2 < \lambda_1$. Then the following decay estimates are valid:*

$$\begin{aligned} E(t) &\leq E(0)\exp\left(1-\frac{t}{C+t_0}\right) \quad \textit{for all} \quad t \geq 0 \quad \textit{with} \quad \rho = \infty, \\ E(t) &\leq E(0)\left[\frac{(t_0+C)(1+\rho)}{t+\rho(t_0+C)}\right]^{\rho} \quad \textit{for all} \quad t \geq 0 \quad \textit{with} \quad \rho \in (2, \infty). \end{aligned}$$

Proof. Multiplying (2.1) by $\phi(t)\;u(t)$ (where $\phi(t) : [0, \infty) \longrightarrow [0, \infty)$ is a nonincreasing function),integrating the result over $\Omega \times [t_1, t_2]$ $(t_0 \leq t_1 \leq t_2)$, we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} \phi(t) \int_\Omega u_{tt}(t)\,u(t)\,dxdt - \int_{t_1}^{t_2} \phi(t) \int_\Omega M(\|\nabla u(t)\|_2^2)\Delta u u(t)\,dxdt \\ &+ \int_{t_1}^{t_2} \phi(t) \int_\Omega \int_0^t h(t-s)\,\Delta u(s)\,u_t ds dx dt \\ &+ a \int_{t_1}^{t_2} \phi(t) \int_\Omega |u_t(t)|^{m-2}\,u(t)\,u_t dx dt \\ = &\int_{t_1}^{t_2} \phi(t) \int_\Omega |u|^{p-2}\,u(t)\,u(t)\,u_t dx dt \end{aligned} \tag{2.19}$$

using Green's formula dans(2.19)

$$\begin{aligned}
&-\int_{t_1}^{t_2} \phi(t) \int_{\Omega} M(\|\nabla u(t)\|_2^2)\Delta u u(t)\, dxdt \\
&= \int_{t_1}^{t_2} \phi(t) \int_{\Omega} M(\|\nabla u(t)\|_2^2)\nabla u(t)\, \nabla u dxdt - \int_{t_1}^{t_2} \phi(t) \int_{\Gamma_0} M(\|\nabla u(t)\|_2^2)\frac{\partial u}{\partial v} u(t)\, dSdt \\
&\int_{t_1}^{t_2} \phi(t)\, M(\|\nabla u(t)\|_2^2)\, \|\nabla u(t)\|_2^2\, dt - \int_{t_1}^{t_2} \phi(t) \int_{\Gamma_0} M(\|\nabla u(t)\|_2^2)\frac{\partial u}{\partial v} u(t)\, dSdt \qquad (2.20)
\end{aligned}$$

using Green's formula

$$\begin{aligned}
&\int_{t_1}^{t_2} \phi(t) \int_{\Omega} \int_0^t h(t-s)\, \Delta u(s)\, u_t dsdxdt \\
&= -\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)\, (\nabla u(s), \nabla u(t))_{L^2(\Omega)}\, dsdt \\
&\quad -\int_{t_1}^{t_2} \phi(t) \int_{\Gamma_0} \int_0^t h(t-s)\, \frac{\partial u(s)}{\partial v} u(t)\, dSdt \qquad (2.21)
\end{aligned}$$

Where :

$$y_t = \int_{\Gamma_0} M(\|\nabla u\|_2^2)\frac{\partial u}{\partial v} dS - \int_{\Gamma_0} \int_0^t h(t-s)\, \frac{\partial u(s)}{\partial v} dS \qquad \Gamma_0 \times (0, \infty)$$

substitant (2.20), (2.21) and y_t whith (2.19)

$$\begin{aligned}
&\int_{t_1}^{t_2} \phi(t)\, (u_{tt}(t), u(t))_{L^2(\Omega)}\, dt + \int_{t_1}^{t_2} \phi(t)\, M(\|\nabla u(t)\|_2^2)\, \|\nabla u(t)\|_2^2\, dt \\
&-\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)\, (\nabla u(s), \nabla u(t))_{L^2(\Omega)}\, dsdt \\
&+a \int_{t_1}^{t_2} \phi(t) \left(|u_t(t)|^{m-2} u_t(t), u(t)\right)_{L^2(\Omega)} dt \\
&= \int_{t_1}^{t_2} \phi(t)\, \|u(t)\|_p^p\, dt + \int_{t_1}^{t_2} \phi(t)\, (y_t(t), u(t))_{L^2(\Gamma_0)}\, dt \qquad (2.22)
\end{aligned}$$

It follows from (A_1) and (2.12) that

$$\begin{aligned}
E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(\overline{M}(\|\nabla u(t)\|_2^2) - \int_0^t h(s)\, ds\, \|\nabla u(t)\|_2^2\right) + \frac{1}{2} h \circ \nabla u \\
&\quad + \frac{1}{2}\int_{\Gamma_0} \beta(x)\, y^2 dS - \frac{1}{P}\|u(t)\|_P^P
\end{aligned}$$

(A_1) :

$$\begin{aligned}
\overline{M}(\|\nabla u(t)\|_2^2) &\leq \|\nabla u(t)\|_2^2\, M(\|\nabla u(t)\|_2^2) \\
\overline{M}(\|\nabla u(t)\|_2^2) - \|u(t)\|_p^p &\leq M(\|\nabla u(t)\|_2^2)\, \|\nabla u(t)\|_2^2 - \|u(t)\|_P^P
\end{aligned}$$

implies that

$$\overline{M}(\|\nabla u(t)\|_2^2) - \|u(t)\|_P^P = 2E(t) - \|u_t(t)\|_2^2 + \int_0^t h(s)\,ds\,\|\nabla u(t)\|_2^2 - h \circ \nabla u - \int_{\Gamma_0} \beta(x)\,y^2 ds + \frac{2}{P}\|u(t)\|_P^P - \|u(t)\|_P^P$$

implies that

$$\begin{aligned} & 2E(t) + \frac{2}{P}\|u(t)\|_p^p - \|u_t(t)\|_2^2 + \int_0^t h(s)\,ds\,\|\nabla u(t)\|_2^2 - h \circ \nabla u(t) \\ & - \int_{\Gamma_0} \beta(x)\,y^2 dS - \|u(t)\|_P^P \\ = \ & \overline{M}(\|\nabla u(t)\|_2^2) - \|u(t)\|_P^P \leq M(\|\nabla u(t)\|_2^2)\,\|\nabla u(t)\|_2^2 - \|u(t)\|_P^P \end{aligned}$$

implies that

$$\begin{aligned} 2E(t) + \frac{2-p}{P}\|u(t)\|_P^P \leq\ & M(\|\nabla u(t)\|_2^2)\,\|\nabla u(t)\|_2^2 - \|u(t)\|_P^P + \int_{\Gamma_0} \beta(x)\,y^2 dS \\ & + \|u_t(t)\|_2^2 - \int_0^t h(s)\,ds\,\|\nabla u(t)\|_2^2 + h \circ \nabla u(t) \end{aligned}$$

Multiplying by $\phi(t)$ and integrating the result (t_1, t_2) weget

$$\begin{aligned} & 2\int_{t_1}^{t_2} \phi(t)\,E(t)\,dt + \frac{2-p}{p}\int_{t_1}^{t_2} \phi(t)\,\|u(t)\|_p^p\,dt \\ \leq\ & \int_{t_1}^{t_2} \phi(t)\left(M\left(\|\nabla u\|_2^2(t)\right)\|\nabla u(t)\|_2^2 - \|u(t)\|_p^p\right)dt + \int_{t_1}^{t_2} \phi(t)\,\|u_t(t)\|_2^2\,dt \\ & + \int_{t_1}^{t_2} \phi(t)\,(h \circ \nabla u)(t)\,dt - \int_{t_1}^{t_2} \phi(t)\int_0^t h(s)\,ds\,\|\nabla u(t)\|_2^2\,dt \\ & + \int_{t_1}^{t_2} \phi(t)\int_{\Gamma_0} \beta(x)\,y^2(t)\,dSdt. \end{aligned} \tag{2.23}$$

Note that

$$\left[\int_{t_1}^{t_2}\int_\Omega \phi(t)\,u_t u dxdt\right]_t = \int_{t_1}^{t_2}\int_\Omega \phi'(t)\,u_t u dxdt + \int_{t_1}^{t_2}\int_\Omega \phi(t)\,u_{tt} u dxdt + \int_{t_1}^{t_2}\int_\Omega \phi(t)\,u_t^2 dxdt$$

implies that

$$\begin{aligned} -\int_{t_1}^{t_2} \phi(t)\,(u_{tt}(t), u(t))_{L^2(\Omega)}\,dt =\ & \int_{t_1}^{t_2} \phi'(t)\,(u_t(t), u(t))_{L^2(\Omega)}\,dt + \int_{t_1}^{t_2} \phi(t)\,\|u_t(t)\|_2^2\,dt \\ & - \phi(t)\,(u_t(t), u(t))_{L^2(\Omega)}\Big|_{t_1}^{t_2} \end{aligned} \tag{2.24}$$

Substituting (2.22) and (2.24) into (2.23), we obtain

$$\begin{aligned}
& 2\int_{t_1}^{t_2}\phi(t)E(t)\,dt+\frac{2-p}{p}\int_{t_1}^{t_2}\phi(t)\|u(t)\|_p^p\,dt \\
\leq\ & 2\int_{t_1}^{t_2}\phi(t)\|u_t(t)\|_2^2\,dt+\int_{t_1}^{t_2}\phi'(t)(u_t(t),u(t))\,dt-\phi(t)(u_t(t),u(t))|_{t_1}^{t_2} \\
& +\int_{t_1}^{t_2}\phi(t)\int_0^t h(t-s)(\nabla u(s),\nabla u(t))\,dsdt-a\int_{t_1}^{t_2}\phi(t)\left(|u_t(t)|^{m-2}u_t(t),u(t)\right)dt \\
& +\int_{t_1}^{t_2}\phi(t)(h\circ\nabla u)(t)\,dt-\int_{t_1}^{t_2}\phi(t)\int_0^t h(s)\,ds\,\|\nabla u(t)\|_2^2\,dt \\
& +\int_{t_1}^{t_2}\phi(t)(y_t(t),u(t))_{\Gamma_0}\,dt+\int_{t_1}^{t_2}\phi(t)\int_{\Gamma_0}\beta(x)y^2(t)\,dsdt. \qquad (2.25)
\end{aligned}$$

Since

$$\begin{aligned}
& \int_{t_1}^{t_2}\phi(t)\int_0^t h(t-s)(\nabla u(s),\nabla u(t))\,dsdt-\int_{t_2}^{t_1}\phi(t)\int_0^t h(s)\,ds\,\|\nabla u(t)\|_2^2 \\
=\ & \int_{t_1}^{t_2}\phi(t)\int_0^t h(t-s)(\nabla u(s)-\nabla u(t),\nabla u(t))\,dsdt,
\end{aligned}$$

we can write (2.25) as

$$\begin{aligned}
& 2\int_{t_1}^{t_2}\phi(t)E(t)\,dt+\frac{2-p}{p}\int_{t_1}^{t_2}\phi(t)\|u(t)\|_p^p\,dt \\
\leq\ & 2\int_{t_1}^{t_2}\phi(t)\|u_t(t)\|_2^2\,dt+\int_{t_1}^{t_2}\phi'(t)(u_t(t),u(t))\,dt-\phi(t)(u_t(t),u(t))|_{t_1}^{t_2} \\
& +\int_{t_1}^{t_2}\phi(t)\int_0^t h(t-s)(\nabla u(s),\nabla u(t))\,dsdt-a\int_{t_1}^{t_2}\phi(t)\left(|u_t(t)|^{m-2}u_t(t),u(t)\right)dt \\
& +\int_{t_1}^{t_2}\phi(t)\int_0^t h(t-s)(\nabla u(s),\nabla u(t))\,dsdt+\int_{t_1}^{t_2}\phi(t)(y_t(t),u(t))_{\Gamma_0}\,dt \\
& +\int_{t_1}^{t_2}\phi(t)\int_{\Gamma_0}\beta(x)y^2(t)\,dsdt. \\
=\ & :J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8. \qquad (2.26)
\end{aligned}$$

By Lemma 2.3, we have

$$2\int_{t_1}^{t_2}\phi(t)E(t)\,dt+\frac{2-p}{p}\int_{t_1}^{t_2}\phi(t)\|u(t)\|_p^p\,dt\geq(2-2k)\int_{t_1}^{t_2}\phi(t)E(t)\,dt.$$

Next, we shall estimate every term of the right-hand side of (2.26). First,by the Young inequality and Lemmas 2.1, we can write

$$\begin{aligned}
|(u_t,u)| &= \int_\Omega u_t u\,dx\leq\frac{1}{2}\left(\int_\Omega u_t\,dx\right)^2+\frac{1}{2}\left(\int_\Omega u\,dx\right)^2 \\
&\leq \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|u\|_2^2\leq\frac{1}{2}\left(\|u_t\|_2^2+c_*^2\|\nabla u\|_2^2\right)
\end{aligned}$$

by Lemma (2.2) we have

$$E(t) \geq \frac{1}{2}\|u_t\|_2^2$$

by Lemma (2.3)

$$E(t) \geq \left(\frac{1}{2}-\frac{1}{p}\right) l \|\nabla u\|_2^2$$

implies that

$$\|\nabla u\|_2^2 \leq \frac{2p}{(p-2)\,l} E(t)$$

applying young inquality and Lemma 2.1

$$|(u_t, u)| \leq \frac{1}{2}\left(\|u_t\|_2^2 + \|u\|_2^2\right) \leq \frac{1}{2}\left(\|u_t\|_2^2 + c_*^2\|\nabla u\|_2^2\right) \leq \left(1+\frac{c_*^2 p}{(p-2)\,l}\right) E(t).$$

which, together with Lemma 2.2 and the definition of $\varphi(t)$, implies

$$|J_2| = \left|\int_{t_1}^{t_2} \phi'(t)\,(u_t(t), u(t))\,dt\right| \leq c_1 \int_{t_1}^{t_2} \phi'(t)\,E(t)\,dt \leq \phi(0)\,c_1 E(t_1). \tag{2.27}$$

$$|J_3| = \left|-\phi(t)\,(u_t(t), u(t))|_{t_1}^{t_2}\right| \leq 2\phi(0)\,c_1 E(t_1), \tag{2.28}$$

where $c_1 = 1 + c_*^2 p/((p-2)\,l)$.

Using the Holder and Young inequalities, we obtain

$$\left|\int_\Omega |u_t|^{m-2} u_t u dx\right| \leq \int_\Omega |u_t|^{m-1}|u|\,dx \leq \left(\int_\Omega \left(|u_t|^{m-1}\right)^{m/(m-1)}\right)^{(m-1)/m}\left(\int_\Omega |u|^m\,dx\right)^{1/m}$$

Young ineqaulity :

$$X\cdot Y \leq \frac{\delta^r}{r}X^r + \frac{\delta^{-q}}{q}Y^q \qquad X, Y>0, \delta>0, \frac{1}{r}+\frac{1}{q}=1$$

$$\begin{aligned}
\|u\|_m \|u_t\|_m^{m-1} &\leq \frac{\delta^m}{m}\left(\|u\|_m\right)^m + \frac{\delta^{\frac{m}{m-1}}}{\frac{m}{m-1}}\left(\|u_t\|_m^{m-1}\right)^{\frac{m}{m-1}} \qquad /\delta = \eta^{\frac{1}{m}} \\
&= \frac{\eta}{m}\|u\|_m^m + \eta^{\frac{-1}{(m-1)}}\frac{m}{m-1}\|u_t\|_m^m \\
&= \frac{\eta}{m}\|u\|_m^m + \frac{m-1}{\eta^{\frac{1}{(m-1)}}m}\|u_t\|_m^m \qquad \text{For } \eta>0
\end{aligned}$$

which, together with Lemmas 2.2, 2.3 implies

$$\begin{aligned}
|J_4| &= \left| -a \int_{t_1}^{t_2} \phi(t) \left(|u_t(t)|^{m-2} u_t(t), u(t) \right) dt \right| \\
&\leq \frac{a\eta}{m} \int_{t_1}^{t_2} \phi(t) \|u(t)\|_m^m dt + \frac{a(m-1)}{\eta^{1/(m-1)} m} \int_{t_1}^{t_2} \phi(t) \|u_t(t)\|_m^m dt \\
&\leq \frac{a\eta c(m)}{m} \int_{t_1}^{t_2} \phi(t) \|\nabla u(t)\|_2^2 dt + \frac{a(m-1)}{\eta^{1/(m-1)} m} \int_{t_1}^{t_2} \phi(t) \|u_t(t)\|_m^m dt \\
&\leq \frac{a\eta c(m)}{m} \frac{2p}{l(p-2)} \int_{t_1}^{t_2} \phi(t) E(t) dt + \frac{m-1}{\eta^{1/(m-1)} m} \int_{t_1}^{t_2} \phi(t) (-E'(t)) dt \\
&= : ac_2\eta \int_{t_1}^{t_2} \phi(t) E(t) dt + \frac{\phi(0)(m-1)}{\eta^{1/(m-1)} m} E(t_1), \qquad (2.29)
\end{aligned}$$

where $c_2 = 2pc(m) / (ml(p-2))$.

Applying (A_2) and Lemma 2.3, we obtain

$$\begin{aligned}
|J_6| &= \int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s) (\nabla u(s) - \nabla u(t), \nabla u(t)) ds dt. \\
&\leq \frac{\eta}{2} \int_{t_1}^{t_2} \phi(t) \|\nabla u(t)\|_2^2 dt + \frac{1}{2\eta} \int_{t_1}^{t_2} \phi(t) \left(\int_0^t h(t-s) \|\nabla u(s) - \nabla u(t)\|_2 ds \right)^2 dt \\
&\leq \frac{\eta}{2} \int_{t_1}^{t_2} \phi(t) \|\nabla u(t)\|_2^2 dt \\
&\quad + \frac{1}{2\eta} \int_{t_1}^{t_2} \phi(t) \left(\int_0^t h^{1-1/\rho}(s) ds \right) \left(\int_0^t h^{1+1/\rho}(t-s) \|\nabla u(s) - \nabla u(t)\|_2^2 ds \right) dt \\
&\leq \frac{p\eta}{l(p-2)} \int_{t_1}^{t_2} \phi(t) E(t) dt - \frac{\phi(0)}{2k\eta} \int_0^\infty h^{1-1/\rho}(s) ds \int_{t_1}^{t_2} h' \circ \nabla u(t) dt \\
&\leq \frac{p\eta}{l(p-2)} \int_{t_1}^{t_2} \phi(t) E(t) dt + \frac{\phi(0)}{k\eta} \int_0^\infty h^{1-1/\rho}(s) ds E(t_1). \qquad (2.30)
\end{aligned}$$

It follows from Lemmas 2.1, 2.2, and 2.3 and Assumption (A_3) that, for all $\eta > 0$, we have

$$\begin{aligned}
|J_7| &= \left| \int_{t_1}^{t_2} \phi(t) (y_t(t), u(t))_{\Gamma_0} dt \right| \leq \frac{1}{2\eta} \int_{t_1}^{t_2} \phi(t) \|y_t(t)\|_{2,\Gamma_0}^2 dt + \frac{\eta}{2} \int_{t_1}^{t_2} \phi(t) \|u(t)\|_{2,\Gamma_0}^2 dt \\
&\leq -\frac{\phi(0)}{2\alpha_0\eta} \int_{t_1}^{t_2} E'(t) dt + \frac{\eta\lambda}{2} \int_{t_1}^{t_2} \phi(t) \|\nabla u\|_2^2 dt \\
&\leq \frac{\phi(0) E(t_1)}{2\alpha_0\eta} + \frac{\eta\lambda p}{l(p-2)} \int_{t_1}^{t_2} \phi(t) E(t) dt =: \frac{\phi(0)}{2\alpha_0\eta} E(t_1) + c_3\eta \int_{t_1}^{t_2} \phi(t) E(t) dt, \qquad (2.31)
\end{aligned}$$

where $c_3 = p\lambda / (l(p-2))$.

Using (2.4), the Holder and Cauchy inequalities, and Lemma 2.1, we obtain

$$\begin{aligned}
\int_{\Gamma_0} \beta(x) y^2(t)\, dS &= \frac{d}{dt}(y(t), u(t))_{\Gamma_0} - (y_t(t), u(t))_{\Gamma_0} - \frac{1}{2}\frac{d}{dt}\left(\alpha(x), y^2(t)\right)_{\Gamma_0}, \\
\phi(t)(y(t), u(t))_{\Gamma_0} &\leq \frac{\phi(0)}{2}\left(\|y(t)\|^2_{2,\Gamma_0} + \|u(t)\|^2_{2,\Gamma_0}\right) \leq \frac{\phi(0)}{2}\left(\|y(t)\|^2_{2,\Gamma_0} + \lambda\|\nabla u(t)\|^2_2\right) \\
&\leq \left[\frac{\lambda p}{l(p-2)} + \frac{1}{\beta_0}\right]\phi(0) E(t),
\end{aligned}$$

which, together with(2.31) and Lemmas 2.2 and 2.3, implies

$$\begin{aligned}
J_8 &= \int_{t_1}^{t_2} \phi(t) \int_{\Gamma_0} \beta(x) y^2(t)\, dSdt \\
&= \phi(t)(y, u)_{\Gamma_0}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \phi'(t)(y, u)_{\Gamma_0} dt - \int_{t_1}^{t_2} \phi(t)(y_t(t), u(t))_{\Gamma_0} dt \\
&\quad -\frac{1}{2}\phi(t)\left(\alpha(x), y^2(t)\right)_{\Gamma_0}\Big|_{t_1}^{t_2} + \frac{1}{2}\int_{t_1}^{t_2} \phi'(t)\left(\alpha(x), y^2(t)\right)_{\Gamma_0} dt \\
&\leq \left|\phi(t)(y, u)_{\Gamma_0}\Big|_{t_1}^{t_2}\right| + \left|\int_{t_1}^{t_2} \phi'(t)(y, u)_{\Gamma_0} dt\right| + \left|\int_{t_1}^{t_2} \phi(t)(y_t(t), u(t))_{\Gamma_0} dt\right| \\
&\quad +\frac{1}{2}\left|\phi(t)\left(\alpha(x), y^2(t)\right)_{\Gamma_0}\Big|_{t_1}^{t_2}\right| \\
&\leq \left[\frac{3\lambda p}{l(p-2)} + \frac{3}{\beta_0} + \frac{1}{2\alpha_0\eta} + \frac{\alpha_1}{\beta_0}\right]\phi(0) E(t_1) + c_3\eta \int_{t_1}^{t_2} \phi(t) E(t)\, dt. \qquad (2.32)
\end{aligned}$$

Let us estimate the first term on the right-and side of (2.11). Multiplying (2.1) by

$$\phi(t)\int_0^t h(t-s)(u(t) - u(s))\, ds$$

and integrating the result over $\Omega \times [t_1, t_2]$, where $t_0 \leq t_1 \geq t_2$, we obtain

$$\begin{aligned}
&-\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)(u_{tt}(t), u(t) - u(s))\, dsdt \\
&= \int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)\left(M\left(\|\nabla u\|^2_2\right)\nabla u(t), \nabla u(t) - \nabla u(s)\right) dsdt \\
&\quad -\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)(y_t, u(t) - u(s))_{\Gamma_0} dsdt \\
&\quad -\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s) \int_0^t h(t-\tau)(\nabla u(\tau), \nabla u(t) - \nabla u(s))\, d\tau dsdt \\
&\quad +a\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)((t)\left(|u_t(t)|^{m-2} u_t, u(t) - u(s)\right) dsdt \\
&\quad -\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)\left(|u(t)|^{p-2} u(t), u(t) - u(s)\right) dsdt. \qquad (2.33)
\end{aligned}$$

Note that

$$
\begin{aligned}
& -\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)\left(u_{tt}(t), u(t)-u(s)\right) dsdt \\
= & \int_{t_1}^{t_2} \phi(t) \int_0^t h'(t-s)\left(u_t(t), u(t)-u(s)\right) dsdt + \int_{t_1}^{t_2} \phi(t) \int_0^t h(s)\, ds \left\|u_t\right\|_2^2 dt \\
& -\phi(t) \int_0^t h(t-s)\left(u_t(t), u(t)-u(s)\right) ds \Big|_{t_1}^{t_2} \\
& + \int_{t_1}^{t_2} \phi'(t) \int_0^t h(t-s)\left(u_t(t), u(t)-u(s)\right) dsdt. \qquad (2.34)
\end{aligned}
$$

Substituting (2.33) into (2.34), we obtain

$$
\begin{aligned}
& \int_{t_1}^{t_2} \phi(t) \int_0^t h(s)\, ds \left\|u_t\right\|_2^2 dt \\
= & \int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)\left(M\left(\|\nabla u\|_2^2\right) \nabla u(t), \nabla u(t)-\nabla u(s)\right) dsdt \\
& -\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)\left(y_t, u(t)-u(s)\right)_{\Gamma_0} dsdt \\
& -\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s) \int_0^t h(t-\tau)\left(\nabla u(\tau), \nabla u(t)-\nabla u(s)\right) d\tau dsdt \\
& +a \int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)\left(\left|u_t(t)\right|^{m-2} u_t(t), u(t)-u(s)\right) dsdt \\
& -\int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s)\left(|u(t)|^{p-2} u(t), u(t)-u(s)\right) dsdt \\
& -\int_{t_1}^{t_2} \phi(t) \int_0^t h'(t-s)\left(u_t(t), u(t)-u(s)\right) dsdt \\
& +\phi(t) \int_0^t h(t-s)\left(u_t(t), u(t)-u(s)\right) ds \Big|_{t_1}^{t_2} \\
& -\int_{t_1}^{t_2} \phi'(t) \int_0^t h(t-s)\left(u_t(t), u(t)-u(s)\right) dsdt \\
& =: T_1+T_2+T_3+T_4+T_5+T_6+T_7+T_8. \qquad (2.35)
\end{aligned}
$$

Now, we estimate the terms on the right-hand side of (2.35). By virtue of the Young inequality,Lemmas 2.2 and 2.3, formula (2.30), and Assumption (A_1) (where M(s) is a positive c^1-function), we have

$$\begin{aligned} |T_1| &= \left| \int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s) \left(M\left(\|\nabla u\|_2^2 \right) \nabla u(t), \nabla u(t) - \nabla u(s) \right) dsdt \right| \\ &\leq \frac{\eta}{2} \int_{t_1}^{t_2} \phi(t) M^2 \left(\|\nabla u\|_2^2 \right) \|\nabla u(t)\|_2^2 dt \\ &\quad + \frac{1}{2\eta} \int_{t_1}^{t_2} \phi(t) \left(\int_0^t h(t-s) \|\nabla u(s) - \nabla u(t)\|_2 ds \right)^2 dt \\ &\quad \eta \frac{c_M^2 P}{l(p-2)} \int_{t1}^{t_2} \phi(t) E(t) dt + \frac{\phi(0)}{k\eta} \int_0^\infty h^{1-1/\rho}(s) ds E(t_1), \end{aligned} \tag{2.36}$$

where $M\left(\|\nabla u\|_2^2\right) \leq cM$.
Using the Holder and Cauchy inequalities and Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned} |T_2| &= \left| - \int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s) (y_t(t), u(t) - u(s))_{\Gamma_0} dsdt \right| \\ &\leq \int_{t_1}^{t_2} \phi(t) \int_0^t h(t-s) \left(\frac{1}{2\eta} \|y_t(t)\|_{2,\Gamma_0}^2 + \frac{\eta\lambda}{2} \|\nabla u(t) - \nabla u(s)\|_2^2 \right) dsdt \\ &\leq \frac{\lambda\eta}{2} \int_{t_1}^{t_2} \phi(t) (h \circ \nabla u)(t) dt + \frac{1}{2\eta} (m_0 - l) \int_{t_1}^{t_2} \phi(t) \|y_t(t)\|_{2,\Gamma_0}^2 dt \\ &\leq \lambda\eta \int_{t_1}^{t_2} \phi(t) E(t) dt + \frac{m_{0-l}}{\alpha_0 \eta} \phi(0) E(t_1) \quad \text{for all} \quad \eta > 0. \end{aligned} \tag{2.37}$$

The Holder and Cauchy inequalities, Lemma 2.2, and (2.30) imply

$$\begin{aligned}
|T_3| &= \left|-\int_{t_1}^{t_2}\phi(t)\int_0^t h(t-s)\int_0^t h(t-\tau)(\nabla u(\tau),\nabla u(t)-\nabla u(s))\,d\tau ds dt\right| \\
&= \left|\int_{t_1}^{t_2}\phi(t)\int_\Omega\left(\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)\left(\int_0^t h(t-\tau)\nabla u(\tau)\,d\tau\right)dxdt\right| \\
&= \left|\int_{t_1}^{t_2}\phi(t)\int_\Omega\left(\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)\right. \\
&\quad \left.\times\left(\int_0^t h(t-\tau)(\nabla u(t)-\nabla u(\tau)-\nabla u(t)\,d\tau)\right)dxdt\right| \\
&\leq \int_{t_1}^{t_2}\phi(t)\int_\Omega\left(\int_0^t h(t-s)|\nabla u(s)-\nabla u(t)|\,ds\right)^2dxdt \\
&\quad +\left|\int_{t_1}^{t_2}\phi(t)\int_0^t h(t-s)\left(\int_0^t h(\tau)\,d\tau\nabla u(t),\nabla u(s)-\nabla u(t)\right)dsdt\right| \\
&\leq \frac{m_0\phi(0)}{k}\int_0^\infty h^{1-1/\rho}(s)\,dsE(t_1) \\
&\quad +\frac{1}{2}\int_{t_1}^{t_2}\phi(t)\int_\Omega\left[\frac{1}{\eta}(\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds)^2+\eta\left(\int_0^t h(\tau)\,d\tau\nabla u(t)\right)^2\right]dxdt \\
&\leq \frac{\phi(0)}{k}\left(m_0+\frac{m_0}{2\eta}\right)\int_0^\infty h^{1-1/\rho}(s)\,dsE(t_1)+\frac{\eta}{2}(m_0-l)^2\int_{t_1}^{t_2}\phi(t)\|\nabla u\|^2dt \\
&\leq \frac{\phi(0)}{k}\left(m_0+\frac{m_0}{2\eta}\right)\int_0^\infty h^{1-1/\rho}(s)\,dsE(t_1)+\frac{p(m_0-l)^2}{l(p-2)}\eta\int_{t_1}^{t_2}\phi(t)E(t)\,dt. \qquad (2.38)
\end{aligned}$$

By the Holder inequality and Lemmas 2.1 and 2.11, we have

$$\begin{aligned}
& a\int_0^t h(t-s)\left(|u_t(t)|^{m-2}u_t(t),u(t)-u(s)\right)ds \\
\leq\;& a\frac{\eta}{m}\int_\Omega(\int_0^t h(t-s)|\nabla u(s)-\nabla u(t)|\,ds)^m dx+a\frac{m-1}{\eta^{1/(m-1)}m}\|u_t\|_m^m \\
\leq\;& a\frac{\eta}{m}\int_\Omega(\int_0^t h^{(m-1)/m}(t-s)h^{1/m}(t-s)|\nabla u(s)-\nabla u(t)|\,ds)^m dx+a\frac{m-1}{\eta^{1/(m-1)}m}\|u_t\|_m^m \\
\leq\;& a\frac{\eta}{m}(\int_0^t h(s)\,ds)^{m-1}\int_0^t h(t-s)\|u(t)-u(s)\|_m^m ds+a\frac{m-1}{\eta^{1/(m-1)}m}\|u_t\|_m^m \\
\leq\;& a\frac{\eta}{m}c_*^m(m_0-l)^{m-1}\left[\frac{4p}{l(p-2)}E(0)\right]^{(m-2)/2}\int_0^t h(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2 ds \\
& +a\frac{m-1}{\eta^{1/(m-1)}m}\|u_t\|_m^m \\
=\;& a\frac{\eta}{m}c_*^m(m_0-l)^{m-1}\left[\frac{4p}{l(p-2)}E(0)\right]^{(m-2)/2}h\circ\nabla u(t)+a\frac{m-1}{\eta^{1/(m-1)}m}\|u_t\|_m^m,
\end{aligned}$$

which, together with Lemma 2.2, implies

$$\begin{aligned}|T_4| \leq\ & a\frac{\eta}{m}c_*^m (m_0-l)^{m-1}\left[\frac{4p}{l(p-2)}E(0)\right]^{(m-2)/2}\int_{t_1}^{t_2}\phi(t)\,h\circ\nabla u(t)\,dt \\ & + a\frac{m-1}{\eta^{1/(m-1)}m}\int_{t_1}^{t_2}\phi(t)\|u_t(t)\|_m^m\,dt \\ & \leq a\tfrac{2\eta}{m}c_*^m (m_0-l)^{m-1}\left[\tfrac{4p}{l(p-2)}E(0)\right]^{(m-2)/2}\int_{t_1}^{t_2}\phi(t)E(t)\,dt \\ & +2\tfrac{m-1}{\eta^{1/(m-1)}m}\phi(0)E(t_1)\quad \text{for all } \eta>0.\end{aligned} \tag{2.39}$$

By the Holder inequality and Lemmas 2.2 and 2.3, we can write

$$\begin{aligned}& \int_0^t h(t-s)\left(|u(t)|^{p-2}u(t),u(t)-u(s)\right)ds \\ \leq\ & \frac{1}{p\eta^{p-1}}\int_\Omega\left(\int_0^t h(t-s)|\nabla u(s)-\nabla u(t)|\,ds\right)^p dx+\frac{p-1}{p}\eta\|u\|_p^p \\ =\ & \frac{1}{p\eta^{p-1}}\int_\Omega\left(\int_0^t h^{(p-1-1/\rho)/p}(t-s)\,h^{(1+1/\rho)/p}(t-s)|\nabla u(s)-\nabla u(t)|\,ds\right)^p dx+\frac{p-1}{p}\eta\|u\|_p^p \\ \leq\ & -\frac{1}{p\eta^{p-1}}\left(\int_0^t h(s)\,ds\right)^{p-1-1/\rho}\int_0^t h'(t-s)\|u(t)-u(s)\|_p^p\,ds+\frac{p-1}{p}\eta\|u\|_p^p \\ \leq\ & -\frac{1}{p\eta^{p-1}}c_*^p(m_0-l)^{p-1-1/\rho}\int_0^t h'(t-s)\|\nabla u(t)-\nabla u(s)\|_2^p\,ds+\frac{p-1}{p}\eta\|u\|_p^p \\ \leq\ & -\frac{1}{p\eta^{p-1}}c_*^p(m_0-l)^{p-1-1/\rho}\left[\frac{4p}{l(p-2)}E(0)\right]^{(p-2)/2}h'\circ\nabla u(t)+\frac{p-1}{p}\eta\|u\|_p^p,\end{aligned}$$

which,together with Lemma 2.2,implies

$$\begin{aligned}|T_5| =\ & \left|-\int_{t_1}^{t_2}\phi(t)\int_0^t h(t-s)\left(|u(t)|^{p-2}u(t),u(t)-u(s)\right)dsdt\right| \\ \leq\ & -\frac{1}{p\eta^{p-1}}c_*^p(m_0-l)^{p-1-1/\rho}\left[\frac{4p}{l(p-2)}E(0)\right]^{(p-2)/2}\int_{t_1}^{t_2}\phi(t)\,h'\circ\nabla u(t)\,dt \\ & +\frac{p-1}{p}\eta\int_{t_1}^{t_2}\phi(t)\|u\|_p^p\,dt \\ \leq\ & \frac{2}{p\eta^{p-1}}c_*^p(m_0-l)^{p-1-1/\rho}\left[\frac{4p}{l(p-2)}E(0)\right]^{(p-2)/2}\phi(0)E(t_1) \\ & +\frac{2\eta k(p-1)}{p-2}\int_{t_1}^{t_2}\phi(t)E(t)\,dt.\end{aligned} \tag{2.40}$$

Using the Holder and Cauchy inequalities and Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}
|T_6| &= \left|-\int_{t_1}^{t_2}\phi(t)\int_0^t h'(t-s)\left(u_t(t),u(t)-u(s)\right)dsdt\right| \\
&\le \frac{\delta}{2}\int_{t_1}^{t_2}\phi(t)\|u\|_2^2\,dt+\frac{c_*^2}{2\delta}\int_{t_1}^{t_2}\phi(t)\int_0^t h'(t-s)\,ds\int_0^t h'(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2\,dsdt \\
&\le \frac{\delta}{2}\int_{t_1}^{t_2}\phi(t)\|u\|_2^2\,dt-\frac{c_*^2h(0)}{2\delta}\int_{t_1}^{t_2}\phi(t)\,h'\circ\nabla u dt \\
&\le \frac{\delta}{2}\int_{t_1}^{t_2}\phi(t)\|u_t\|_2^2\,dt+\frac{c_*^2h(0)\,\phi(0)}{\delta}E(t_1).
\end{aligned} \tag{2.41}$$

Using the Holder and Cauchy inequalities and (2.12), we see that

$$\begin{aligned}
\int_0^t h(t-s)\left(u_t(t),u(t)-u(s)\right)ds &\le \frac{1}{2}\int_0^t h(t-s)\left(\|u_t(t)\|^2+\|u(t)-u(s)\|^2\right)ds \\
&\le \frac{1}{2}(m_0-l)\|u_t(t)\|^2+\frac{1}{2}c_*^2h\circ\nabla u\le\left(m_0-l+c_*^2\right)E(t),
\end{aligned}$$

which, together with Lemma 2.3, implies

$$|T_7|=\left|\phi(t)\int_0^t h(t-s)\left(u_t(t),u(t)-u(s)\right)ds\Big|_{t_1}^{t_2}\right|\le 2\left(m_0-l+c_*^2\right)\phi(0)\,E(t_1), \tag{2.42}$$

$$|T_8|=\left|-\int_{t_1}^{t_2}\phi'(t)\int_0^t h(t-s)\left(u_t(t),u(t)-u(s)\right)dsdt\right|\le\left(m_0-l+c_*^2\right)\phi(0)\,E(t_1). \tag{2.43}$$

Remark 2.1 and the inequality $t_1>t_0$ imply

$$\int_{t_1}^{t_2}\int_0^t h(s)\,ds\,\|u_t\|_2^2\,dt\ge h_0\int_{t_1}^{t_2}\|u_t\|_2^2\,dt. \tag{2.44}$$

Combining (2.35)–(2.42) and choosing $\delta=h_0$, we conclude that

$$J_1=\int_{t_1}^{t_2}\phi(t)\|u_t\|_2^2\,dt\le\frac{2}{h_0}c_4\eta\int_{t_1}^{t_2}\phi(t)\,E(t)\,dt+\frac{2}{h_0}c_5\phi(0)\,E(t_1), \tag{2.45}$$

where

$$c_4=\frac{C_M^2p}{l(p-2)}+\lambda+\frac{p(m_0-l)^2}{l(p-2)}+a\frac{2}{m}c_*^m(m_0-l)^{m-1}\left(\frac{4p}{l(p-2)}E(0)\right)^{(m-2)/2}+\frac{2k(p-1)}{p-2},$$

$$\begin{aligned}
c_5 &= \frac{1}{k\eta}\int_0^\infty h^{1-1/\rho}(s)\,ds+\frac{m_0-l}{\alpha_0\eta}+\frac{1}{k}\left(m_0+\frac{m_0}{2\eta}\right)\int_0^\infty h^{1-1/\rho}(s)\,ds+2\frac{m-1}{\eta^{1/(m-1)}m} \\
&\quad+\frac{2}{p\eta^{p-1}}c_*^p(m_0-l)^{p-1-1/\rho}\left[\frac{4p}{l(p-2)}E(0)\right]^{(P-2)/2}+\frac{c_*^2h(0)}{\delta}+3\left(m_0-l+c_*^2\right).
\end{aligned}$$

Combining Lemma 2.3, (2.26)–(2.32), and (2.45), we obtain

$$(2-2k)\int_{t_1}^{t_2}\phi(t)\,E(t)\,dt \le c_6\eta\int_{t_1}^{t_2}\phi(t)\,E(t)\,dt + c_7\phi(0)\,E(t_1) + \int_{t_1}^{t_2}\phi(t)\,h\circ\nabla u(t)\,dt, \quad (2.46)$$

where

$$\begin{aligned} c_6 &= \frac{2}{h_0}c_4 + ac_2 + \frac{p}{l(p-2)} + 2c_3, \\ c_7 &= \frac{2}{h_0}c_5 + 3c_1 + \frac{m-1}{\eta^{1/(m-1)}m} + \frac{1}{k\eta}\int_0^\infty h^{1-1/\rho}(s)\,ds + \frac{1}{2\alpha_0\eta} + \frac{3\lambda p}{l(p-2)} + \frac{3}{\beta_0} + \frac{1}{2\alpha_0\eta} + \frac{\alpha_1}{\beta_0}. \end{aligned}$$

Then, choosing $\eta > 0$ small enough, we conclude from (2.46) that

$$\int_{t_1}^{t_2}\phi(t)\,E(t)\,dt \le C_1\phi(0)\,E(t_1) + \int_{t_1}^{t_2}\phi(t)\,(h\circ\nabla u)(t)\,dt, \quad (2.47)$$

where C_1 is a positive constant.

Case $1:\rho=\infty$. Setting $\phi(t)=1$ in (3.37), we see from Lemma 3.1 that

$$\int_{t_1}^{t_2} h\circ\nabla u(t)\,dt \le -\varepsilon\int_{t_1}^{t_2} h'\circ\nabla u(t)\,dt \le 2\varepsilon E(t_1),$$

that is,

$$\int_{t_1}^{t_2} E(t)\,dt \le CE(t_1). \quad (2.48)$$

Now, letting $t_2 \longrightarrow \infty$ in (2.47), we obtain

$$\int_{t_1}^{\infty} E(t)\,dt \le CE(t_1) \qquad \text{for all} \qquad t_1 > t_0.$$

Therefore, by Lemma 2.4, we have

$$E(t) \le E(0)\exp\left(1-\frac{t}{C+t_0}\right) \qquad \text{for} \qquad t\ge 0.$$

Case 2: $\rho\in(2,\infty)$. Setting $\phi(t)=E^{m/\rho}(t)$, $n\ge 1$ in (2.46), taking into account Assumption (A_2),and using the Holder inequality and Lemma 2.2, we obtain

$$\int_{t_1}^{t_2} E^{m/\rho}(t)\,h\circ\nabla u(t)\,dt$$

$$\begin{aligned} = \int_{t_1}^{t_2} E^{m/\rho}(t)\int_0^t & h^{(m-1)/(\rho+m)}(t-s)\,\|\nabla u(s)-\nabla u(t)\|_2^{2m/(\rho+m)} \\ & \times h^{1-(m-1)/(\rho+m)}(t-s)\,\|\nabla u(s)-\nabla u(t)\|_2^{2p/(\rho+m)}\,dsdt \end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_1}^{t_2} E^{m/\rho}(t)\left(\int_0^t h^{1-1/m}(t-s)\|\nabla u(s)-\nabla u(t)\|_2^2\,ds\right)^{m/(\rho+m)} \\
&\quad\times\left(\int_0^t h^{1+1/\rho}(t-s)\|\nabla u(s)-\nabla u(t)\|_2^2\,ds\right)^{\rho/(\rho+m)} dt \\
&\leq \left(\int_{t_1}^{t_2} E^{1+m/\rho}(t)\left(\int_0^t h^{1-1/m}(t-s)\|\nabla u(s)-\nabla u(t)\|_2^2\,ds\right)dt\right)^{m/(\rho+m)} \\
&\quad\times\left(\int_{t_1}^{t_2}\left(\int_0^t h^{1+1/\rho}(t-s)\|\nabla u(s)-\nabla u(t)\|_2^2\,ds\right)dt\right)^{\rho/(\rho+m)} \\
&\leq \frac{1}{\varepsilon^{\rho/(\rho+m)}}\left(\int_{t_1}^{t_2} E^{1+m/\rho}(t)\left(\int_0^t h^{1-1/m}(t-s)\|\nabla u(s)-\nabla u(t)\|_2^2\,ds\right)dt\right)^{m/(\rho+m)} \\
&\quad\times\left(-\int_{t_1}^{t_2}(h'\circ\nabla u)(t)\,dt\right)^{\rho/(\rho+m)} \\
&\leq \left(\frac{2}{\varepsilon}\right)^{\rho/(\rho+m)}\left(\int_{t_1}^{t_2} E^{1+m/\rho}(t)\left(\int_0^t h^{1-1/m}(t-s)\|\nabla u(s)-\nabla u(t)\|_2^2\,ds\right)dt\right)^{m/(\rho+m)} \\
&\quad\times E^{\rho/(\rho+m)}(t_1).
\end{aligned} \tag{2.49}$$

For any $m\geq 1$, we define

$$\varphi_m(t):=\int_0^t h^{1-1/m}(t-s)\|\nabla u(s)-\nabla u(t)\|_2^2\,ds,$$

so that $\varphi_m(t)$ is bounded. Indeed, thanks to Remark 2.2, we know that $h^{1/2}\in L^1(0,\infty)$, since $\rho>2$. Hence, recalling Lemma 2.3, we obtain

$$|\varphi_2(t)|\leq C\int_0^t h^{1/2}(t-s)(E(s)+E(t))\,ds\leq 2C\int_0^\infty h^{1/2}ds E(0)\quad\text{for all } t\geq 0,$$

Thus,

$$\|\varphi_2\|_\infty\leq cE(0), \tag{2.50}$$

as claimed.

Owing to (2.49) and (2.50), we have

$$\begin{aligned}
\int_{t_1}^{t_2} E^{2/\rho}(t)\,h\circ\nabla u(t)\,dt &\leq c\left(\int_{t_1}^{t_2} E^{2/\rho}(t)\,dt\right)^{2/(\rho+2)}\|\varphi_2\|_\infty^{2/(\rho+2)}\|\varphi_2\|_\infty^{2/(\rho+2)}E^{\rho/(\rho+2)}(t_1) \\
&\leq \eta\int_{t_1}^{t_2} E^{1+2/\rho}(t)\,dt+c(\eta)\|\varphi_2\|_\infty^{2/\rho}E(t_1).
\end{aligned} \tag{2.51}$$

Thanks to (2.51), for $\eta > 0$ small enough, (2.47) with $\varphi(t) = E^{2/\rho}(t)$becomes

$$\int_{t_1}^{t_2} E^{1+2/\rho}(t)\,dt \leq cE^{2/\rho}(0)\,E(t_1) \quad \text{for all } \ t_1 > t_0. \tag{2.52}$$

Letting $t_2 \to \infty$ in (2.52) and applying Lemma 2.5, we obtain

$$E(t) \leq E(0)\left[\frac{(t_0 + c)(2 + \rho)}{2t + \rho(t_0 + c)}\right]^{\rho/2} \quad \text{for all } \ t \geq 0. \tag{2.53}$$

Recalling Lemma 2.3 and the fact that$\rho > 2$ and using (2.53), we obtain

$$|\varphi_1(t)| \leq \int_0^t \|\nabla u(s) - \nabla u(t)\|_2^2\,ds \leq \frac{4}{l(p-2)}\left(\int_0^\infty E(s)\,ds + tE(t)\right) \leq cE(0)\,. \tag{2.54}$$

Owing to (2.49) and (2.54), we have

$$\begin{aligned}\int_{t_1}^{t_2} E^{1/\rho}(t)\,h \circ \nabla u(t)\,dt &\leq C\left(\int_{t_1}^{t_2} E^{1+1/\rho}(t)\,dt\right)^{1/(\rho+1)} \|\varphi_1\|_\infty^{1/(\rho+1)} E^{\rho/(\rho+1)}(t_1) \\ &\leq \eta\int_{t_2}^{t_2} E^{1+1/\rho}(t)\,dt + c(\eta)\,\|\varphi_1\|_\infty^{1/\rho} E(t_1)\,. \end{aligned} \tag{2.55}$$

Thanks to (2.55), for $\eta > 0$ small enough, we know that (3.37) with $\phi(t) = E^{1/\rho}(t)$ becomes

$$\int_{t_1}^{t_2} E^{1+1/\rho}(t)\,dt \leq cE^{1/\rho}(0)\,E(t_1) \quad \text{for all} \quad t_1 > t_0. \tag{2.56}$$

As above, letting $t_2 \to \infty$ in (3.46), we can write

$$\int_{t_1}^{\infty} E^{1+1/\rho}(t)\,dt \leq cE^{1/\rho}(0)\,E(t_1) \quad \text{for all} \quad t_1 > t_0.$$

Letting $t_2 \to \infty$in $(2.56)$and applying Lemma 2.3, we obtain

$$E(t) \leq E(0)\left[\frac{(t_0 + c)(1 + \rho)}{t + \rho(t_0 + c)}\right]^{\rho} \quad \text{for all } t \geq 0.$$

■

Conclusion

The study of the dynamic properties over time of the solutions of nonlinear evolution equations with acoustic control boundary conditions has aroused the interest of many mathematicians for a long time. Instead of using a lyapunov-type technique for some pertubed energy, a new active method, due to komornik is presented so as to achieve arbitrily large decay rate for the energy of the problem.

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