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By : **Hamza MEDEKHEL**

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An asymptotic behavior of positive solutions for a new class of parabolic systems involving of (p(x),q(x))-Laplacian systems

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Dedication

I dedicate this thesis to my precious parents may Allah protect and joy them.

To the soul of my brother **Ali**, May Allah bless his pure soul.

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Abstract

In this thesis, the deals with an asymptotic behavior of positive solution for a new classe of parabolic system involving of (p(x),q(x))- Laplacian system of partial differential equations using a new method which is a sub and super solution according to some ([13]-[44]) which treated the stationary case, this idea is new for evolutionary case of this kind of problem. The purpose of our this thesis will provide a framework for image restoration. Furthermore, fuild modeling electrolysis is widely considered as an important application that treats non-homogenous Laplace operators. In the last century, many studies of the experimental side have been studied on various materials that rely on this advanced theory, as they are important in electrical fluids, which states that viscosity relates to the electric field in a certain liquid.

Keywords

Parabolic differential equations-(p(x)-q(x))-Laplacian-Positive solutions- Sub-super solution- Asymptotic behavior.

Résumé

Dans cette thèse, le comportement de la présence de la solution positive a été prouvé ainsi que son comportement convergent pour une nouvelle classe d'équations parabolique (système de Laplace d'équations aux dérivées partielles parabolique), en utilisant une nouvelle méthode, qui est la méthode des solutions partielles considérant quelques des conditions aux limites données dans les articles de recherche précédents liés aux équations aux dérivées partielles elliptiques, Et nos résultats sont une extension de notre publication précédente dans ([13]-[44]), qui traitait de l'état stationnaire qui n'est pas lié au temps variable, et cette idée est un nouveau cas évolutif pour ce type de problème, de nombreuses expériences ont été étudiées sur différents problèmes physiques sur la base de cette étude mathématique présentée, car ils sont importants dans les électro-fluides.

Mots clés

équations différentielles- parabolique-(p(x)-q(x))- système Laplacian-solutions positive-sub-super solution- comportement asymptotique.

اَلملخّص

في هذه الأَطروحة، تم إِثبَات سلوك وجود الحل الإِيجَابي إِلَى جَانب سلوكه المتقَارب لفئِة جديدة من المعَادلَات المكَافئِة (نظَام لَابلَاس لِلمعَادلَات التفَاضلية المكَافئِة) بإِستخدَام طريقة جديدة و هي طريقة الحلول الجزئِية بإِعتبَار بعض الشروط الحدية المعطَاة في أورَاق بحثية سَابقة تِتعلق بَالمعَادلَات التفَاضلية الجزئِية النَاقصية، و نتَائِجهَا هي إِمتدَاد لنشرنَا السَابق في $([13]-[44])$ ، الذي عَالج الحَالة الثَابتة و التي لَا تِتعلق بَالمتغير الزمني، و هذه الفكرة تعتبر حَالة تطورية جديدة لهذَا النوع من المسَائِل، تم درَاسة العديد من التجَارب علَى مسَائِل فيزيَائِية مختلفة تعتمد علَى هذه الدرَاسة الريَاضية المقدمة، حيث إِنهَا مهمة في السوَائِل الكهربَائِية.

اَلكلمَات المفتَاحية

المعَادلة التفَاضلية المكَافئِة ـ نظَام $(p(x)-q(x))-$ لَابلَاسيَان ـ اَلحلول الموجبة ـ الحلول الجزئِية ـ سلوك تقَاربي

Contents

Notation

Ω : bounded domain in $\mathbb{R}^2$.

Γ : topological boundary of Ω.

$x = (x_1, x_2)$: generic point of $\mathbb{R}^2$.

$dx = dx_1 dx_2$: Lebesgue measuring on Ω.

∇u : gradient of u.

Δu : Laplacien of u.

$divu$: diverge of u.

$\mathcal{D}(\Omega)$: space of differentiable functions with compact support in Ω.

$\mathcal{D}'(\Omega)$: distribution space.

$C^k(\Omega)$: space of functions k-times continuously differentiable in Ω.

$L^p(\Omega)$: space of functions p-th power integrated on with measure of dx.

$\|f\|_p = \left(\int\limits_{\Omega} (|f|^p) \right)^{\frac{1}{p}}$.

$W^{1,p}(\Omega) = \{u \in L^p(\Omega),\ \nabla u \in L^p(\Omega)\}$.

H : Hilbert space.

$H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

$H_0^m(\Omega) = W_0^{1,m}(\Omega)$.

$Q_T = (0, T) \times \Omega$, $T > 0$

<u>If X is a Banach space</u>

$L^p(0,\ T;\ X) = \left\{ f : (0,T) \longrightarrow X \text{ is measurable; } \int\limits_0^T \|f(t)\|_X^p \ dt < \infty \right\}$.

$L^\infty(0,\ T;\ X) = \left\{ f : (0,T) \longrightarrow X \text{ is measurable; } ess - \sup\limits_{t\ \in[0,\ T]} \|f(t)\|_X^p \ < \infty \right\}$.

$C^k([0,\ T];\ X)$:Space of functions $k-$times continuously differentiable for $[0,\ T] \longrightarrow X$.

$\mathcal{D}([0,\ T];\ X)$: Space of functions continuously differentiable with compact support in $[0,\ T]$.

Euler's schema method

Euler's Method assumes our solution is written in the form of a Taylor's Series.We'll have a function of the form:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(x)}{2!} + \frac{h^2 f'''(x)}{3!} + \frac{h^2 f^{(iv)}(x)}{4!} + ...$$

This gives us a reasonably good approximation if we take plenty of terms, and if the value of h is reasonably small.

For Euler's Method, we just take the first 2 terms only.

$$f(x+h) = f(x) + hf'(x)$$

The last term is just h times our $\frac{df(x)}{dx}$expression, so we can write Euler's Method as follows:

$$f(x+h) = f(x) + hf'(x)$$

Introduction

Partial differential equations are of crucial importance in modelization and description of natural phenomena in physics, mechanics, chemistry, biology ...etcs.
Several physical phenomena : Fluid dynamics, continuum mechanics, simulation of airplane, calculator charts and time prediction are modelized by various systems of partial differential equations.

The authors in their paper in [77] studied the existence of positively solution for the following stationary problem:

$$\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)} f(v) \quad \text{in } \Omega, \\ -\Delta_{q(x)}v = \lambda^{q(x)} g(u) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where, we have the following condition:

$$\lim_{u\to+\infty} \frac{f(M(g(u))^{\frac{1}{p-1}})}{u^{p-1}} = 0 \text{ for all } M > 0,$$

and the author did not consider any condition of symmetric and without any sign initial condition on $g(0)$ and $f(0)$. Then they studied the existence of positively solution of the last stationary problem, in this theoretical of the thesis, we will extend the previous study into the following evolutionary problem: find $u \in L^2(0, T, H_0^1(\Omega))$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_{p(x)}u = \lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)\right] \quad \text{in} \quad Q_T = (0, T) \times \Omega, \\ \\ \dfrac{\partial v}{\partial t} - \Delta_{q(x)}v = \lambda^{q(x)} \left[\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)\right] \quad \text{in} \quad Q_T = (0, T) \times \Omega, \\ \\ u = v = 0 \quad \text{on} \quad \partial Q_T = (0, T) \times \partial\Omega, \\ \\ u(x, 0) = \varphi(x), \end{cases}$$

We assume also $\Omega \subset \mathbb{R}^N$ is a bounded domain, and the functions $p(x), q(x)$ belong to $C^1(\overline{\Omega})$ and satisfying the following conditions:

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{\Omega} p(x) < \infty, 1 < q^- := \inf_{x\in\Omega} q(x) \leq q^+ := \sup_{x\in\Omega} q(x) < \infty$$

and satisfy some natural growth condition at $u = \infty$.
$\Delta_{p(x)}$ is given by $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, the parameters $\lambda, \lambda_1, \lambda_2, \mu_1$ and μ_2 are positive with a, b, c, d are regular functions. In addition we did not consider any sign condition on $f(0), g(0), h(0), \tau(0)$.
The linear and nonlinear stationary equations with operators of quasilinear homogeneous type as p-Laplace operator can be carried out according to the standard Sobolev spaces theory of $W^{m,p}$, and thus we can find the weak solutions. The last spaces consist of functions having weak derivatives which verify some conditions of integrability. Thus, we can have the nonhomogeneous case of $p(.)$-Laplace operators in this last condition. We will use Sobolev spaces of the exponential variable in our standard framework, so that $L^{p(\cdot)}(\Omega)$ will be used instead of Lebesgue spaces $L^p(\Omega)$.
Also, we will denote the new Sobolev space by $W^{m,p}(\Omega)$ and if we replace $L^p(\Omega)$ by $L^{p(\cdot)}(\Omega)$,the Sobolev spaces becomes $W^{m,p(\cdot)}(\Omega)$.Several Sobolev spaces properties have been extended to spaces of Orlicz-Sobolev, particularly by O'Neill in the reference ([61]). The spaces $W^{m,p(\cdot)}(\Omega)$ and $L^{p(\cdot)}(\Omega)$ have been carefully studied by many researchers team (see the references [29]-[30], [50]-[56],[70]).
Here, in our study we consider the boundedness condition in domain Ω, because many results for $p(x)$-Laplacian theory are not usually verified for the $p(x)$-Laplacian theory, for that in ([37]) the quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx}$$

becomes 0 generally. Then $\lambda_{p(x)}$ can be positive only for some given conditions In fact, the first eigenvalue of $p(x)$-Laplacian and its associated eigenfunction cannot exist, the existence of the positive first eigenvalue λ_p and getting its eigenfunction are very important in the p-Laplacian problem study. Therefore, the study of existence of solutions of our problems have more meaning.
Many studies of the experimental side have been studied on various materials that rely on this advanced theory, as they are important in electrical fluids, which states that viscosity relates to the electric field in a certain liquid.
We shall introduce the existence of positively solution of the parabolic partial differential equation and will be proved according to the conditions of symmetry, using super-solution and sub-solution.

The outline of the thesis is as follows:
- In the first chapter, we introduce some of the basic concepts of functional spaces, and we present a brief description
of those aspects of the Hilbert space, Banach space, continuous function spaces, and functional analysis, the L^p space and Sobolev spaces, which lie at the heart of the modern theory of Partial Differential Equations (PDE).
- In the second chapter, we introduce a elliptic boundary value problems system for $(p, q, r)-$Laplacien, we can be applied in evolutionary boundary value problems.
- In the third chapter we prove that model for parabolic problem involving $(p(x), q(x))$-Laplacien system, we shall study is problems we prove the existence of positive solutions by sup-super solutions methods. Finally we will study the asymptotics behavior of that models.
-In fourth chapter we provide a existence of positive solutions of Kirchhoff parabolic systems involving of $(p(x), q(x))$-Laplacien systems with multiple parameter, she is nouvels models. Where are apply the previous theories by existence of positive solutions and results.
During the period of the thesis study, we were able to publish the following article:
1. Medekhel, H.; Boulaaras, S; Guefaifia, R. Existence of positive solutions for a class of Kirchhoff parabolic systems with multiple parameters. *Appl. Math. E-Not.(18)*, 295–306, **2018**. (index in Scopus)
2. H. Medekhel, S. Boulaaras, K.Zennir and A. Allahem , Existence of Positive Solutions and Its Asymptotic Behavior of $(p(x), q(x))$-Laplacian Parabolic System, 11(3), 332, *Symmetry,* **2019.** https://doi.org/10.3390/sym11030332. (index in ISI)

Chapter 1

Preliminary and functional analysis

1- Continuous function spaces
2-Banach spaces
3-Hilbert spaces
4- L^p Spaces
5- Functional analysis
6- Sobolev Spaces
7- Maximum principle
8- Eigenvalue problem
9- Comparison lemma

In this chapter we shall introduce and state some necessary materials needed in the proof of our results, and shortly the basic results which concerning continuous spaces, Banach spaces, Hilbert space, the L^p space, Sobolev spaces, Maximum principe and other theorems. The knowledge of all this notations and results are important for our study.

1.1 Continuous function spaces

We give here some notations and conventions used in the following.

Let $x=(x_1,x_2,...,x_n)$ denote the generic point of an open set Ω of $\mathbb{R}^n$. Let u be a function defined from Ω to $\mathbb{R}^n$, we designate by $D^i u(x)=\dfrac{\partial u(x)}{\partial x_i}$ the partial derivative of u with respect to $x_i\,(1\leq i\leq n)$. Let's also define the gradient and the p-Laplacian from u, respectively as following

$$\nabla u=(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},...,\frac{\partial u}{\partial x_n})^{\ T} \quad \text{and} \quad |\nabla u|^2=\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^2 \tag{1.1}$$

$$\Delta_p u(x)=div\left(|\nabla u|^{p-2}\nabla u\right)(x). \tag{1.2}$$

Note by $C(\Omega)$ the space of continuous functions from Ω to $\mathbb{R}$, $(C(\Omega),\mathbb{R}^m)$ the space of continuous functions from Ω to $\mathbb{R}^m$ and $C_b\left(\overline{\Omega}\right)$ the space of all continuous and bounded functions on $\overline{\Omega}$, it is equipped with the norm $\|.\|_\infty$:

$$\|u\|_\infty=\sup_{x\in\overline{\Omega}}|u(x)| \tag{1.3}$$

For $k\geq 1$ integer, $C^k(\Omega)$ is the space of functions u which are k times derivable and whose derivation of order k is continuous on Ω. $C_c^k(\Omega)$ is the set of functions of $C^k(\Omega)$, whose support is compact and contained in Ω.
We are also define $C^k\left(\overline{\Omega}\right)$, as the set of restrictions to $\overline{\Omega}$ of elements from $C^k(\mathbb{R}^n)$ or as being the set of functions of $C^k(\Omega)$, such that for all $0\leq j\leq k$, and for all $x_0\in\partial\Omega$, the limit $\lim\limits_{x\to x_0} D^j u(x)$ exists and depends only on x_0.
$C_0^\infty(\Omega)$ or $\mathfrak{D}(\Omega)$, is the space of the infinitely differentiable functions, with compact supports called test function space.

1.2 Banach Spaces: Definition and Properties

We first review some basic facts from calculus in the most important class of linear spaces "Banach spaces".

Definition 1.2.1 *A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \longrightarrow \mathbb{R}$.*

Proposition 1.2.1 *([72]) X' equipped with the norm*

$$\|f\|_{X'} = \sup\left\{|f(u)| : \|u\|_X \leq 1\right\},$$

is also a Banach space.

Definition 1.2.2 *Let X be a Banach space, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X. Then u_n converges strongly to u in X if and only if*

$$\lim_{n\longrightarrow\infty} \|u_n - u\|_X = 0,$$

and this is denoted by $u_n \longrightarrow u$, or $\lim\limits_{n\longrightarrow\infty} u_n = u$

Definition 1.2.3 *A sequence (u_n) in X is weakly convergent to u if and only if*

$$\lim_{n\longrightarrow\infty} f(u_n) = f(u),$$

for every $f \in X'$ and this is denoted by $\lim\limits_{n\longrightarrow\infty} u_n = u$.

1.3 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analysis, known as Hilbert spaces. Then, we must give some important results on these spaces here.

Definition 1.3.1 *A Hilbert space H is a vectorial space supplied with inner product (u, v) such that $\|u\| = \sqrt{(u,\ u)}$ is the norm which let H complete.*

The Cauchy-Schwarz inequality Every inner product satisfies the Cauchy-Schwarz inequality

$$|(x_1,\ x_2)| \leq \|x_1\|\,\|x_2\|\,.$$

The equality sign holds if and only if x_1 and x_2 are dependent.

Corollary 1.3.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence which converges to u, in the weak topology and $(v_n)_{n\in\mathbb{N}}$ is an other sequence which converge weakly to v, then*

$$\lim_{n\longrightarrow\infty}(v_n, u_n) = (v, u)\,.$$

1.4 Functional Spaces

1.4.1 The $L^p(\Omega)$ spaces

Now we define Lebesgue spaces and collect some properties of these spaces. We consider $\mathbb{R}^2$ with the Lebesgue-measure μ.
If $\Omega \subset \mathbb{R}^2$ is a measurable set, two measurable functions $f, g : \Omega \longrightarrow \mathbb{R}$ are called equivalent, if $f = g$ a.e. (almost every where) in Ω.
An element of a Lebesgue space is an equivalence class.

Definition 1.4.1 *Let $1 \leq p < \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$. Define the standard Lebesgue space $L^p(\Omega)$, by*

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ is measurable; } \int_\Omega |f(x)|^p \ dx < \infty \right\}.$$

Notation 1.4.1 *For $p \in \mathbb{R}$, and $1 \leq p < \infty$ denote by*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \ dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$L^\infty(\Omega) = \left\{ \begin{array}{l} f : \Omega \longrightarrow \mathbb{R} \text{ is measurable and there exist a constant } C, \\ \text{ such that, ; } |f(x)| < C \ \text{a.e on } \Omega. \end{array} \right\}$$

Also, we denote by

$$\|f\|_{L^\infty} = ess\sup_{t\in\Omega} |f(x)| = \inf\ \{C,\ |f(x)| < C \ a.e \ on \ \Omega\}\,.$$

Theorem 1.4.1 *([72])* $\left(L^p(\Omega), \|.\|_p\right), (L^\infty(\Omega), \|.\|_\infty)$ *are a Banach spaces.*

Remark 1.4.1 *In particularly, when* $p = 2$, $L^2(\Omega)$ *equipped with the inner product*

$$(f,\ g)_{L^2(\Omega)} = \int_\Omega f(x) .g(x)\, dx,$$

is a Hilbert space.

1.4.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Theorem 1.4.2 *([72]) (Hölder's inequality)*
Let $1 \leq p < \infty$. *Assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$, *then,* $fg \in L^1(\Omega)$ *and*

$$\int_\Omega |f.g|\ \ dx \leq \|f\|_{L^p(\Omega)}\ \ \|g\|_{L^q(\Omega)}\ .$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.4.1 *(Minkowski inequality)*
For $1 \leq p < \infty$, *we have*

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}\,.$$

1.5 Sobolev spaces

1.5.1 Weak derivative

Definition 1.5.1 *Let* Ω *be an open set of* $\mathbb{R}$, *and* $1 \leq i \leq n$. *a function* $u \in L^1_{loc}(\Omega)$ *has an* i^{th} *weak derivative in* $L^1_{loc}(\Omega)$ *if there exists* $f_i \in L^1_{loc}(\Omega)$ *such that for all* $\varphi \in C_0^\infty(\Omega)$ *we have*

$$\int_\Omega u(x)\, \partial_i \varphi(x)\, dx = -\int_\Omega f_i(x)\, \varphi(x)\, dx$$

This leads to say that the i^{th} *derivative within the meaning of distributions of* u *belongs to* $L^1_{loc}(\Omega)$, *we write*

$$\partial_i u = \frac{\partial u}{\partial x_i} = f_i$$

1.5.2 $W^{1,p}(\Omega)$ spaces

Let Ω be a bounded or unbounded open set of $\mathbb{R}^n$, and $p \in \mathbb{R}$, $1 \leq p \leq +\infty$, the space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega)\,;\ \text{such that } \partial_i u \in L^p(\Omega), 1 \leq i \leq n\} \tag{1.4}$$

where ∂_i is the i^{th} weak derivative of $u \in L^1_{loc}(\Omega)$

Theorem 1.5.1 *The space $W^{1,p}(\Omega)$ is continuously embedded into $L^\infty(\Omega)$ ($W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$) for all $1 \leq p \leq +\infty$, i.e that there is a constant C (depending only on Ω) such as*

$$\|u\|_{L^\infty} \leq C\|u\|_{W^{1,p}}\,, \quad \forall u \in W^{1,p}(\Omega)$$

furthermore if Ω is bounded we have

$$W^{1,p}(\Omega) \hookrightarrow C(\Omega) \text{ with compact imbedding }, 1 < p \leq +\infty,$$
$$W^{1,1}(\Omega) \hookrightarrow L^q(\Omega) \text{ with compact imbedding }, 1 \leq q < +\infty.$$

Corollary 1.5.1 *Suppose that Ω an unbounded open set of $\mathbb{R}^n$, and let $u \in W^{1,p}(\Omega)$. Then*

$$\lim_{\substack{|x|\to+\infty \\ x\in\Omega}} u(x) = 0$$

1.5.3 $W^{m,p}(\Omega)$ Spaces

Let Ω be an open set of $\mathbb{R}^n, m \geq 2$ and p a real number such that $1 \leq p \leq +\infty$, we define the space $W^{m,p}(\Omega)$ as following

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \text{ such that } \partial_i u \in L^p(\Omega),\ \forall\alpha, |\alpha| \leq m\}$$

where $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$ the length of α and $\partial_i u = \partial_1^{\alpha_1}...\partial_n^{\alpha_n}$ is the weak derivative of a function $u \in L^1_{loc}(\Omega)$ in the sense of definition 1.5.1.
The space $W^{m,p}(\Omega)$ is equiped with the norm

$$\|u\|_{W^{m,p}} = \|u\|_{L^p} + \sum_{0<|\alpha|\leq m} \|\partial_i u\|_{L^p}$$

1.5.4 $W_0^{1,p}(\Omega)$ Spaces

Definition 1.5.2 *For $1 \leq p < +\infty$ we define the space $W_0^{1,p}(\Omega)$ as being the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$, and we write*

$$W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}}$$

Definition 1.5.3 *$H_0^m(\Omega)$ is given by the completion of $\mathcal{D}(\Omega)$ with respect to the norm $\|.\|_{H^m(\Omega)}$.*

Remark 1.5.1 *Clearly $H_0^m(\Omega)$ is a Hilbert space with respect to the norm $\|.\|_{H^m(\Omega)}$.*
The dual space of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega) = [H_0^m(\Omega)]^*$.

Lemma 1.5.1 *Since $\mathcal{D}(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω, and we have*

$$\mathcal{D}(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

1.6 The $L^p(0,\ T;\ X)$ spaces

Definition 1.6.1 *[51] Let X be a Banach space, denote by $L^p(0,\ T; X)$ the space of measurable functions*

$$f :]0,\ T[\longrightarrow X$$

$$t \longrightarrow f(t),$$

such that

$$\int_0^T (\|f(t)\|_X^p)^{\frac{1}{p}}\, dt = \|f\|_{L^p(0,\ T,\ X)} < \infty.$$

If $p = \infty$

$$\|f\|_{L^\infty(0,\ T,\ X)} = \sup_{t\ \in]0,\ T[} ess\, \|f(t)\|_X.$$

Theorem 1.6.1 *([58],[72])The space $L^p(0,\ T,\ X)$ is a Banach space.*

Lemma 1.6.1 *Let $f \in L^p(0,\ T,\ X)$ and $\dfrac{\partial f}{\partial t} \in L^p(0,\ T,\ X)$ for $1 \leq p \leq \infty$, then the function f is continuous from $[0,\ T]$ to X. i. e. $f \in C^1(0, T, X)$.*

Proof. see of [51], [58]. ■

1.6.1 Green's formula

Proposition 1.6.1 *([58]) Let Ω be an open subset of $\mathbb{R}^d$, with a Lipschitz boundary. Then for all $u, v \in H^1(\Omega)$ we have*

$$\int_{\Omega} (\frac{\partial u}{\partial x_i} v + \frac{\partial v}{\partial x_i} u) \, dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)\eta_i ds, \qquad i = 1, ..., d.$$

where η_i is the $i-th$ component of the outward normal vector η.

1.7 Maximum principle

A large number of results of existence or uniqueness of solutions to boundary problems (elliptic or parabolic), can be established using the maximum principle. Here we give some variants of this result.

Let Ω be an open set of $\mathbb{R}^n$, $a(.) = (a_{ij}(.))_{1\leq i,j\leq n}$ a matrix, $b(.) = (b_i(.))_{1\leq i\leq n}$ a vector and c a function. We consider the second-order symmetric operator L defined by

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{i=1}^{n} b_i\partial_i u + cu \tag{1.5}$$

It is assumed that the square matrix a satisfies the coercive (or elliptic) condition.

$$\exists\alpha > 0,\ \forall\xi \in \mathbb{R}^n,\ a(x)\xi.\xi = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad \text{a.e on } \Omega, \tag{1.6}$$

where $|\xi|$ designates the Euclidean norm of ξ in $\mathbb{R}^n$

Theorem 1.7.1 (Classical maximum principle) *[48] Let Ω a bounded and connected open set, and L as in (1.5) . We suppose that $c \geq 0$, (1.6) is satisfied and $a_{ij}, b_i,\ c \in C(\overline{\Omega})$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ verify $Lu \leq 0$ then we have*

$$\sup_{x\in\overline{\Omega}} u(x) \leq \sup_{\sigma\in\partial\Omega} u^+(\sigma) \qquad \textit{where} \ \ u^+(\sigma) = \max(u(\sigma), 0)$$

Theorem 1.7.2 (Hopf maximum principle) *[48] Let Ω a bounded and connected open set, and L as in (1.5). We suppose that $c \geq 0$, (1.6) is satisfied and $a_{ij}, b_i,\ c \in C(\overline{\Omega})$.*
If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ verify $Lu \leq 0$ and if u reaches a maximum ≥ 0 in the interior of Ω, then u is constant on Ω.

Theorem 1.7.3 (Aleksandrov maximum principle) *[48] Let Ω a bounded and connected open set, and L as in (1.5). We suppose that $c \geq 0$, (1.6) is satisfied and $a_{ij}, b_i,\ c \in C\left(\overline{\Omega}\right)$ and $f \in L^N(\Omega)$. There exists $C > 0$ depending on N, $\|b\|_{L^N(\Omega)}$ and the diameter of Ω such that if $u \in W^{2,N}_{loc}(\Omega) \cap C\left(\overline{\Omega}\right)$ verify $Lu \leq f$ then we have*

$$\sup_{x\in\overline{\Omega}} u(x) \leq \sup_{\sigma\in\partial\Omega} u(\sigma) + C\|f\|_{L^N(\Omega)}$$

Lemma 1.7.1 (boundary point lemma) *[68] Suppose u is continuous on Ω; $Lu \geq 0$ (resp. $Lu \leq 0$) on Ω , and u reaches its maximum (resp. minimum) at a point $p \in \partial\Omega$. So, all outward directional drifts from u to point p are positive (resp. negative).*

1.8 Eigenvalue problem

Definition 1.8.1 *We say that $u \in W^{1,p}_0(\Omega), u \neq 0$, is an eigenfunction of the operator $-\triangle_p u$ if:*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla\varphi dx = \lambda \int_\Omega |u|^{p-2} u.\varphi dx \tag{1.7}$$

for all $\varphi \in C^\infty_0(\Omega)$. The corresponding real number λ is called eigenvalue.

Let λ_1 defined by

$$\lambda_1 = \inf_{u\in W^{1,p}_0(\Omega), u\neq 0} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \tag{1.8}$$

equivalent to

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla u|^p dx; \int_\Omega |u|^p dx = 1, u \in W^{1,p}_0(\Omega), u \neq 0\right\}$$

λ_1 is the first eigenvalue of the p-Laplacian operator with null Dirichlet conditions at the edge.

Lemma 1.8.1 *λ_1 is isolated, i.e there exists $\delta > 0$ such that in the interval $(\lambda_1, \lambda_1 + \delta)$, there is no other eigenvalues of (1.7).*

Lemma 1.8.2 *a) Let $p \geq 2$, then for all $\xi_1, \xi_2 \in \mathbb{R}^n$*

$$|\xi_2|^p \geq |\xi_1|^p + p|\xi_1|^{p-2}\langle\xi_1, \xi_2 - \xi_1\rangle + C(p)|\xi_1 - \xi_2|^p,$$

b)Let $p < 2$, then for all $\xi_1, \xi_2 \in \mathbb{R}^n$

$$|\xi_2|^p \geq |\xi_1|^p + p\,|\xi_1|^{p-2}\,\langle \xi_1, \xi_2 - \xi_1\rangle + C\,(p)\,\frac{|\xi_1 - \xi_2|^p}{(|\xi_2| + |\xi_1|)^{2-p}},$$

where $C\,(p)$ is constant depending only on p.

Lemma 1.8.3 *The first eigenvalue λ_1 is simple, i.e, if u, v are two eigenfunctions associated with λ_1, then, there exists k such that $u = kv$.*

Lemma 1.8.4 *Let u an eigenfunction associated with the eigenvalue λ_1, then u does not change sign on Ω, further if $u \in C^{1,\alpha}$, $\forall x \in \overline{\Omega}$. $u\,(x) \neq 0$*

Proof. By lemma *1.7.1*, we can suppose that u, v are positive on Ω, and taking

$$\begin{aligned} \varphi_1 &= \frac{(u^p - v^p)}{u^{p-1}}, \\ \varphi_2 &= \frac{(v^p - u^p)}{v^{p-1}}, \end{aligned}$$

two test functions in the weak formulation 1.7, we get

$$\begin{aligned} \int\limits_{\Omega} |\nabla u|^{p-2}\,\nabla u \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) dx &= \lambda \int\limits_{\Omega} |u|^{p-2}\,u\left(\frac{u^p - v^p}{u^{p-1}}\right) dx \\ \int\limits_{\Omega} |\nabla v|^{p-2}\,\nabla v \nabla\left(\frac{v^p - u^p}{v^{p-1}}\right) dx &= \lambda \int\limits_{\Omega} |v|^{p-2}\,v\left(\frac{v^p - u^p}{v^{p-1}}\right) dx \end{aligned} \tag{1.9}$$

The addition of these two formulas gives

$$0 = \int\limits_{\Omega} |\nabla u|^{p-2}\,\nabla u \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) dx + \int\limits_{\Omega} |\nabla v|^{p-2}\,\nabla v \nabla\left(\frac{v^p - u^p}{v^{p-1}}\right) dx \tag{1.10}$$

And using the identities

$$\begin{aligned} \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) &= \nabla u - p\frac{v^{p-1}}{u^{p-1}}\nabla v + (p-1)\,\frac{v^p}{u^p}\nabla u, \\ \nabla\left(\frac{v^p - u^p}{v^{p-1}}\right) &= \nabla v - p\frac{u^{p-1}}{v^{p-1}}\nabla u + (p-1)\,\frac{u^p}{v^p}\nabla v, \end{aligned} \tag{1.11}$$

we get the first term of 1.10

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\left(\frac{u^p-v^p}{u^{p-1}}\right)dx &= \int_\Omega |\nabla u|^p\,dx - p\int_\Omega \frac{v^{p-1}}{u^{p-1}}|\nabla u|^{p-2}\nabla v\nabla u dx \\ &+\int_\Omega (p-1)\frac{v^p}{u^p}|\nabla u|^p\,dx \\ &= \int_\Omega |\nabla \ln u|^p u^p dx - p\int_\Omega v^p|\nabla \ln u|^{p-2}\langle\nabla \ln u,\nabla \ln v\rangle dx \\ &+\int_\Omega (p-1)|\nabla \ln u|^p v^p dx\end{aligned} \tag{1.12}$$

We have a similar expression for the second term of 1.10. Then the formula 1.10 becomes

$$\begin{aligned}0 &= \int_\Omega (u^p-v^p)\left(|\nabla \ln u|^p-|\nabla \ln v|^p\right)dx \\ &-p\int_\Omega v^p\left(|\nabla \ln u|^{p-2}\langle\nabla \ln u,\nabla \ln v-\nabla \ln u\rangle\right)dx \\ &-p\int_\Omega u^p\left(|\nabla \ln v|^{p-2}\langle\nabla \ln v,\nabla \ln u-\nabla \ln v\rangle\right)dx\end{aligned} \tag{1.13}$$

Taking $\xi_1=\nabla\ln u$ and $\xi_2=\nabla\ln v$ and using lemma *1.6.1* we get, for $p\geq 2$

$$0\geq \int_\Omega C(p)\,|\nabla \ln u-\nabla \ln v|\,(u^p+v^p)\,dx \tag{1.14}$$

or

$$0=|\nabla \ln u-\nabla \ln v| \tag{1.15}$$

then $u=kv$. ■

Theorem 1.8.1 (Dominated convergence theorem) *[48] Let $\{f_n\}_{n\geq 1}$ a series of functions of $L^1(\Omega)$ converging almost everywhere to a measurable function f. It is assumed that there exists $g\in L^1(\Omega)$ such that for all $n\geq 1$, we get $|f_n|\leq g$ a.e on Ω. Then $f\in L^1(\Omega)$ and*

$$\lim_{n\to+\infty}\|f_n-f\|_{L^1}=0, \text{ and } \int_\Omega f(x)\,dx=\lim_{n\to+\infty}\int_\Omega f_n(x)\,dx$$

Definition 1.8.2 *[48] Let ω be a part of a Banach space X and $F:\omega\rightarrow\mathbb{R}$. Si $u\in\omega$, we says that F is **Gâteaux** differentiable (or G-differentiable) at u, if there exists $l\in X'$ such that in each direction $z\in X$ where $F(u+tz)$ exists for $t>0$ small enough, the directional derivative $F'_z(u)$ exists and we have*

$$\lim_{t\rightarrow0^+}\frac{F(u+tz)-F(u)}{t}=\langle l,z\rangle .$$

we write $F'(u)=l$.

Theorem 1.8.2 *Let $\Omega\subset\mathbb{R}^n, n\geq 3$, an open set, for $p\in(1,+\infty)$ we define a functional $J:W_0^{1,p}(\Omega)\rightarrow\mathbb{R}$ by*

$$J(u)=\int_\Omega|\nabla u|^p\,dx$$

then J is differentiable in $W_0^{1,p}(\Omega)$ and

$$J'(u)(v)=p\int_\Omega|\nabla u|^{p-2}\nabla u.\nabla v dx,\forall v\in W_0^{1,p}(\Omega)$$

Proof. We consider the function $\varphi:\mathbb{R}^n\rightarrow\mathbb{R}$, defined by $\varphi(x)=|x|^p$, it is a function of class C^1, and $\nabla\varphi=p|x|^{p-2}x$,
then for all $x,y\in\mathbb{R}^n$,

$$\lim_{t\rightarrow0}\frac{\varphi(x+ty)-\varphi(x)}{t}=p|x|^{p-2}x.y$$

as a consequence

$$\lim_{t\rightarrow0}\frac{|\nabla u(x)+t\nabla v(x)|^p-|\nabla u(x)|^p}{t}=p|\nabla u(x)|^{p-2}\nabla u(x).\nabla v(x)$$

by Mean value theorem, for almost every $x\in\Omega$ and for $t>0$, there exists a function θ that takes its values in $]0,1[$ and we can write

$$\begin{aligned}&|\nabla u(x)+t\nabla v(x)|^p-|\nabla u(x)|^p-tp|\nabla u(x)|^{p-2}\nabla u(x).\nabla v(x)\\=\;&tp|\nabla u(x)+\theta(t,x)t\nabla v(x)|^{p-2}(\nabla u(x)+\theta(t,x)t\nabla v(x)).\nabla v(x)\\&-tp|\nabla u(x)|^{p-2}\nabla u(x).\nabla v(x)\end{aligned}\tag{1.16}$$

By dividing by t, we get for almost every x :

$$\lim_{t\to 0}\frac{|\nabla(u+tv)(x)|^p-|\nabla u(x)|^p-tp|\nabla u(x)|^{p-2}\nabla u(x).\nabla v(x)}{t}=0.$$

On the other hand, one can see that the second member of the equality 1.16 devided by t is bounded by

$$h(x)=2|\nabla v(x)|(|\nabla u(x)|+|\nabla v(x)|)^{p-1}$$

Then using the Holder inequality we have:

$$|h|\le C\|\nabla v\|_p\left(\|\nabla u\|_p^{p-1}+\|\nabla v\|_p^{p-1}\right).$$

One can apply the Dominated convergence theorem and conclude

$$J'(u)(v)=p\int_\Omega|\nabla u|^{p-2}\nabla u.\nabla v dx,\forall v\in W_0^{1,p}(\Omega)$$

then J is Gâteaux differentiable. ■

Lemma 1.8.5 (*Comparison lemma*)*[4] Let $u,v\in W_0^{1,p}(\Omega)$ satisfying*

$$\int_\Omega|\nabla u|^{p-2}\nabla u.\nabla\varphi dx\le\int_\Omega|\nabla v|^{p-2}\nabla v.\nabla\varphi dx \tag{1.17}$$

for all $\varphi\in W_0^{1,p}(\Omega),\varphi\ge 0,$ then $u\le v$ a.e in Ω.

Proof. Our proof is based on the arguments presented in [8, 9]. We start by defining the function $J:W_0^{1,p}(\Omega)\to\mathbb{R}$ by the formula

$$J(u)=\frac{1}{p}\int_\Omega|\nabla u|^p dx \tag{1.18}$$

it is clear that the functional J is Gâteaux differentiable and continuous and its derivative at $u\in W_0^{1,p}(\Omega)$ is the function $J'(u)\in W_0^{-1,p}(\Omega)$ i.e

$$J'(u)(\varphi)=\int_\Omega|\nabla u|^{p-2}\nabla u.\nabla\varphi dx,\varphi\in W_0^{1,p}(\Omega). \tag{1.19}$$

$J'(u)$ is continuous and bounded. We will show that $J'(u)$ is strictly monotonic in $W_0^{1,p}(\Omega)$. Indeed, for all $u,v\in W_0^{1,p}(\Omega),u\neq v$ without loss of generality, we can suppose that

$$\int_\Omega|\nabla u|^p dx\ge\int_\Omega|\nabla v|^p dx$$

Using the Cauchy inequality we have

$$\nabla u.\nabla v \leq |\nabla u|\,|\nabla v| \leq \frac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2\right) \tag{1.20}$$

from formula (1.18) we deduce

$$\int\limits_{\Omega} |\nabla u|^p\,dx - \int\limits_{\Omega} |\nabla u|^{p-2}\,\nabla u.\nabla v dx \geq \frac{1}{2}\int\limits_{\Omega} |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right)dx \tag{1.21}$$

$$\int\limits_{\Omega} |\nabla v|^p\,dx - \int\limits_{\Omega} |\nabla v|^{p-2}\,\nabla v.\nabla u dx \geq \frac{1}{2}\int\limits_{\Omega} |\nabla v|^{p-2}\left(|\nabla v|^2 - |\nabla u|^2\right)dx \tag{1.22}$$

If $|\nabla u| \geq |\nabla v|$, By using (1.18)-(1.20), we get

$$\begin{aligned}
I_1(u) \quad &= J'(u)(u) - J'(u)(v) - J'(v)(u) + J'(v)(v) \\
&= \left(\int\limits_{\Omega} |\nabla u|^p\,dx - \int\limits_{\Omega} |\nabla u|^{p-2}\,\nabla u.\nabla v dx\right) \\
&- \left(\int\limits_{\Omega} |\nabla v|^{p-2}\,\nabla v.\nabla u dx - \int\limits_{\Omega} |\nabla v|^p\,dx\right) \\
&\geq \int\limits_{\Omega} \tfrac{1}{2}\,|\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right)dx \qquad (1.23) \\
&- \tfrac{1}{2}\int\limits_{\Omega} |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right)dx \\
&= \tfrac{1}{2}\int\limits_{\Omega} \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right)dx \\
&\geq \tfrac{1}{2}\int\limits_{\Omega} \left(|\nabla u|^{p-2} - |\nabla v|^{p-2}\right)\left(|\nabla u|^2 - |\nabla v|^2\right)dx
\end{aligned}$$

if $|\nabla v| \geq |\nabla u|$, by changing the role of u and v in (1.18)-(1.20) we have

$$\begin{aligned} I_2(v) &= J'(v)(v) - J'(v)(u) - J'(u)(v) + J'(u)(u) \\ &= \left(\int_\Omega |\nabla v|^p dx - \int_\Omega |\nabla v|^{p-2} \nabla v.\nabla u dx \right) \\ &- \left(\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla v dx - \int_\Omega |\nabla u|^p dx \right) \\ &\geq \frac{1}{2} \int_\Omega |\nabla v|^{p-2} \left(|\nabla v|^2 - |\nabla u|^2 \right) dx \\ &- \frac{1}{2} \int_\Omega |\nabla v|^{p-2} \left(|\nabla v|^2 - |\nabla u|^2 \right) dx \\ &= \frac{1}{2} \int_\Omega \left(|\nabla v|^{p-2} - |\nabla u|^{p-2} \right) \left(|\nabla v|^2 - |\nabla u|^2 \right) dx \\ &\geq \frac{1}{2} \int_\Omega \left(|\nabla v|^{p-2} - |\nabla u|^{p-2} \right) \left(|\nabla v|^2 - |\nabla u|^2 \right) dx \end{aligned} \tag{1.24}$$

from (1.21)-(1.22), we have

$$(J'(u) - J'(v))(u - v) = I_1 = I_2 \geq 0, \forall u, v \in W_0^{1,p}(\Omega)$$

in addition, if $u \neq v$ and $(J'(u) - J'(v))(u - v) = 0$, then we have

$$\int_\Omega \left(|\nabla u|^{p-2} - |\nabla v|^{p-2} \right) \left(|\nabla u|^2 - |\nabla v|^2 \right) dx = 0,$$

if $|\nabla u| = |\nabla v|$ in Ω, we deduce that

$$\begin{aligned} (J'(u) - J'(v))(u - v) &= J'(u)(u - v) - J'(v)(u - v) \\ &= \int_\Omega |\nabla u|^{p-2} |\nabla u - \nabla v|^2 dx = 0, \end{aligned} \tag{1.25}$$

i.e $u - v$ is a constant, given $u = v = 0$ on $\partial\Omega$ we are getting $u = v$, which is contrary with $u \neq v$. Then $(J'(u) - J'(v))(u - v) > 0$ et $J'(u)$ is strictly monotonic in $W_0^{-1,p}(\Omega)$. Let u, v two functions such that (1.19) is satisfied, let's take $\varphi = (u - v)^+$, the positive part of $u - v$ as a test function in (1.19), we get that

$$(J'(u) - J'(v))(\varphi) = \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi dx - \int_\Omega |\nabla v|^{p-2} \nabla v.\nabla \varphi dx \leq 0. \tag{1.26}$$

Relationships (1.23) and (1.24) imply that $u \leq v$. ■

Chapter 2

Results on existence and non-existence of positive weak solutions for 3×3 p-Laplacian elliptic systems

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In mathematics, in the field of partial differential equations, a **boundary value problem** is a differential equation together with a set of additional constraints, called the boundary conditions following:

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu = g & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where Ω is an open domain in $\mathbb{R}^N$,and $\Gamma = \partial\Omega$ is the boundary of Ω.

A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. It's called the strong solution of the problem, and (2.1) is called the strong formulation of the problem.

A side from the boundary condition, boundary value problems are also classified according to the type of differential operator involved. For an elliptic operator, one discusses *elliptic boundary value problems* and for an parabolic operator, one discusses parabolic boundary value problems.

In most cases it is not possible to find analytical solutions of these problems i.e. that the explicit computation of the exact solution of such equations is often out to be achieved. Therefore, in general, the exact problem is the solution weak positive by a discrete problem that can be solved by sub-super solution methods.

During the past few years, the treatise of positive solutions of singular partia differential equations or systems has been an extremely active research areas. The singular nonlinear problems emerge naturally and they take a main role in the interdisciplinary eld between analysis, biology, geometry, mathematical physics, elasticity, etc.

We will explain in this chapter the main for the solution weak by sub-super solution methods that will be used later.

Consider in this chapter for elliptic system problem the following :

$$\begin{cases} -\Delta_p u = \lambda \alpha (x) f (u, v, w) \ \text{ in } \Omega, \\ \\ -\Delta_q v = \mu \beta (x) g (u, v, w) \ \text{ in } \Omega, \\ \\ -\Delta_r w = \nu \gamma (x) h (u, v, w) \ \text{ in } \Omega, \\ \\ u = v = w = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.2}$$

where $\Delta_\sigma z = \ div \left(|\nabla z|^{\sigma-2} \nabla z \right), \sigma \geq 1, \lambda, \mu$ and ν are functions on $L^\infty (\Omega)$ and Ω is a bounded domain of $\mathbb{R}^N$ with a bounded border $\partial\Omega$. we prove the existence of a positive weak solution for λ , μ and ν big enough under the following condition

$$\lim_{t\to+\infty}\frac{f(t,t,t)}{t^{p-1}}=\lim_{t\to+\infty}\frac{g(t,t,t)}{t^{q-1}}=\lim_{t\to+\infty}\frac{h(t,t,t)}{t^{r-1}}=0.$$

2.1 Definitions and notations

Let X be the Cartesian product of the 3 spaces $W_0^{1,p}(\Omega), W_0^{1,q}(\Omega)$ and $W_0^{1,r}(\Omega)$, i.e,

$$X=W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)\times W_0^{1,r}(\Omega).$$

Let's start by defining the weak solution, the weak sub-solution and the weak super-solution of problem (2.2)

Definition 2.1.1 *We say that $(u_1,u_2,u_3)\in X$ is a weak positive solution of (2.2) if*

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\phi_1 dx=\lambda\int_\Omega \alpha(x) f(u,v,w)\phi_1 dx,$$

$$\int_\Omega |\nabla v|^{q-2}\nabla v.\nabla\phi_2 dx=\mu\int_\Omega \beta(x) g(u,v,w)\phi_2 dx,$$

$$\int_\Omega |\nabla w|^{r-2}\nabla w.\nabla\phi_3 dx=\nu\int_\Omega \gamma(x) h(u,v,w)\phi_3 dx,$$

for all $\phi=(\phi_1,\phi_2,\phi_3)\in X$ with $\phi_i\geq 0$.

Definition 2.1.2 *We say that $(\psi_1,\psi_2,\psi_3),(z_1,z_2,z_3)\in X$ are respectively sub-solution and positive super-solution of (2.2), if the following formulas are satisfied*

$$\int_\Omega |\nabla \psi_1|^{p-2}\nabla \psi_1.\nabla\phi_1 dx\leq\lambda\int_\Omega \alpha(x) f(\psi_1,\psi_2,\psi_3)\phi_1 dx,$$

$$\int_\Omega |\nabla \psi_2|^{q-2}\nabla \psi_2.\nabla\phi_2 dx\leq\mu\int_\Omega \beta(x) g(\psi_1,\psi_2,\psi_3)\phi_2 dx,$$

$$\int_\Omega |\nabla \psi_3|^{r-2}\nabla \psi_3.\nabla\phi_3 dx\leq\nu\int_\Omega \gamma(x) h(\psi_1,\psi_2,\psi_3)\phi_3 dx,$$

respectively

$$\int_\Omega |\nabla z_1|^{p-2} \nabla z_1.\nabla \phi_1 dx \geq \lambda \int_\Omega \alpha(x) f(z_1, z_2, z_3) \phi_1 dx,$$

$$\int_\Omega |\nabla z_2|^{q-2} \nabla z_2.\nabla \phi_2 dx \geq \mu \int_\Omega \beta(x) g(z_1, z_2, z_3) \phi_2 dx,$$

$$\int_\Omega |\nabla z_3|^{r-2} \nabla z_3.\nabla \phi_3 dx \geq \nu \int_\Omega \gamma(x) h(z_1, z_2, z_3) \phi_3 dx,$$

with $0 \leq \psi_i \leq z_i$ *, for all* $\phi = (\phi_1, \phi_2, \phi_3) \in X$ *with* $\phi_i \geq 0, 1 \leq i \leq 3$.

We suppose that f, g and $h : [0, \infty[\times [0, \infty[\times [0, \infty[\to \mathbb{R}$ are respectively in $L^{p^*}(\Omega)$, respectively $L^{q^*}(\Omega)$ et $L^{r^*}(\Omega)$, where $p^* = \frac{Np}{N-p}, q^* = \frac{Nq}{N-q}$ et $r^* = \frac{Nr}{N-r}$, verify the assumption :

$$\begin{aligned} &1) \quad f, g, h : [0, \infty[\times [0, \infty[\times [0, \infty[\to \mathbb{R} \text{ monotonic of class } C^1, \\ &2) \quad \lim_{t_1,t_2,t_3 \to \infty} f(t_1, t_2, t_3) = \lim_{t_1,t_2,t_3 \to \infty} g(t_1, t_2, t_3) = \lim_{t_1,t_2,t_3 \to \infty} h(t_1, t_2, t_3) = +\infty, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &3) \quad \exists k_0 > 0 : f(t_1, t_2, t_3), g(t_1, t_2, t_3), h(t_1, t_2, t_3) \geq -k_0 \text{ pour tout } t_1, t_2, t_3 \geq 0, \\ &4) \quad \exists \alpha_0, \beta_0, \gamma_0, \alpha_1, \beta_1, \gamma_1, > 0 : \begin{cases} \alpha_0 \leq \alpha(x) \leq \alpha_1 \\ \beta_0 \leq \beta(x) \leq \beta_1 \\ \gamma_0 \leq \gamma(x) \leq \gamma_1 \end{cases} \end{aligned} \tag{2.4}$$

$$\lim_{t \to +\infty} \frac{f(t,t,t)}{t^{p-1}} = \lim_{t \to +\infty} \frac{g(t,t,t)}{t^{q-1}} = \lim_{t \to +\infty} \frac{h(t,t,t)}{t^{r-1}} = 0 \tag{2.5}$$

$$\exists \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \nu_1, \nu_2, \nu_3 > 0 : \begin{cases} f(t_1, t_2, t_3) \leq \xi_1 t_1^{p-1} + \eta_1 t_2^{q\left(\frac{p-1}{p}\right)} + \zeta_1 t_3^{r\left(\frac{p-1}{p}\right)} \\ g(t_1, t_2, t_3) \leq \xi_2 t_1^{p\left(\frac{q-1}{q}\right)} + \eta_2 t_2^{q-1} + \zeta_2 t_3^{r\left(\frac{q-1}{q}\right)} \\ h(t_1, t_2, t_3) \leq \xi_3 t_1^{p\left(\frac{p-1}{p}\right)} + \eta_3 t_2^{q\left(\frac{r-1}{r}\right)} + \zeta_3 t_3^{r-1} \end{cases} \tag{2.6}$$

Let λ_1, μ_1 and ν_1 respectively the first eigenvalues of $-\Delta_p$,$-\Delta_q$,$-\Delta_r$, with the homogeneous Dirichlet conditions at the boundary, φ_p, φ_q and φ_r the corresponding positive eigenfunctions with $\|\varphi_p\|_\infty = \|\varphi_q\|_\infty = \|\varphi_r\|_\infty = 1$,et $m_p, m_q, m_r, \delta, \alpha_0, \beta_0, \gamma_0, \alpha_1, \beta_1, \gamma_1 > 0$ real numbers verifying

$$\begin{cases} |\nabla\varphi_p|^p - \lambda_1\varphi_p^p \geq m_p \\ \\ |\nabla\varphi_q|^q - \mu_1\varphi_q^q \geq m_q \quad \text{in } \overline{\Omega}_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\} \\ \\ |\nabla\varphi_r|^r - \nu_1\varphi_r^r \geq m_r \end{cases} \tag{2.7}$$

Note by

$$\theta_1 = \left(\frac{\alpha_1}{(p-1)}(\xi_1 + \xi_2) + \beta_1\eta_1 + \gamma_1\zeta_1\right),$$

$$\theta_2 = \left(\frac{\beta_1}{(p-1)}(\eta_1 + \eta_2) + \alpha_1\xi_1 + \gamma_1\zeta_2\right),$$

$$\theta_3 = \left(\frac{\gamma_1}{(p-1)}(\zeta_1 + \zeta_2) + \alpha_1\xi_2 + \beta_1\eta_2\right),$$

$$\lambda_0 = \frac{p\lambda_1}{(p-1)\max\limits_{i=1,2,3}(\theta_i)}$$
$$\mu_0 = \frac{q\mu_1}{(q-1)\max\limits_{i=1,2,3}(\theta_i)}$$
$$\nu_0 = \frac{r\nu_1}{(r-1)\max\limits_{i=1,2,3}(\theta_i)}$$

2.2 Existence result

Theorem 2.2.1 *Assume that (2.3) and (2.4) are true, then for $\lambda, \mu,$ and ν large enough, system (2.2) admits a weak positive solution(u, v, w).*

Proof. choose $(\psi_1,\psi_2,\psi_3)\in X$, as following

$$\psi_1=(\tfrac{\lambda\alpha_0k_0}{m_p})^{\frac{1}{p-1}}\left(\tfrac{p-1}{p}\right)\varphi_p^{\frac{p}{p-1}},$$

$$\psi_2=(\tfrac{\mu\beta_0k_0}{m_q})^{\frac{1}{q-1}}\left(\tfrac{q-1}{q}\right)\varphi_q^{\frac{q}{q-1}},$$

$$\psi_3=(\tfrac{\nu\gamma_0k_0}{m_r})^{\frac{1}{r-1}}\left(\tfrac{r-1}{r}\right)\varphi_r^{\frac{r}{r-1}},$$

and see that it is a sub-solution of (2.2) for λ,μ and υ large enough.
Let $\phi=(\phi_1,\phi_2,\phi_3)\in X$ with $\phi_i\geq 0, 1\leq i\leq 3$. A simple calculation shows that

$$\begin{aligned}\int\limits_\Omega -\Delta_p\psi_1\phi_1dx &= \int\limits_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla\phi_1dx\\
&=\tfrac{\lambda\alpha_0k_0}{m_p}\int\limits_\Omega \varphi_p|\nabla\varphi_p|^{p-2}\nabla\varphi_p.\nabla\phi_1dx\\
&=\tfrac{\lambda\alpha_0k_0}{m_p}\left\{\int\limits_\Omega |\nabla\varphi_p|^{p-2}\nabla\varphi_p\nabla\left(\varphi_p\phi_1\right)dx-\int\limits_\Omega|\nabla\varphi_p|^p\phi_1dx\right\}\\
&=\tfrac{\lambda\alpha_0k_0}{m_p}\int\limits_\Omega\left(\lambda_1\varphi_p^p-|\nabla\varphi_p|^p\right)\phi_1dx.\end{aligned}$$

$$\int\limits_\Omega -\Delta_q\psi_2\phi_2dx=\tfrac{\mu\beta_0k_0}{m_q}\int\limits_\Omega\left(\mu_1\varphi_q^q-|\nabla\varphi_q|^q\right)\phi_2dx.$$

$$\int\limits_\Omega -\Delta_r\psi_3\phi_3dx=\tfrac{\nu\gamma_0k_0}{m_r}\int\limits_\Omega\left(\mu_1\varphi_r^r-|\nabla\varphi_r|^r\right)\phi_3dx.$$

Now, in $\overline{\Omega}_\delta$ we have

$$|\nabla\varphi_p|^p-\lambda_1\varphi_p^p\geq m_p,$$

$$|\nabla\varphi_q|^q-\mu_1\varphi^q\geq m_q,$$

$$|\nabla\varphi_r|^r-\nu_1\varphi^r\geq m_r.$$

Which imply that

$$\frac{\alpha_0 k_0}{m_p}\left(\lambda_1 \varphi_p^p - |\nabla \varphi_p|^p\right) - \alpha(x) f(\psi_1, \psi_2, \psi_3) \leq k_0 (\alpha_0 - \alpha(x)) \leq 0,$$

$$\frac{\beta_0 k_0}{m_q}\left(\mu_1 \varphi_q^q - |\nabla \varphi_q|^q\right) - \beta(x) g(\psi_1, \psi_2, \psi_3) \leq k_0 (\beta_0 - \beta(x)) \leq 0,$$

$$\frac{\gamma_0 k_0}{m_r}\left(\upsilon_1 \varphi_r^r - |\nabla \varphi_r|^r\right) - \gamma(x) h(\psi_1, \psi_2, \psi_3) \leq k_0 (\gamma_0 - \gamma(x)) \leq 0.$$

Whereas in $\Omega \backslash \overline{\Omega}_\delta$, we have $\varphi_p \geq \sigma_p$, $\varphi_q \geq \sigma_q$ and $\varphi_r \geq \sigma_r$ for σ_p, σ_q and $\sigma_r \geq 0$, and then for λ, μ and ν large enough

$$\alpha(x) f(\psi_1, \psi_2, \psi_3) \geq \frac{\alpha_0 k_0}{m_p}\lambda_1 \geq \frac{\alpha_0 k_0}{m_p}\left(\lambda_1 \varphi_p^p - |\nabla \varphi_p|^p\right)$$

$$\beta(x) g(\psi_1, \psi_2, \psi_3) \geq \frac{\beta_0 k_0}{m_q}\mu_1 \geq \frac{\beta_0 k_0}{m_q}\left(\mu_1 \varphi_q^q - |\nabla \varphi_q|^q\right)$$

$$\gamma(x) h(\psi_1, \psi_2, \psi_3) \geq \frac{\gamma_0 k_0}{m_r}\nu_1 \geq \frac{\gamma_0 k_0}{m_r}\left(\nu_1 \varphi_r^r - |\nabla \varphi_r|^r\right)$$

And consequently

$$\int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 . \nabla \phi_1 dx \leq \lambda \int_\Omega \alpha(x) f(\psi_1, \psi_2, \psi_3) \phi_1 dx,$$

$$\int_\Omega |\nabla \psi_2|^{q-2} \nabla \psi_2 . \nabla \phi_2 dx \leq \mu \int_\Omega \beta(x) g(\psi_1, \psi_2, \psi_3) \phi_2 dx,$$

$$\int_\Omega |\nabla \psi_3|^{r-2} \nabla \psi_3 . \nabla \phi_3 dx \leq \nu \int_\Omega \gamma(x) h(\psi_1, \psi_2, \psi_3) \phi_3 dx,$$

i.e. (ψ_1, ψ_2, ψ_3) is a sub-solution of (2.2).
Let e_p, e_q and e_r the solutions of the following problems:

$$\begin{cases} -\Delta_p e_p = 1 \text{ in } \Omega, \\ e_p = 0 \text{ on } \partial\Omega. \end{cases}, \quad \begin{cases} -\Delta_q e_q = 1 \text{ in } \Omega, \\ e_q = 0 \text{ on } \partial\Omega. \end{cases} \text{ and } \begin{cases} -\Delta_r e_r = 1 \text{ in } \Omega, \\ e_r = 0 \text{ on } \partial\Omega. \end{cases}$$

Let

$$z_1 = \frac{C}{\|e_p\|_\infty}\lambda^{\frac{1}{p-1}}e_p,$$

$$z_2 = (\mu)^{\frac{1}{q-1}}\left(g\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\right)^{\frac{1}{q-1}}e_q,$$

$$z_3 = (\nu)^{\frac{1}{r-1}}\left(h\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\right)^{\frac{1}{r-1}}e_r.$$

where C is a large enough positive number. We are going to check that (z_1, z_2, z_3) is a super-solution of (2.2) for λ, μ and υ large enough.
by (2.3) and (2.4), we can choose C large enough so that

$$\begin{aligned}\left(C\lambda^{\frac{1}{p-1}}\right)^{q-1} &\geq \|e_q\|_\infty^{q-1}\mu g\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\\ &\geq \mu g\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)e_q^{q-1} = z_2^{q-1}\end{aligned}$$

which implies

$$C\lambda^{\frac{1}{p-1}} \geq z_2$$

and consequently

$$\begin{aligned}\left(C\lambda^{\frac{1}{p-1}}\right)^{r-1} &\geq \|e_r\|_\infty^{r-1}\nu h\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\\ &\geq \nu h\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)e_r^{r-1}\end{aligned}$$

from which we deduce that

$$C\lambda^{\frac{1}{p-1}} \geq z_3$$

which implies then

$$\begin{aligned}\left(C\lambda^{\frac{1}{p-1}}\right)^{p-1} &\geq \|e_p\|_\infty^{p-1}\lambda f\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\\ &\geq \|e_p\|_\infty^{p-1}\lambda f\left(\frac{C}{\|e_p\|_\infty}\lambda^{\frac{1}{p-1}}\|e_p\|_\infty, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\\ &\geq \|e_p\|_\infty^{p-1}\lambda f\left(\frac{C}{\|e_p\|_\infty}\lambda^{\frac{1}{p-1}}e_p, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}\right)\\ &= \|e_p\|_\infty^{p-1}\lambda f(z_1, z_2, z_3)\end{aligned}$$

then we have

$$\begin{aligned}
\int_{\Omega} |\nabla z_1|^{p-2} \nabla z_1 . \nabla \phi_1 dx &= \lambda \left(\frac{C}{\|e_p\|_\infty} \right)^{p-1} \int_{\Omega} |\nabla e_p|^{p-2} \nabla e_p . \nabla \phi_1 dx \\
&= \lambda \left(\frac{C}{\|e_p\|_\infty} \right)^{p-1} \int_{\Omega} \phi_1 dx \\
&\geq \lambda \int_{\Omega} \alpha_1 f(z_1, z_2, z_3) \phi_1 dx \\
&\geq \lambda \int_{\Omega} \alpha(x) f(z_1, z_2, z_3) \phi_1 dx
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_{\Omega} |\nabla z_2|^{q-2} \nabla z_2 . \nabla \phi_2 dx &= \mu \beta_1 g\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}} \right) \int_{\Omega} |\nabla e_q|^{q-2} \nabla e_q . \nabla \phi_2 dx \\
&\geq \mu \int_{\Omega} \beta_1 g\left(\frac{C}{\|e_p\|_\infty} \lambda^{\frac{1}{p-1}} \|e_p\|_\infty, C\lambda^{\frac{1}{p-1}} \right) \phi_2 dx \\
&\geq \mu \int_{\Omega} \beta_1 g(z_1, z_2, z_3) \phi_2 dx \geq \mu \int_{\Omega} \beta(x) g(z_1, z_2, z_3) \phi_2 dx
\end{aligned}$$

and with the same way, we get

$$\begin{aligned}
\int_{\Omega} |\nabla z_3|^{r-2} \nabla z_3 . \nabla \phi_3 dx &= \nu \gamma_1 h\left(C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}} \right) \int_{\Omega} |\nabla e_r|^{r-2} \nabla e_r . \nabla \phi_3 dx \\
&\geq \nu \int_{\Omega} \gamma_1 h\left(\frac{C}{\|e_p\|_\infty} \lambda^{\frac{1}{p-1}} \|e_p\|_\infty, C\lambda^{\frac{1}{p-1}}, C\lambda^{\frac{1}{p-1}} \right) \phi_3 dx \\
&\geq \nu \int_{\Omega} \gamma(x) h(z_1, z_2, z_3) \phi_3 dx
\end{aligned}$$

i.e (z_1, z_2, z_3) is a super-solution of (2.2) with $z_i \geq \psi_i$ for C large enough, $i = 1, 2, 3$. Hence the existence of a weak solution (u, v, w) of (2.2) with $\psi_1 \leq u \leq z_1, \psi_2 \leq v \leq z_2$ and $\psi_3 \leq w \leq z_3$.

■

2.3 Non-existence Result

Theorem 2.3.1 *Assume that f, g and h verify (2.6) and*

$$f(0,0,0) = g(0,0,0) = h(0,0,0) = 0,$$

then for

$$0 < \lambda < \lambda_0, 0 < \mu < \mu_0 \text{ and } 0 < \nu < \nu_0. \tag{2.8}$$

system (2.2) admits only the trivial solution.
λ_1, μ_1 are ν_1 are respectively the first eigenvalues of the operators $-\Delta_p$,$-\Delta_q$ and $-\Delta_r$.

Proof. Let's multiply the first equation by u, and integrating on Ω, using Young's inequality, we get

$$\|\nabla u\|_p^p = \int_\Omega \lambda\alpha(x) f(u,v,w)\,udx \leq \lambda\alpha_1 \int_\Omega \left(\xi_1 u^{p-1} + \eta_1 v^{q\left(\frac{p-1}{p}\right)} + \zeta_1 w^{r\left(\frac{p-1}{p}\right)}\right) udx$$

$$\leq \lambda\alpha_1 \int_\Omega \left(\xi_1 u^p + \tfrac{\eta_1}{p}\left(u^p + (p-1)v^q\right) + \tfrac{\zeta_1}{p}\left(u^p + (p-1)w^r\right)\right) dx$$

$$\leq \lambda\alpha_1 \int_\Omega \left(\xi_1 + \frac{\eta_1+\zeta_1}{p} u^p + \left(\tfrac{p-1}{p}\right)\eta_1 v^q + \left(\tfrac{p-1}{p}\right)\zeta_1 w^r\right) dx$$

$$= \frac{\lambda\alpha_1}{p}\left(p\xi_1 + \eta_1 + \zeta_1\right)\|u\|_p^p + \lambda\alpha_1\left(\tfrac{p-1}{p}\right)\eta_1\|v\|_q^q + \lambda\alpha_1\left(\tfrac{p-1}{p}\right)\zeta_1\|w\|_r^r$$

then we have

$$\|\nabla u\|_p^p \leq \frac{\lambda\alpha_1}{p}\left(p\xi_1+\eta_1+\zeta_1\right)\|u\|_p^p + \frac{\lambda\alpha_1(p-1)}{p}\eta_1\|v\|_q^q + \frac{\lambda\alpha_1(p-1)}{p}\zeta_1\|w\|_r^r$$

$$\|\nabla v\|_q^q \leq \tfrac{\mu\beta_1}{q}\left(\xi_2 + q\eta_2 + \zeta_2\right)\|v\|_q^q + \tfrac{\mu\beta_1(q-1)}{q}\xi_2\|u\|_p^p + \frac{\mu\beta_1(q-1)}{q}\zeta_2\|w\|_r^r \tag{2.9}$$

$$\|\nabla w\|_r^r \leq \tfrac{\nu\gamma_1}{r}\left(\xi_3+\eta_3+r\zeta_3\right)\|w\|_r^r + \frac{\nu\gamma_1(r-1)}{r}\xi_3\|u\|_p^p + \frac{\nu\gamma_1(r-1)}{r}\eta_3\|v\|_q^q$$

On the other hand

$$\lambda_1 = \inf\frac{\|\nabla u\|_p^p}{\|u\|_p^p}\ ,\mu_1 = \inf\frac{\|\nabla v\|_q^q}{\|v\|_q^q} \text{ et } \nu_1 = \inf\frac{\|\nabla w\|_r^r}{\|w\|_r^r} \tag{2.10}$$

combine (2.9) and (2.10) we get

$$(\lambda_1 - \lambda_0) \|u\|_p^p + (\mu_1 - \mu_0) \|v\|_q^q + (\nu_1 - \nu_0) \|w\|_r^r \leq 0.$$

which contradicts (2.8). So (2.2) does not admit weak solutions other than the trivial solution $(u = v = w = 0)$. ■

2.4 Applications

Theorem 2.4.1 *For the system :*

$$\begin{cases} -\Delta_p u = \lambda u^{m_1} v^{n_1} w^{l_1} \ \ in\ \Omega, \\ -\Delta_q v = \mu\ u^{m_2} v^{n_2} w^{l_2} in\ \Omega, \\ -\Delta_r w = \upsilon u^{m_3} v^{n_3} w^{l_3} \ \ in\ \Omega, \\ u = v = w = 0 \ \ on\ \partial\Omega, \end{cases} \tag{2.11}$$

1) If

$$\begin{aligned} m_1 + n_1 + l_1 &< p - 1, \\ m_2 + n_2 + l_2 &< q - 1, \\ m_3 + n_3 + l_3 &< r - 1. \end{aligned} \tag{2.12}$$

System (2.11) admits a large positive weak solution.
2) If

$$\begin{aligned} qrm_1 + prn_1 + pql_1 &= qr\,(p - 1)\,, \\ qrm_2 + prn_2 + pql_2 &= pr\,(q - 1)\,, \\ qrm_3 + prn_3 + pql_3 &= pq\,(r - 1)\,. \end{aligned} \tag{2.13}$$

and

$$0 < \lambda < \lambda_1, 0 < \mu < \mu_1 \text{ and } 0 < \nu < \nu_1. \tag{2.14}$$

system (2.11) admits only the trivial solution

Proof. 1) (2.5) implies that (2.12) is verified. So by theorem (2.2.1) the system (2.11) admits a weak positive solution.

2) The first equation in (2.13) implies that

$$\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3} = \frac{1}{\left(\frac{p-1}{m_1}\right)} + \frac{1}{\frac{q}{p}\left(\frac{p-1}{n_1}\right)} + \frac{1}{\frac{r}{p}\left(\frac{p-1}{l_1}\right)} = 1 \tag{2.15}$$

Using the generalized Young inequality, we get

$$\begin{aligned} f_1(u,v,w) = u^{m_1} v^{n_1} w^{l_1} &\le \frac{1}{\theta_1} u^{m_1\theta_1} + \frac{1}{\theta_2} v^{n_1\theta_2} + \frac{1}{\theta_3} w^{l_1\theta_3} \\ = \frac{1}{\theta_1} u^{p-1} + \frac{1}{\theta_2} v^{q\left(\frac{p-1}{p}\right)} + \frac{1}{\theta_3} w^{r\left(\frac{p-1}{p}\right)} \end{aligned} \tag{2.16}$$

The assumption (2.6) is satisfied.

Let

$$\lambda_0 = \frac{1}{p}\left(\lambda\left(m_1+1\right) + \mu m_2 + \nu m_3\right) < \lambda_1,$$

$$\mu_0 = \frac{1}{q}\left(\lambda n_1 + \mu\left(n_2+1\right) + \nu n_3\right) < \mu_1,$$

$$\nu_0 = \frac{1}{r}\left(\lambda l_1 + \mu l_2 + \nu\left(l_3+1\right)\right) < \nu_1.$$

Then

$$p\left(\lambda - \lambda_1\right) + q\left(\mu - \mu_1\right) + r\left(\nu - \nu_1\right) < 0. \tag{2.17}$$

Therefore, the system (2.11) does not admit non trivial positive weak solutions. ■

Theorem 2.4.2 *For λ large, the problem*

$$\begin{cases} -\Delta_p^3 u = \lambda^3 \gamma(x)\, H\left(u, -\Delta_p u, \Delta_p^2 u\right) \ \ in\ \Omega, \\ \\ u = \Delta_p u = \Delta_p^2 u = 0 \ \ on\ \partial\Omega, \end{cases} \tag{2.18}$$

admits a positive weak solution.

Here Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, λ is a positive real parameter, $\gamma \in L^{\infty}(\Omega)$ and

$$\begin{aligned} & H : ([0,\infty[)^3 \rightarrow \mathbb{R} \text{ is of class } C^1, \\ & H(t_1,t_2,t_3) \text{ is increasing compared to à } t_1, t_3, \\ & H(t_1,t_2,t_3) \text{ is decreasing compared to } t_2, \\ & \lim_{t\rightarrow+\infty} \frac{H(t,-\lambda t,\lambda^2 t)}{t^{p-1}} = 0, p > 2 \\ & \exists k_0 > 0 : H(t_1,t_2,t_3) \geq -k_0, \forall (t_1,t_2,t_3) \in ([0,+\infty[)^3 \end{aligned} \tag{2.19}$$

Proof. The problem (2.18) can be written in the following form

$$\begin{cases} -\Delta_p u = \lambda v \text{ in } \Omega, \\ -\Delta_p v = \lambda w \text{ in } \Omega, \\ -\Delta_p w = \lambda\gamma(x) H(u,-\lambda v,\lambda^2 w) \text{ in } \Omega, \\ u = v = w = 0 \text{ on } \partial\Omega, \end{cases}$$

in this case, the assumptions of theorem (2.2.1) are satisfied. ■

Chapter 3

Existence of positive solutions and its asymptotic behavior of $(p(x), q(x))$-Laplacian parabolic systems.

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1) Preliminaire results problems and assumption
2) The Semi-Discrete problem
3) Existence results of $(p(x), q(x))$-Laplacian parabolic systems
4) Asymptotic behavior of the $(p(x), q(x))$-Laplacian parabolic systems.

In this chapter deals with the study of existence of positively solution and its asymptotic behavior for parabolic system of $(p(x), q(x))$-Laplacian system of partial differential equations using a method sub and super solution according to some given boundary conditions. We will study an extension of Boulaaras's [13],[15, 45], that is which studie the stationary case, we will study idea is new for evolutionary case of this kind of problem for $(p(x), q(x))$-Laplacian parabolic system.

We consider the following evolutionary problem: find $u \in L^2(0,T,H_0^1(\Omega))$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_{p(x)} u = \lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)\right] \quad \text{in} \quad Q_T = (0,T) \times \Omega, \\ \\ \dfrac{\partial v}{\partial t} - \Delta_{q(x)} v = \lambda^{q(x)} \left[\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)\right] \quad \text{in} \quad Q_T = (0,T) \times \Omega, \\ \\ u = v = 0 \quad \text{on} \quad \partial Q_T = (0,T) \times \partial\Omega, \\ \\ u(x,0) = \varphi(x), \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and the functions $p(x), q(x)$ belong to $C^1(\overline{\Omega})$ and satisfying the following conditions:

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{\Omega} p(x) < \infty, 1 < q^- := \inf_{x\in\Omega} q(x) \leq q^+ := \sup_{x\in\Omega} q(x) < \infty \tag{3.2}$$

and satisfy some natural growth condition at $u = \infty$.
$\Delta_{p(x)}$ is given by $\Delta_{p(x)} u = div(|\nabla u|^{p(x)-2} \nabla u)$ is called $p(x)$-Laplacian, the parameters $\lambda, \lambda_1, \lambda_2, \mu_1$ and μ_2 are positive with a, b, c, d are regular functions. In addition we did not consider any sign condition on $f(0), g(0), h(0), \tau(0)$.
The linear and nonlinear stationary equations with operators of quasilinear homogeneous type as p-Laplace operator can be carried out according to the standard Sobolev spaces theory of $W^{m,p}$, and thus we can find the weak solutions. The last spaces consist of functions having weak derivatives which verify some conditions of integrability. Thus, we can have the nonhomogeneous case of $p(.)$-Laplace operators in this last condition. We will use Sobolev spaces of the exponential variable in our standard framework, so that $L^{p(\cdot)}(\Omega)$ will be used instead of Lebesgue spaces $L^p(\Omega)$.
We denote new Sobolev space by $W^{m,p}(\Omega)$, if we replace $L^p(\Omega)$ by $L^{p(\cdot)}(\Omega)$, the Sobolev spaces becomes $W^{m,p(\cdot)}(\Omega)$. Several Sobolev spaces properties have been extended to spaces

of Orlicz-Sobolev, particularly by O'Neill in the reference ([61]). The spaces $W^{m,p(.)}(\Omega)$ and $L^{p(.)}(\Omega)$ have been carefully studied by many researchers team (see the references ([13] and [30, 39, 40]).
Here, in our study we consider the boundedness condition in domain Ω, because many results under p-Laplacian theory are not usually verified for the $p(x)$-Laplacian theory; for that in ([14]) the quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx} \tag{3.3}$$

becomes 0 generally. Then $\lambda_{p(x)}$ can be positive only for some given conditions. In fact, the first eigenvalue of $p(x)$-Laplacian and its associated eigenfunction cannot exist, the existence of the positive first eigenvalue λ_p and getting its eigenfunction are very important in the p-Laplacian problem study. Therefore, the study of existence of solutions of our problems have more meaning. Many studies of the experimental side have been studied on various materials that rely on this advanced theory, as they are important in electrical fluids, which states that viscosity relates to the electric field in a certain liquid.
Recently, in ([13, 14, 44]), we have proved the existence of positive solutions of many classes of $(p(x), q(x))$-Laplacian stationary problems by using the sub-super solution concept. The current results are an extension of our previous stationary study to the parabolic case, where we follow-up the same procedures mathematical proofs similar to that in ([13, 16]) by using difference time scheme taking into consideration the stability analysis of the used scheme and the same conditions which have given in references mentioned earlier. Our result is an extension for our previous study in ([13, 16, 45]) which studied the stationary case, this idea is new for evolutionary case of this kind of problem.
The outline of chapter consists as follow: In first section we give some definitions, basic theorems and necessarily propositions in the functional analysis which will be used in our study. Then in Section 3.4, we prove our main result.

3.1 Preliminaries Results and Assumptions

In order to discuss problem (3.1), we need some theories on $W_0^{1,p(x)}(\Omega)$ which we call variable exponent Sobolev space. Firstly we state some basic properties of spaces $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [74]).

Let us define

$$L^{p(x)}(\Omega)=\left\{u: u \text{ is a measurable real-valued function such that } \int_{\Omega}|u(x)|^{p(x)}\,dx<\infty\right\}.$$

We introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u(x)|_{L^{p(x)}}=\inf\left\{\lambda>0:\int_{\Omega}\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\leq 1\right\}$$

and

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega);|\nabla u|\in L^{p(x)}(\Omega)\right\},$$

with the norm

$$\|u\|=|u|_{L^{p(x)}}+|\nabla u|_{L^{p(x)}},\forall u\in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$.

We introduce in this applying for problem (2), we will assume that:

(H_1) $p,q\in C^1(\overline{\Omega})$ and $1<p_-<p_+,1<q_-<q_+$;

(H_2) f,g,h and $\tau:[0,+\infty[\to\mathbb{R}$ are C^1,monotone functions, such that

$$\lim_{u\to+\infty}f(u_k)=+\infty\ \lim_{u\to+\infty}g(u_k)=+\infty,\ \lim_{u\to+\infty}h(u_k)=+\infty,\ \lim_{u\to+\infty}\tau(u_k)=+\infty,$$

(H_3) $\lim\limits_{u\to+\infty}\dfrac{f(M(g(u_k))^{\frac{1}{q^--1}})}{u_k^{p^--1}}=0$, for all $M>0$;

(H_4) $\lim\limits_{u\to+\infty}\dfrac{h(u_k)}{u_k^{p^--1}}=0$, and $\lim\limits_{u\to+\infty}\dfrac{\tau(u_k)}{u_k^{p^--1}}=0$;

(H_5) $a,b,c,d:\overline{\Omega}\to(0,+\infty)$ are contionous functions, such that

$$a_1=\min_{x\in\overline{\Omega}}a(x),b_1=\min_{x\in\overline{\Omega}}b(x),c_1=\min_{x\in\overline{\Omega}}c(x),d_1=\min_{x\in\overline{\Omega}}d(x),$$

$$a_2=\max_{x\in\overline{\Omega}}a(x),b_2=\max_{x\in\overline{\Omega}}b(x),c_2=\max_{x\in\overline{\Omega}}c(x),d_2=\max_{x\in\overline{\Omega}}d(x).$$

3.2 The Semi-Discrete problem

We discrete the problem (3.1) by difference time scheme, we obtain the following problems

$$\begin{cases} u_k - \tau' \Delta_{p(x)} u_k = \tau' \lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u_k)\right] + u_{k-1} \text{ in } \Omega, \\ v_k - \tau' \Delta_{q(x)} v = \tau' \lambda^{q(x)} \left[\lambda_2 b(x) g(u_k) + \mu_2 d(x) \tau(v)\right] + v_{k-1} \text{ in } \Omega, \\ u_k = v = 0 \text{ on } \partial\Omega, \\ u_0 = \varphi_0, \end{cases} \tag{3.4}$$

where $N\tau' = T$, $0 < \tau' < 1$, and for $1 \leq k \leq N$.
We define

$$\langle L(u_k), v \rangle = \int_\Omega |\nabla u_k|^{p(x)-2} \nabla u_k \nabla v dx, \forall u_k, v \in W_0^{1,p(x)}(\Omega).$$

According to([15] in Theorem 3.1), the bounded operator $L : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)^*$ is a continuous and strictly monotone, and it is a homeomorphism.
We considere mapping $A : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)^*$ as

$$\langle A(u_k), \varphi \rangle = \int_\Omega \left(|\nabla u_k|^{p(x)-2} \nabla u_k \nabla \varphi + h(x, u_k)\varphi\right) dx, \text{for all } u_k, v \in W_0^{1,p(x)}(\Omega),$$

where $h(x, u_k)$ is continuous on $\overline{\Omega} \times \mathbb{R}$, and $h(x, .)$ is increasing function.It is easy to verify that A is a continuous bounded mapping. By the proof ([73]).

3.3 Existence of positive solutions of $(p(x), q(x))$-Laplacian parabolic systems

An weak solution to discretized problems (P_k) is a sequence $(u_k, v)_{0 \leq k \leq N}$ such that $u_0 = \varphi_0$ and (u_k, v) is defined by

$$\begin{cases} u_k - \tau' \Delta_{p(x)} u_k = \tau' \lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u_k)\right] + u_{k-1} \quad \text{in } \Omega, \\ v_k - \tau' \Delta_{q(x)} v = \tau' \lambda^{q(x)} \left[\lambda_2 b(x) g(u_k) + \mu_2 d(x) \tau(v)\right] + v_{k-1} \quad \text{in } \Omega, \\ u_k = v = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

such that

$$\begin{cases} -\Delta_{p(x)} u_k = \lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u_k)\right] - \frac{u_k - u_{k-1}}{\tau'} \quad \text{in } \Omega, \\ \\ -\Delta_{q(x)} v = \lambda^{q(x)} \left[\lambda_2 b(x) g(u_k) + \mu_2 d(x) \tau(v)\right] - \frac{v_k - v_{k-1}}{\tau'} \quad \text{in } \Omega, \\ \\ u_k = v = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.5}$$

We have the following:

(1) If $(u_k, v) \in \left(W_0^{1.p(x)}(\Omega) \times W_0^{1.q(x)}(\Omega)\right)$, (u_k, v) is called a weak solution of (3.5) if it satisfies

$$\int_\Omega |\nabla u_k|^{p(x)-2} \nabla u_k . \nabla \varphi dx = \int_\Omega \left[\lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u_k)\right] - \frac{u_k - u_{k-1}}{\tau'}\right] \varphi dx,$$

$$\int_\Omega |\nabla v|^{q(x)-2} \nabla v . \nabla \psi dx = \int_\Omega \left[\lambda^{q(x)} \left[\lambda_2 b(x) g(u_k) + \mu_2 d(x) \tau(v)\right] - \frac{v_k - v_{k-1}}{\tau'}\right] \psi dx. \tag{3.6}$$

for all

$$(\varphi, \psi) \in \left(W_0^{1.p(.)}(\Omega) \times W_0^{1.q(.)}(\Omega)\right)$$

with $(\varphi, \psi) \geqslant 0$.

(2) We say called a sub solution (respectively a super solution) of (3.1) if

$$\int_\Omega |\nabla u_k|^{p(x)-2} \nabla u_k . \nabla \varphi dx \leq (\text{respectively } \geqslant) \int_\Omega \left[\lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u_k)\right] - \frac{u_k - u_{k-1}}{\tau'}\right] \varphi dx,$$

$$\int_\Omega |\nabla v|^{q(x)-2} \nabla v . \nabla \psi dx \leq (\text{respectively } \geqslant) \int_\Omega \left[\lambda^{q(x)} \left[\lambda_2 b(x) g(u_k) + \mu_2 d(x) \tau(v)\right] - \frac{v_k - v_{k-1}}{\tau'}\right] \psi dx.$$

Lemma 3.3.1 *(Comparison principle) Let $u_k, v \in W_0^{1,p(x)}(\Omega)$ verify $Au_k - Av \geqslant 0$ in $\left(W_0^{1,p(x)}(\Omega)\right)^*$, and $\varphi(x) = \min\{u_k(x) - v(x), 0\}$. If $\varphi(x) \in W_0^{1,p(x)}(\Omega)$ (i.e., $u_k \geqslant v$ on $\partial\Omega$),then $u_k \geqslant v$ a.e in Ω .*

Here, we will use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$ to denote the distance of Ω.

Denote $d(x) = d(x, \partial\Omega)$ and $\partial\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}$.

Since $\partial\Omega$ is C^2 regularly, there exists a constant $\delta \in (0,1)$ such that $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$ and $|\nabla d(x)| = 1$.

Denote also

$$v_1(x) = \begin{cases} \gamma d(x), \quad d(x) < \delta, \\ \\ \gamma\delta + \int_\delta^{d(x)} \gamma(\frac{2\delta - t}{\delta})^{\frac{2}{p^- - 1}} (\lambda_1 a_1 + \mu_1 c_1)^{\frac{2}{p^- - 1}} dt, \quad \delta \le d(x) \le 2\delta, \\ \\ \gamma\delta + \int_\delta^{2\delta} \gamma(\frac{2\delta - t}{\delta})^{\frac{2}{p^- - 1}} (\lambda_1 b_1 + \mu_1 d_1)^{\frac{2}{p^- - 1}} dt, \quad 2\delta \le d(x) \end{cases}$$

and

$$v_2(x) = \begin{cases} \gamma d(x), \quad d(x) < \delta, \\ \\ \gamma\delta + \int_\delta^{d(x)} \gamma(\frac{2\delta - t}{\delta})^{\frac{2}{p^- - 1}} (\lambda_2 a_2 + \mu_2 c_2)^{\frac{2}{q^- - 1}} dt, \quad \delta \le d(x) \le 2\delta, \\ \\ \gamma\delta + \int_\delta^{2\delta} \gamma(\frac{2\delta - t}{\delta})^{\frac{2}{p^- - 1}} (\lambda_2 b_2 + \mu_2 d_2)^{\frac{2}{q^- - 1}} dt, \quad 2\delta \le d(x). \end{cases}$$

Obviously,

$$0 \le v_1(x), v_2(x) \in C^1(\overline{\Omega}).$$

Considering

$$\begin{cases} -\Delta_{p(x)} w(x) = \eta \quad \text{in } \Omega \\ \\ w = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Lemma 3.3.2 *([32]), If positive parameter η is large enough and w is the unique solution of (3.7), then we have*
(i) For any $\theta \in (0, 1)$ there exists a positive constant C_1, such that

$$C_1 \eta^{\frac{1}{p^+ - 1 + \theta}} \le \max_{x \in \overline{\Omega}} w(x).$$

(ii) There exists a positive constant C_2, such that

$$\max_{x \in \overline{\Omega}} w(x) \le C_2 \eta^{\frac{1}{p^- - 1}}$$

3.4 Existence result

In the following, once we have no misunderstanding, we always use C_i to denote the positive constants.

Theorem 3.4.1 *Assume that the conditions (H_1)–(H_5) are statisfied. Then, problem (3.1) has a positive solution when λ is large enough.*

Proof. We establish Theorem 3.4.1 by constructing a positive subsolution (ϕ_{k_1}, ϕ_{k_2}) and supersolution (z_{k_1}, z_{k_2}) of (3.1) such that $\phi_{k_1} \leq z_{k_1}$ and $\phi_{k_2} \leq z_{k_2}$, that is (ϕ_{k_1}, ϕ_{k_2}) and (z_{k_1}, z_{k_2}) satisfies

$$\begin{aligned}
\int_\Omega |\nabla \phi_{k_1}|^{p(x)-2} \nabla \phi_{k_1}.\nabla \varphi dx &\leq \int_\Omega \left[\lambda^{p(x)} \left[\lambda_1 a(x) f(\phi_{k_2}) + \mu_1 c(x) h(\phi_{k_1})\right] - \frac{\phi_{k_1} - \phi_{k_1-1}}{\tau'}\right] \varphi dx, \\
\int_\Omega |\nabla \phi_{k_2}|^{q(x)-2} \nabla \phi_{k_2}.\nabla \psi dx &\leq \int_\Omega \left[\lambda^{q(x)} \left[\lambda_2 b(x) g(\phi_{k_1}) + \mu_2 d(x) \tau(\phi_{k_2})\right] - \frac{\phi_{k_1} - \phi_{k_1-1}}{\tau'}\right] \psi dx,
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |\nabla z_{k_1}|^{p(x)-2} \nabla z_{k_1}.\nabla \varphi dx &\geq \int_\Omega \left[\lambda^{p(x)} \left[\lambda_1 a(x) f(z_{k_2}) + \mu_1 c(x) h(z_{k_1})\right] - \frac{z_{k_1} - z_{k_1-1}}{\tau'}\right] \varphi dx, \\
\int_\Omega |\nabla z_{k_2}|^{q(x)-2} \nabla z_{k_2}.\nabla \psi dx &\geq \int_\Omega \left[\lambda^{q(x)} \left[\lambda_2 b(x) g(z_{k_1}) + \mu_2 d(x) \tau(z_{k_2})\right] - \frac{z_{k_1} - z_{k_1-1}}{\tau'}\right] \psi dx,
\end{aligned}$$

for all $(\varphi, \psi) \in \left(W_0^{1.p(x)}(\Omega) \times W_0^{1.q(x)}(\Omega)\right)$ with $(\varphi, \psi) \geqslant 0$. According to the sub-super solution method for $(p(x), q(x))$-Laplacian systems see ([32, 45]), the problem (3.1) has a positive solution.

Step 1. We will construct a subsolution of (3.1). Let $\sigma \in (0, \delta)$ is small enough. Denote

$$\phi_{k_1}(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ e^{kd(x)} - 1 + \int_\delta^{d(x)} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} dt, \quad \sigma \leq d(x) < 2\delta, \\ e^{kd(x)} - 1 + \int_\sigma^{2\delta} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} dt, \quad 2\delta \leq d(x) \end{cases}$$

and

$$\phi_{k_2}(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ e^{kd(x)} - 1 + \int_\delta^{d(x)} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{q^- - 1}} dt, \quad \sigma \leq d(x) < 2\delta, \\ e^{kd(x)} - 1 + \int_\sigma^{2\delta} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{q^- - 1}} dt, \quad 2\delta \leq d(x). \end{cases}$$

It easy to see that $\phi_{k_1}, \phi_{k_2} \in C^1(\overline{\Omega})$.

Denote

$$\alpha = \min\left\{\frac{\inf p(x)-1}{4(\sup|\nabla p(x)+1|)}, \frac{\inf q(x)-1}{4(\sup|\nabla q(x)+1|)}, 1\right\}$$

and

$$\xi = \min\left\{\lambda_1 a_1 f(0) + \mu_1 c_1 h(0), \lambda_2 b_1 g(0) + \mu_2 d_1 \sigma(0), -1\right\}.$$

By some simple computations we obtain

$$-\Delta_{p(x)}\phi_{k_1} = \begin{cases} -k(e^{kd(x)})^{p(x)-1}\left[(p(x)-1) + (d(x)+\frac{\ln k}{k})\nabla p \nabla d + \frac{\Delta d}{k}\right],\ d(x)<\sigma \\ \left\{\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^- -1} - \left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^- -1}}\nabla p \nabla d + \Delta d\right]\right\} \\ \times\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^- -1}-1},\ \sigma \le d(x) < 2\delta, \\ 0,\ 2\delta \le d(x) \end{cases}$$

and

$$-\Delta_{p(x)}\phi_{k_2} = \begin{cases} -k(e^{kd(x)})^{q(x)-1}\left[(q(x)-1) + (d(x)+\frac{\ln k}{k})\nabla q \nabla d + \frac{\Delta d}{k}\right],\ d(x)<\sigma, \\ \left\{\frac{1}{2\delta-\sigma}\frac{2(q(x)-1)}{q^- -1} - \left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{q^- -1}}\nabla q \nabla d + \Delta d\right]\right\} \\ \times\left(Ke^{k\sigma}\right)^{q(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\dfrac{2(q(x)-1)}{q^- -1}-1},\ \sigma \le d(x) < 2\delta, \\ 0,\ 2\delta \le d(x). \end{cases}$$

From (H_3) there exists a positive constant $M>1$ such that

$$\begin{aligned} f(M-1) &\geqslant 1,\ g(M-1) \geqslant 1, \\ h(M-1) &\geqslant 1,\ \sigma(M-1) \geqslant 1. \end{aligned}$$

Let $\sigma = \frac{1}{k}\ln M$, then

$$\sigma k = \ln M. \tag{3.8}$$

If k is sufficiently large, from (3.8), we have

$$-\Delta_{p(x)}\phi_{k_1} \le -k^{p(x)}\alpha,\ d(x)<\sigma. \tag{3.9}$$

Let $\lambda\xi = k\alpha$, then

$$k^{p(x)}\alpha \geqslant -\lambda^{p(x)}\xi.$$

From (3.9), we have

$$\begin{cases} -\Delta_{p(x)}\phi_{k_1} \le \lambda^{p(x)}\xi \le \lambda^{p(x)}(\lambda_1 a_1 f(0) + \mu_1 c_1 h(0)) \\ \\ \le \lambda^{p(x)}(\lambda_1 a(x) f(\phi_{k_2}) + \mu_1 c(x) h(\phi_{k_1})), \quad d(x) < \sigma. \end{cases} \tag{3.10}$$

Since $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, there exists a positive constant C_3, such that

$$\begin{aligned} -\Delta_{p(x)}\phi_{k_1} &\le \left(Ke^{k\sigma}\right)^{p(x)-1} \left(\frac{2\delta - d}{2\delta-\sigma}\right)^{\dfrac{2(p(x)-1)}{p^- - 1} - 1} (\lambda_1 a_1 + \mu_1 c_1) \\ &\quad \times \left| \left\{ \frac{1}{2\delta - \sigma} \frac{2(p(x)-1)}{p^- - 1} - \left(\frac{2\delta-d}{2\delta-\sigma}\right) \right.\right. \\ &\quad \left.\left. \times \left[\left(\ln ke^{k\sigma}\right) \left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^- - 1}} \nabla p \nabla d + \Delta d \right] \right\} \right| \\ &\le C_3 \left(Ke^{k\sigma}\right)^{p(x)-1} (\lambda_1 a_1 + \mu_1 c_1) \ln k, \quad \sigma \le d(x) < 2\delta. \end{aligned}$$

If k is sufficiently large, let $\lambda\xi = k\alpha$, then we have

$$\begin{aligned} C_3 \left(Ke^{k\sigma}\right)^{p(x)-1} (\lambda_1 a_1 + \mu_1 c_1) \ln k &= C_3 (kM)^{p(x)-1} (\lambda_1 a_1 + \mu_1 c_1) \ln k \\ &\le \lambda^{p(x)}(\lambda_1 a_1 + \mu_1 c_1), \end{aligned}$$

then

$$-\Delta_{p(x)}\phi_{k_1} \le \lambda^{p(x)}(\lambda_1 a_1 + \mu_1 c_1), \quad \sigma \le d(x) < 2\delta \tag{3.11}$$

Since $\phi_{k_1}(x)$, $\phi_{k_2}(x)$ and f, h are monotone, when λ is large enough, we have

$$-\Delta_{p(x)}\phi_{k_1} \le \lambda^{p(x)}(\lambda_1 a(x) f(\phi_{k_2}) + \mu_1 c(x) h(\phi_{k_1})), \ \sigma \le d(x) < 2\delta$$

and

$$\begin{aligned} -\Delta_{p(x)}\phi_{k_1} = 0 \le \lambda^{p(x)}(\lambda_1 a_1 + \mu_1 c_1) \le \lambda^{p(x)}(\lambda_1 a(x) f(\phi_{k_2}) \\ + \mu_1 c(x) h(\phi_{k_1})), \ 2\delta \le d(x). \end{aligned} \tag{3.12}$$

Combining (3.10), (3.12) and (3.13), we can deduce that

$$-\Delta_{p(x)}\phi_{k_1} \leq \lambda^{p(x)}(\lambda_1 a(x) f(\phi_{k_2}) + \mu_1 c(x) h(\phi_{k_1})), a.e. \text{ on } \Omega. \tag{3.13}$$

Similarly

$$-\Delta_{q(x)}\phi_{k_2} \leq \lambda^{q(x)}(\lambda_2 b(x) g(\phi_{k_1}) + \mu_2 d(x) \tau(\phi_{k_2})), a.e. \text{ on } \Omega \tag{3.14}$$

From (3.13) and (3.14), we can see that (ϕ_{k_1}, ϕ_{k_2}) is a subsolution of problem (3.1).
Step 2. We will construct a supersolution of problem (3.1), we consider

$$\begin{cases} -\Delta_{p(x)} z_{k_1} = \lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu \quad \text{in } \Omega, \\ \\ -\Delta_{q(x)} z_{k_2} = \lambda^{q+}(\lambda_1 b_2 + \mu_1 d_2) g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu)) \quad \text{in } \Omega, \\ \\ z_{k_1} = z_{k_2} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where

$$\beta = \beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu) = \max_{x\in\overline{\Omega}} z_{k_1}(x).$$

We shall prove that (z_{k_1}, z_{k_2}) is a supersolution of problem (3.1).
From Lemma 3.3.2, we have

$$\max_{x\in\overline{\Omega}} z_{k_1}(x) \leq C_2 \left[\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu\right]^{\frac{1}{p^- -1}}$$

and

$$\max_{x\in\overline{\Omega}} z_{k_2}(x) \leq C_2 \left[\lambda^{q+}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu))\right]^{\frac{1}{q^- -1}}.$$

For $\psi \in W_0^{1,q(x)}(\Omega)$ with $\psi \geqslant 0$, it is easy to see that

$$\int_\Omega |\nabla z_{k_2}|^{q(x)-2} \nabla z_{k_2}.\nabla\psi dx = \int_\Omega \lambda^{q+}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu))\psi dx \geqslant$$

$$\int_\Omega \lambda^{q+}\lambda_2 b(x) g(z_{k_1})\psi dx + \int_\Omega \lambda^{q+}\mu_2 d(x) g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu))\psi dx.$$

By (H_4), for μ a large enough, using Lemma 3.3.2, we have

$$\begin{aligned} & g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu)) \\ & \geqslant \tau(C_2\left[\lambda^{q+}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu))\right]^{\frac{1}{q^- - 1}}) \\ & \geqslant \tau(z_{k_2}). \end{aligned} \tag{3.15}$$

Hence

$$\int_\Omega |\nabla z_{k_2}|^{q(x)-2} \nabla z_{k_2}.\nabla \psi dx \geqslant \int_\Omega \lambda^{q+}\lambda_2 b(x) g(z_{k_1})\psi dx + \int_\Omega \lambda^{q+}\mu_2 d(x)\tau(z_{k_2})\psi dx. \tag{3.16}$$

Also, for $\varphi \in W^{1,p(x)}(\Omega)$ with $\varphi \geq 0$, it is easy to see that

$$\int_\Omega |\nabla z_{k_1}|^{p(x)-2} \nabla z_{k_1}.\nabla \varphi dx = \int_\Omega \lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu\varphi dx.$$

By $(H_3), (H_4)$ and Lemma 3.3.2, when μ is sufficiently large, we have

$$\begin{aligned} (\lambda_1 a_2 + \mu_1 c_2)\mu & \geqslant \frac{1}{\lambda^{p+}}\left[\frac{1}{C_2}\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu)\right]^{p^- - 1} \\ & \geqslant \mu_1 h(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu)) \\ & \quad + \lambda_1 f\left(C_2\left[\lambda^{q+}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu))\right]^{\frac{1}{q^- - 1}}\right). \end{aligned}$$

Then

$$\int_\Omega |\nabla z_{k_1}|^{p(x)-2} \nabla z_{k_1}.\nabla \varphi dx \geqslant \int_\Omega \lambda^{p+}\lambda_1 a(x) f(z_{k_2})\varphi dx + \int_\Omega \lambda^{p+}\mu_1 c(x) h(z_{k_1})\varphi dx. \tag{3.17}$$

According to (3.16) and (3.17), we can conclude that (z_{k_1}, z_{k_2}) is a supersolution of problem (3.1). It only remains to prove that $\phi_{k_1} \leq z_{k_1}$ and $\phi_{k_2} \leq z_{k_2}$.
In the definition of $v_1(x)$, let

$$\gamma = \frac{2}{\delta}\left(\max_{\overline{\Omega}} \phi_{k_1}(x) + \max_{\overline{\Omega}} |\nabla \phi_{k_1}|(x)\right).$$

We claim that

$$\phi_{k_1}(x) \leq v_1(x), \quad \forall x \in \Omega. \tag{3.18}$$

From the definition of v_1,it is easy to see that

$$\phi_{k_1}(x) \leq 2 \max_{\overline{\Omega}} \phi_{k_1}(x) \leq v_1(x), \text{ when } d(x) = \delta,$$

$$\phi_{k_1}(x) \leq 2 \max_{\overline{\Omega}} \phi_{k_1}(x) \leq v_1(x), \text{ when } \ d(x) \geqslant \delta$$

and

$$\phi_{k_1}(x) \leq v_1(x) \quad \text{ when } d(x) < \delta.$$

Since $v_1 - \phi_{k_1} \in C^1(\overline{\partial\Omega_\delta})$, there exists a point $x_0 \in \overline{\partial\Omega_\delta}$, such that

$$v_1(x_0) - \phi_{k_1}(x_0) = \min_{x_0 \in \overline{\partial\Omega_\delta}} (v_1(x_0) - \phi_{k_1}(x_0)).$$

If $v_1(x_0) - \phi_{k_1}(x_0) < 0$, It is easy to see that $0 < d(x) < \delta$ and then

$$\nabla v_1(x_0) - \nabla\phi_{k_1}(x_0) = 0.$$

From the definition of v_1,we have

$$|\nabla v_1(x_0)| = \gamma = \frac{2}{\delta}\left(\max_{\overline{\Omega}} \phi_{k_1}(x_0) + \max_{\overline{\Omega}} |\nabla\phi_{k_1}|(x_0)\right) > |\nabla\phi_{k_1}|(x_0).$$

It is a contradiction to

$$\nabla v_1(x_0) - \nabla\phi_{k_1}(x_0) = 0.$$

Thus, (3.18) is valid.
Obviously, there exists a positive constants C_3, such that $\gamma \leq C_3\lambda$.
Since $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, according to the proof of Lemma 3.3.2, there exists a positive constant C_4, such that

$$-\Delta_{p(x)} v_1(x) \leq C_* \gamma^{p(x)-1+\theta} \leq C_4 \lambda^{p(x)-1+\theta} \text{ a.e } \ \Omega, \text{ where } \theta \in (0,1).$$

Since $\eta \geqslant \lambda^{p+}$ is large enough, we have $-\Delta_{p(x)} v_1(x) \leq \eta$.
Under the comparaison principle, we have

$$v_1(x) \leq w(x), \text{ for all } \ x \in \Omega. \tag{3.19}$$

From (3.18) and (3.19), when $\eta \geqslant \lambda^{p+}$ and $\lambda \geqslant 1$ is sufficiently large, we have

$$\phi_{k_1}(x) \leq v_1(x) \leq w(x), \text{ for all } x \in \Omega. \tag{3.20}$$

According to the comparaison principle, when μ is large enough, we have

$$v_1(x) \leq w(x) \leq z_{k_1}(x), \text{ for all } x \in \Omega.$$

Combining the definition of $v_1(x)$ and (3.20), it is easy to see that

$$\phi_{k_1}(x) \leq v_1(x) \leq w(x) \leq z_{k_1}(x), \quad \text{for all} \quad x \in \Omega.$$

When $\mu \geqslant 1$ and λ is a large enough, from Lemma 3.3.2, we can note that $\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu$ is large enough, then

$$\lambda^{q+}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu))$$

is a large enough. Similarly, we have $\phi_{k_2}(x) \leq z_{k_2}(x)$. This completes the proof. ■

3.5 Asymptotic behavior of the $(p(x), q(x))$-Laplacian parabolic systems

Definition 3.5.1 *A measurable funtion $u : Q_T \to \mathbb{R}$ is an weak solution to parabolic systems involving of $(p(x), q(x))-$ Laplacien (3.1) in Q_T if $u(.,0) = u_0$ in Ω,*

$$u \in C(0,T; L^2(\Omega)) \cap L^p(0,T; H_0^1(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^2(Q_T), \nabla u \in \left(L^2(Q_T)\right)^N$$

and for all $\varphi \in C^1(Q_T)$ and $\psi \in C^1(Q_T)$, we have

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t} \varphi dxdt + \int_0^T \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dxdt + \int_0^T \int_\Omega (-\lambda^{p(x)} \mu_1 c(x) h(u)) \varphi dxdt$$
$$= \int_0^T \int_\Omega \lambda^{p(x)} \lambda_1 a(x) f(v) \varphi dxdt \tag{3.21}$$

Lemma 3.5.1 *([53])*

$$\int_0^T \int_\Omega \frac{\partial v}{\partial t}\psi dxdt + \int_0^T \int_\Omega |\nabla v|^{q(x)-2} \nabla v \nabla \psi dxdt + \int_0^T \int_\Omega (-\lambda^{q(x)}\lambda_2 b(x)g(u))\psi dxdt$$

$$= \int_0^T \int_\Omega \lambda^{q(x)}\mu_2 d(x)\sigma(v)\psi dxdt$$

Lemma 3.5.2 *([53]) Let $\underline{u}, \overline{u}$ be the solutions of (3.1) with $\underline{u}(x,0) = \varphi_1, \overline{u}(x,0) = \varphi_2$. Then $\underline{u}(x,t)$ is nondercreasing in t, $\overline{u}(x,t)$ is nonincreasing and $\overline{u} > \underline{u}$ for all $t \geq 0, x \in \Omega$*

Theorem 3.5.1 *Let hypotheses $(H_1), (H_2)$ and (H_3) be satisfied. and let $u(x,t)$ the solution of a new class of parabolic systems (3.1) with $\Psi \in S^*$ then*

$$\lim_{t\to\infty} u(x,t) = \begin{cases} \underline{u}_s(x) & \textit{if } \widehat{u}_s \leq \Psi \leq \underline{u}_s \\ \overline{u}_s(x) & \textit{if } \overline{u}_s \leq \Psi \leq \widetilde{u}_s \end{cases}$$

Proof. The pair $(\underline{u}_s, \widehat{u}_s)$ and the pair $(\widetilde{u}_s, \overline{u}_s)$ are both sub-super solutions of (4.3), the maximale and minimale property of $\overline{u}_s$ and $\underline{u}_s$ in S^* ensures that: $\underline{u}_s$ is the unique solution in $[\widehat{u}_s, \underline{u}_s]$ and $\overline{u}_s$ is the unique solution in $[\overline{u}_s, \widetilde{u}_s]$. ■

Chapter 4

Study of existence the positive solutions for a class of Kirchhoff parabolic systems with multiple parameters.

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1) Statement of the problems and assumption

2) Existence results.

3) Application methods of the existence positive of Kirchhoff parabolic systems.

In this chapter, we introduce the problems of a new class of Kirchhoff parabolics systems, we will study the existence of weak positive solution by using sub-super solutions method for a class of Kirchhoff parabolic systems in bounded domains with multiple parameters. This results are natural extensions from the previous ones in [11] and [39].

4.1 Statement of the problems and assumption

In this chapter, we consider the following system of parabolic differential equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - A\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle u = \lambda_1 \alpha(x) f(v) + \mu_1 \beta(x) h(u) \text{ in } Q_T = \Omega \times [0, T], \\ \dfrac{\partial v}{\partial t} - B\left(\int\limits_{\Omega} |\nabla v|^2 \, dx\right) \triangle v = \lambda_2 \gamma(x) g(u) + \mu_2 \eta(x) \tau(v) \text{ in } Q_T = \Omega \times [0, T], \\ u = v = 0 \text{ on } \partial Q_T, \\ u(x,0) = \varphi(x), \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, and A, B : $\mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions, $\alpha, \beta, \gamma, \eta \in C\left(\overline{\Omega}\right)$, $\lambda_1, \lambda_2, \mu_1$, and μ_2 are non negative parameters.

Since the first equation in (4.1) contains an integral over Ω, it is no longer a pointwise identity, Therefore, it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [74]. Moreover, problem (4.1) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int\limits_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{4.2}$$

presented by Kirchhoff in *1883* (see [49]). This equation is an extension of the classical d'Alembert's wave equation by considering the effect of the changes in the length of the string during the vibrations. The parameters in (4.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

By using Euler time scheme on (4.1), we obtain the following problems

$$\begin{cases} u_k - \tau' A\left(\int\limits_{\Omega} |\nabla u_k|^2 \, dx\right) \triangle u = \tau' \left[\lambda_1 \alpha(x) f(v) + \mu_1 \beta(x) h(u_k)\right] + u_{k-1} \text{ in } \Omega, \\ v_k - \tau' B\left(\int\limits_{\Omega} |\nabla v|^2 \, dx\right) \triangle v = \tau' \left[\lambda_2 \gamma(x) g(u_k) + \mu_2 \eta(x) \tau(v)\right] + v_{k-1} \text{ in } \Omega, \\ u_k = v_k = 0 \text{ on } \partial\Omega, \\ u_0 = \rho, \end{cases} \tag{4.3}$$

where $N\tau' = T$, $0 < \tau' < 1$, and for $1 \leq k \leq N$.

In recent years, problems involving Kirchhoff type operators have been studied in many papers as ([13], [59], [74], [17]-[35], [75]). In this thesis chapter, we have used different methods to get the existence of solutions for (4.1) in the single equation case. Z. Zhang in ([59] and [74]) studied the existence of nontrivial sign-changing solutions for system (4.1) where $A(t) = B(t) = 1$ via sub-supersolution method. Our of the thesis is motivated by the recent results in [10], [11], [16], [40], [44] and [45] . Azzouz and Bensedik (Theorem 2 in [11]) investigated the existence of a positive solution for the nonlocal problem of the form

$$\begin{cases} -M\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle u = |u|^{p-2} u + \lambda f(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.4}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N, N \geq 3$ and $p > 1$, *i.e.* the nonlinear term at infinity and f is a sign-changing function.

Using the sub and supersolution method combining a comparison principle introduced in [10], in this chapter we established the existence of a positive solution for (4.4), where the parameter $\lambda > 0$ is small enough. In the present chapter, we consider system (4.1) in the case when the nonlinearities are "sublinear" at infinity, see the condition (H_3). We are inspired by the ideas in the interesting paper [40], in which the authors considered system (4.1) in the case $A(t) = B(t) = 1$. More precisely, under suitable conditions on f and g, we shall show that system (4.1) has a positive solution for $\lambda > \lambda^*$. To our best knowledge, this is a new research topic for nonlocal problems (see [59] and [74]). In the current in this thesis, motivated by previous works in ([11], [40]) and by using the sub and supersolutions method, we study the existence of weak positive solution for a class of Kirchhoff parabolic systems in bounded domains with multiple parameters.

4.2 Existence result

Lemma 4.2.1 *([10]) Assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nonincreasing function satisfying*

$$M(s) > m_0, \text{ for all } s \geq s_0, \tag{4.5}$$

where m_0 is a positive constant and assume that u, v are two non-negative functions such that

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right) \triangle u \geq -M\left(\int_\Omega |\nabla v|^2 dx\right) \triangle v \ \ in\ \Omega, \\ \\ u = v = 0 \ \ on\ \partial\Omega, \end{cases} \tag{4.6}$$

then $u \geq v$ a.e. in Ω.

Proof. (Thanks to [10]) Suppose further that the function $H(t) = tM(t^2)$, $t \geq 0$ is a increasing on $\mathbb{R}^+$.
We follow along the lines of Alves' work in [10]. Multiplying both sides of the inequality by v and u and integrating, we get

$$\frac{M\left(\|u\|^2\right)\|u\|^2}{M\left(\|v\|^2\right)} \geq (u, v) \geq \frac{M\left(\|v\|^2\right)\|v\|^2}{M\left(\|u\|^2\right)}$$

and so

$$M\left(\|u\|^2\right)\|u\| \geq M\left(\|v\|^2\right)\|v\|$$

i.e.,

$$H(\|u\|) \geq H(\|v\|).$$

Since H is increasing, we obtain

$$\|u\| \geq \|v\|,$$

then

$$M\left(\|u\|^2\right) \leq M\left(\|v\|^2\right). \tag{4.7}$$

Because M is nonincreasing. On the other hand, by application of the maximum principle to (4.4), we get

$$M\left(\|u\|^2\right)u \geq M\left(\|v\|^2\right)v.$$

This with (4.7), yield $u \geq v$. This ends the proof. ■

In this chapter, we shall state and prove the main result of this thesis. Let us assume the following assumptions:

$(H1)$ Assume that $A, B : \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous and increasing functions and there exist $a_i, b_i > 0, i = 1, 2$, such that

$$a_1 \leq A(t) \leq a_2, \qquad b_1 \leq B(t) \leq b_2 \text{ for all } t \in \mathbb{R}^+,$$

$(H2)$ $\alpha, \beta, \gamma, \eta \in C\left(\overline{\Omega}\right)$ and

$$\alpha(x) \geq \alpha_0 > 0, \beta(x) \geq \beta_0 > 0, \gamma(x) \geq \gamma_0 > 0, \eta(x) \geq \eta_0 > 0,$$

for all $x \in \Omega$,

$(H3)$ f, g, h, and τ are continuous on $[0, +\infty[$, C^1 on $(0, +\infty)$, and increasing functions such that

$$\lim_{t\to+\infty} f(t) = +\infty, \quad \lim_{t\to+\infty} g(t) = +\infty, \quad \lim_{t\to+\infty} h(t) = +\infty, \quad \lim_{t\to+\infty} \tau(t) = +\infty,$$

$(H4)$ It holds that

$$\lim_{t\to+\infty} \frac{f(K(g(t)))}{t} = 0, \text{ for all } K > 0,$$

$(H5)$

$$\lim_{t\to+\infty} \frac{h(t)}{t} = 0, \quad \lim_{t\to+\infty} \frac{\tau(t)}{t} = 0.$$

4.3 Application methods of the existence positive of Kirchhoff parabolic systems.

Theorem 4.3.1 *Assume that the conditions $(H1)-(H5)$ hold, we assumption A, B are continuous functions $\mathbb{R}^+ \to \mathbb{R}^+$. Then for $\lambda_1\alpha_0 + \mu_1\beta_0$ and $\lambda_2\gamma_0 + \mu_2\eta_0$ are large then problem (4.1) has a large positive weak solution.*

We give the following two definitions before we give our main result.

Definition 4.3.1 *Let $(u_k, v) \in (H_0^1(\Omega) \times H_0^1(\Omega))$, (u_k, v) is said a weak solution of (4.3) if it satisfies*

$$A\left(\int_\Omega |\nabla u_k|^2 dx\right)\int_\Omega \nabla u_k \nabla \phi dx = \int_\Omega \left[\lambda_1 \alpha(x) f(v) + \mu_1 \beta(x) h(u_k) - \frac{u_k - u_{k-1}}{\tau'}\right]\phi dx \text{ in } \Omega,$$

$$B\left(\int_\Omega |\nabla v|^2 dx\right)\int_\Omega \nabla v \nabla \psi dx = \int_\Omega \left[\lambda_2 \gamma(x) g(u_k)\psi + \mu_2 \eta(x)\tau(v) - \frac{v_k - v_{k-1}}{\tau'}\right]\psi dx \text{ in } \Omega$$

for all $(\phi, \psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$.

Definition 4.3.2 *A pair of nonnegative functions $(\underline{u_k}, \underline{v})$, $(\overline{u_k}, \overline{v})$ in $(H_0^1(\Omega) \times H_0^1(\Omega))$ are called a weak subsolution and supersolution of (4.1) if they satisfy $(\underline{u_k}, \underline{v})$, $(\overline{u_k}, \overline{v}) = (0,0)$ on $\partial\Omega$*

$$A\left(\int_\Omega |\nabla \underline{u_k}|^2 dx\right)\int_\Omega \nabla \underline{u_k} \nabla \phi dx \leq \int_\Omega \left[\lambda_1 \alpha(x) f(\underline{v}) + \mu_1 \beta(x) h(\underline{u_k}) - \frac{u_k - u_{k-1}}{\tau'}\right]\phi \, dx \text{ in } \Omega,$$

$$B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_\Omega \nabla \underline{v} \nabla \psi dx \leq \int_\Omega \left[\lambda_2 \gamma(x) g(\underline{u_k}) + \mu_2 \eta(x)\tau(\underline{v}) - \frac{v_k - v_{k-1}}{\tau'}\right]\psi \, dx \text{ in } \Omega$$

and

$$A\left(\int_\Omega |\nabla \overline{u_k}|^2 dx\right)\int_\Omega \nabla \overline{u_k} \nabla \phi dx \geq \int_\Omega \left[\lambda_1 \alpha(x) f(\overline{v}) + \mu_1 \beta(x) h(\overline{u_k}) - \frac{u_k - u_{k-1}}{\tau'}\right]\phi \, dx \text{ in } \Omega,$$

$$B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right)\int_\Omega \nabla \overline{v} \nabla \psi dx \geq \int_\Omega \left[\lambda_2 \gamma(x) g(\overline{u_k}) + \mu_2 \eta(x)\tau(\overline{v}) - \frac{u_k - u_{k-1}}{\tau'}\right]\psi \, dx \text{ in } \Omega$$

for all $(\phi, \psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$.

Proof. of theorem 4.3.1. Let σ be the first eigenvalue of $\triangle$ with Dirichlet boundary conditions and ϕ_1 the corresponding positive eigenfunction with $\|\phi_1\|_\infty = 1$.
Let $k_0, m_0, \delta > 0$ such that $f(t), g(t), h(t), \tau(t) \geq -k_0$ for all $t \in \mathbb{R}^+$ and $|\nabla \phi_1|^2 - \sigma \phi_1^2 \geq m_0$ on $\overline{\Omega_\delta} = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$. For each $\lambda_1 \alpha_0 + \mu_1 \beta_0$ and $\lambda_2 \gamma_0 + \mu_2\, \eta_0$ large, let us define

$$\underline{u_k} = \left(\frac{(\lambda_1 \alpha_0 + \mu_1 \beta_0)\, k_0}{2 m_0 a_1}\right)\phi_1^2$$

and

$$\underline{v} = \left(\frac{(\lambda_2 \gamma_0 + \mu_2 \eta_0)\, k_0}{2 m_0 b_1} \right) \phi_1^2,$$

where a_1 and b_1 are given by the condition $(H1)$. We shall verify that $(\underline{u_k}, \underline{v})$ is a subsolution of problem (4.1) for $\lambda_1 \alpha_0 + \mu_1 \beta_0$ and $\lambda_2 \gamma_0 + \mu_2\ \eta_0$ large enough. Indeed, let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω. By $(H1) - (H3)$, a simple calculation shows that

$$\begin{aligned}
A\left(\int_\Omega \left|\nabla \underline{u_k}\right|^2 dx\right) \int_{\overline{\Omega}_\delta} \nabla \underline{u_k}.\nabla \phi dx &= A\left(\int_\Omega \left|\nabla \underline{u_k}\right|^2 dx\right) \frac{(\lambda_1 \alpha_0 + \mu_1 \beta_0)\, k_0}{m_0 a_1} \int_{\overline{\Omega}_\delta} \phi_1 \nabla \phi_1 . \nabla \phi dx \\
&= \frac{(\lambda_1 \alpha_0 + \mu_1 \beta_0)\, k_0}{m_0 a_1} A\left(\int_\Omega \left|\nabla \underline{u_k}\right|^2 dx\right) \times \\
&\quad \left\{ \int_{\overline{\Omega}_\delta} \nabla \phi_1 \nabla (\phi_1 . \phi)\, dx - \int_{\overline{\Omega}_\delta} \left|\nabla \phi_1\right|^2 \phi dx \right\} \\
&= \frac{(\lambda_1 \alpha_0 + \mu_1 \beta_0)\, k_0}{m_0 a_1} A\left(\int_\Omega \left|\nabla \underline{u_k}\right|^2 dx\right) \int_{\overline{\Omega}_\delta} \left(\sigma \phi_1^2 - \left|\nabla \phi_1\right|^2\right) \phi dx.
\end{aligned}$$

On $\overline{\Omega}_\delta$, we have $\left|\nabla \phi_1\right|^2 - \sigma \phi_1^2 \geq m_0$, then by using $(H3)$

$$f(\underline{v}),\ h(\underline{u_k}),\ g(\underline{u_k}),\ \tau(\underline{v}) \geq \frac{k_0}{m_0},$$

thus

$$\begin{aligned}
A\left(\int_\Omega \left|\nabla \underline{u_k}\right|^2 dx\right) \int_{\overline{\Omega}_\delta} \nabla \underline{u_k} \nabla \phi dx &\leq \frac{(\lambda_1 \alpha_0 + \mu_1 \beta_0)\, k_0}{m_0} \int_{\overline{\Omega}_\delta} \left(\sigma \phi_1^2 - \left|\nabla \phi_1\right|^2\right) \phi dx \\
&\leq \int_\Omega \left[\lambda_1 \alpha(x) f(\underline{v}) + \mu_1 \beta(x) h(\underline{u_k}) - \frac{u_k - u_{k-1}}{\tau'} \right] \phi\ dx.
\end{aligned} \tag{4.8}$$

Next, on $\Omega \backslash \overline{\Omega}_\delta$, we have $\phi_1 \geq r$ for some $r > 0$. Therefore, under the conditions $(H1) - (H3)$

and the definition of $\underline{v}$, it follows that

$$\int_{\Omega}\left[\lambda_1\alpha(x)f(\underline{v})+\mu_1\beta(x)h(\underline{u_k})-\frac{u_k-u_{k-1}}{\tau'}\right]\phi\,dx \geq (\lambda_1\alpha_0+\mu_1\beta_0)\frac{k_0a_2}{m_0a_1}\sigma\int_{\Omega\setminus\overline{\Omega}_\delta}\phi dx$$

$$\geq (\lambda_1\alpha_0+\mu_1\beta_0)\frac{k_0}{m_0a_1}A\left(\int_{\Omega\setminus\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\sigma\int_{\Omega\setminus\overline{\Omega}_\delta}\phi dx$$

$$\geq (\lambda_1\alpha_0+\mu_1\beta_0)\frac{k_0}{m_0a_1}A\left(\int_{\Omega\setminus\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\int_{\Omega\setminus\overline{\Omega}_\delta}\left(\sigma\phi_1^2-\left|\nabla\phi_1\right|^2\right)\phi dx$$

$$= A\left(\int_{\Omega\setminus\overline{\Omega}_\delta}\left|\nabla\underline{u_k}\right|^2dx\right)\int_{\Omega\setminus\overline{\Omega}_\delta}\nabla\underline{u_k}\nabla\phi dx, \tag{4.9}$$

for $\lambda_1\alpha_0+\mu_1\beta_0>0$ large enough.
Relations (4.8) and (4.9) imply that

$$A\left(\int_{\Omega}\left|\nabla\underline{u_k}\right|^2dx\right)\int_{\Omega}\nabla\underline{u_k}\nabla\phi dx \leq \int_{\Omega}\left[\lambda_1\alpha(x)f(\underline{v})+\mu_1\beta(x)h(\underline{u_k})-\frac{u_k-u_{k-1}}{\tau'}\right]\phi dx \text{ in } \Omega, \tag{4.10}$$

for $\lambda_1\alpha_0+\mu_1\beta_0>0$ large enough and any $\phi\in H_0^1(\Omega)$ with $\phi\geq 0$ in Ω.
Similarly,

$$B\left(\int_{\Omega}\left|\nabla\underline{v}\right|^2dx\right)\int_{\Omega}\nabla\underline{v}\nabla\psi dx \leq \int_{\Omega}\left[\lambda_2\gamma(x)g(u_k)\psi+\mu_2\eta(x)\tau(v)-\frac{v_k-v_{k-1}}{\tau'}\right]\psi dx \text{ in } \Omega, \tag{4.11}$$

for $\lambda_2\gamma_0+\mu_2\eta_0>0$ large enough and any $\psi\in H_0^1(\Omega)$ with $\psi\geq 0$ in Ω. From (4.10) and (4.11), $(\underline{u_k},\underline{v})$ is a subsolution of problem (4.3). Moreover, we have $\underline{u_k}>0$, $\underline{v}>0$ in Ω, $\underline{u}\to+\infty$ and $\underline{v}\to+\infty$ also $\lambda_1\alpha_0+\mu_1\beta_0\to+\infty$ and $\lambda_2\gamma_0+\mu_2\eta_0\to+\infty$.
Next, we shall construct a supersolution of problem (4.3). Let e be the solution of the following problem:

$$\begin{cases} -\triangle e = 1 \text{ in } \Omega, \\ e = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.12}$$

Let

$$\overline{u_k} = Ce,\ \overline{v} = \left(\frac{\lambda_2 \|\gamma\|_\infty + \mu_2 \|\eta\|_\infty}{b_1}\right) \left[g\left(C \|e\|_\infty\right)\right] e,$$

where e is given by (4.12) and $C > 0$ is a large positive real number to be chosen later. We shall verify that $(\overline{u_k}, \overline{v})$ is a supersolution of problem (4.3). Let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω. Then, we obtain from (4.12) and the condition $(H1)$ that

$$\begin{aligned} A\left(\int_\Omega |\nabla \overline{u_k}|^2 dx\right) \int_\Omega \nabla \overline{u_k}.\nabla \phi dx &= A\left(\int_\Omega |\nabla \overline{u_k}|^2 dx\right) C \int_\Omega \nabla \omega.\nabla \phi dx \\ &= A\left(\int_\Omega |\nabla \overline{u_k}|^2 dx\right) C \int_\Omega \phi dx \\ &\geq a_1 C \int_\Omega \phi dx. \end{aligned}$$

By using $(H4)$ and $(H5)$, we can choose C large enough, thus

$$a_1 C \geq \lambda_1 \|\alpha\|_\infty f\left(\left[\frac{\lambda_2 \|\gamma\|_\infty + \mu_2 \|\eta\|_\infty}{b_1}\right] g\left(C \|e\|_\infty\right) \|e\|_\infty\right) + \mu_1 \|\beta\|_\infty h\left(C \|e\|_\infty\right).$$

Therefore,

$$\begin{aligned} & A\left(\int_\Omega |\nabla \overline{u_k}|^2 dx\right) \int_\Omega \nabla \overline{u_k}.\nabla \phi dx \\ & \geq \left[\lambda_1 \|\alpha\|_\infty f\left(\left[\frac{\lambda_2 \|\gamma\|_\infty + \mu_2 \|\eta\|_\infty}{b_1}\right] g\left(C \|e\|_\infty\right) \|e\|_\infty\right) + \mu_1 \|\beta\|_\infty h\left(C \|e\|_\infty\right)\right] - \int_\Omega \frac{u_k - u_{k-1}}{\tau'} \phi dx \\ & \geq \lambda_1 \|\alpha\|_\infty \int_\Omega f\left(\left[\frac{\lambda_2 \|\gamma\|_\infty + \mu_2 \|\eta\|_\infty}{b_1}\right] g\left(C \|e\|_\infty\right) \|e\|_\infty\right) \phi dx + \mu_1 \int_\Omega h\left(C \|e\|_\infty\right) \phi dx - \int_\Omega \frac{u_k - u_{k-1}}{\tau'} \phi dx \\ & \geq \int_\Omega \left[\lambda_1 \alpha(x) f\left(\underline{v}\right) + \mu_1 \beta(x) h\left(\underline{u_k}\right) - \frac{u_k - u_{k-1}}{\tau'}\right] \phi dx. \end{aligned} \tag{4.13}$$

Also, we have

$$B\left(\int_{\Omega}|\nabla\overline{v}|^{2}\,dx\right)\int_{\Omega}\nabla\overline{v}\nabla\psi dx\geq\left(\lambda_{2}\left\|\gamma\right\|_{\infty}+\mu_{2}\left\|\eta\right\|_{\infty}\right)\int_{\Omega}g\left(C\left\|e\right\|_{\infty}\right)\psi dx$$

$$\geq\lambda_{2}\int_{\Omega}\gamma\left(x\right)g\left(\overline{u_{k}}\right)\psi dx+\mu_{2}\int_{\Omega}\eta\left(x\right)g\left(C\left\|e\right\|_{\infty}\right)\psi dx-\int_{\Omega}\frac{v_{k}-v_{k-1}}{\tau'}\psi dx. \tag{4.14}$$

Again by using $(H4)$ and $(H5)$ for C large enough, we have

$$g\left(C\left\|e\right\|_{\infty}\right)\geq\tau\left[\frac{\left(\lambda_{2}\left\|\gamma\right\|_{\infty}+\mu_{2}\left\|\eta\right\|_{\infty}\right)}{b_{1}}g\left(C\left\|e\right\|_{\infty}\right)\left\|e\right\|_{\infty}\right]\geq\tau\left(\overline{v}\right). \tag{4.15}$$

From (4.14) and (4.15), we have

$$B\left(\int_{\Omega}|\nabla\overline{v}|^{2}\,dx\right)\int_{\Omega}\nabla\overline{v}\nabla\psi dx\geq\lambda_{2}\int_{\Omega}\gamma\left(x\right)g\left(\overline{u_{k}}\right)\psi dx+\mu_{2}\int_{\Omega}\eta\left(x\right)\tau\left(\overline{v}\right)\psi dx-\int_{\Omega}\frac{v_{k}-v_{k-1}}{\tau'}\psi dx. \tag{4.16}$$

From (4.13) and (4.16), we have $(\overline{u},\overline{v})$ is a subsolution of problem (4.1) with $\underline{u}\leq\overline{u}$ and $\underline{v}\leq\overline{v}$ for C large enough.

In order to obtain a weak solution of problem (4.3), we shall use the arguments by Azzouz and Bensedik [11] (observe that f, g, h, and τ does not depend on x). For this purpose, we define a sequence $\{(u_n,v_n)\}\subset\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$ as follows: $u_0=\overline{u},v_0=\overline{v}$ and (u_n,v_n) is the unique solution of the system

$$\begin{cases}-A\left(\int_{\Omega}|\nabla u_{n}|^{2}\,dx\right)\triangle u_{n}=\lambda_{1}\alpha\left(x\right)f\left(v_{n-1}\right)+\mu_{1}\beta\left(x\right)h\left(u_{n-1}\right)-\dfrac{u_{k}-u_{k-1}}{\tau'}\text{ in }\Omega,\\ -B\left(\int_{\Omega}|\nabla v_{n}|^{2}\,dx\right)\triangle v_{n}=\lambda_{2}\gamma\left(x\right)g\left(u_{n-1}\right)+\mu_{2}\eta\left(x\right)\tau\left(v_{n-1}\right)-\dfrac{v_{k}-v_{k-1}}{\tau'}\text{ in }\Omega,\\ u_{n}=v_{n}=0\text{ on }\partial\Omega.\end{cases} \tag{4.17}$$

We have $(u_{n-1},v_{n-1})\in\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$, in the sense that, the right hand sides of (4.17) is independent on u_n and v_n.

Setting

$$A\left(t\right)=tA\left(t^{2}\right),B\left(t\right)=tB\left(t^{2}\right).$$

Since $A(\mathbb{R})=\mathbb{R}$, $B(\mathbb{R})=\mathbb{R}$, $f(v_{n-1})$, $h(u_{n-1})$, $g(u_{n-1})$, and $\tau(v_{n-1})\in L^2(\Omega)$, we deduce from the results in [10], that system (4.17) has a unique solution $(u_n,v_n)\in\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$. By using (4.17) and the fact that (u_0,v_0) is a supersolution of (4.1), we have

$$\begin{cases} -A\left(\int\limits_\Omega |\nabla u_0|^2\,dx\right)\triangle u_0 \geq \lambda_1\alpha(x)\,f(v_0)+\ \mu_1\beta(x)\,h(u_0)-\dfrac{u_k-u_{k-1}}{\tau'} \\ \qquad\qquad = -A\left(\int\limits_\Omega |\nabla u_1|^2\,dx\right)\triangle u_1, \\ \\ -B\left(\int\limits_\Omega |\nabla v_0|^2\,dx\right)\triangle v_0 \geq \lambda_2\gamma(x)\,g(u_0)+\mu_2\eta(x)\,\tau(v_0)-\dfrac{v_k-v_{k-1}}{\tau'} \\ \qquad\qquad = -B\left(\int\limits_\Omega |\nabla v_1|\,dx\right)\triangle v_1 \end{cases}$$

and by using Lemma 4.2.1, we also have $u_0\geq u_1$ and $v_0\geq v_1$. In addition, since $u_0\geq\underline{u}$, $v_0\geq\underline{v}$ and under the monotonicity condition of f, h, g, and τ, we can deduce

$$\begin{aligned} -A\left(\int\limits_\Omega |\nabla u_1|^2\,dx\right)\triangle u_1 &= \lambda_1\alpha(x)\,f(v_0)+\ \mu_1\beta(x)\,h(u_0)-\frac{u_k-u_{k-1}}{\tau'} \\ &\geq \lambda_1\alpha(x)\,f(\underline{v})+\ \mu_1\beta(x)\,h(\underline{u})-\frac{u_k-u_{k-1}}{\tau'} \\ &\geq -A\left(\int\limits_\Omega |\nabla\underline{u}|^2\,dx\right)\triangle\underline{u} \end{aligned}$$

and

$$\begin{aligned} -B\left(\int\limits_\Omega |\nabla v_1|^2\,dx\right)\triangle v_1 &= \lambda_2\gamma(x)\,g(u_0)+\mu_2\eta(x)\,\tau(v_0)-\frac{v_k-v_{k-1}}{\tau'} \\ &\geq \lambda_2\gamma(x)\,g(\underline{u})+\mu_2\eta(x)\,\tau(\underline{v})-\frac{v_k-v_{k-1}}{\tau'} \\ &\geq -B\left(\int\limits_\Omega |\nabla\underline{v}|^2\,dx\right)\triangle\underline{v}. \end{aligned}$$

According to Lemma 4.2.1, we have $u_1\geq\underline{u}$, $v_1\geq\underline{v}$ for any u_2,v_2, thus we can write

$$
\begin{aligned}
-A\left(\int\limits_{\Omega}\left|\nabla u_1\right|^2 dx\right)\triangle u_1 &= \lambda_1\alpha\left(x\right)f\left(v_0\right)+\mu_1\beta\left(x\right)h\left(u_0\right)-\frac{u_k-u_{k-1}}{\tau'}\\
&\geq \lambda_1\alpha\left(x\right)f\left(v_1\right)+\mu_1\beta\left(x\right)h\left(u_1\right)-\frac{u_k-u_{k-1}}{\tau'}\\
&= -A\left(\int\limits_{\Omega}\left|\nabla u_2\right|^2 dx\right)\triangle u_2,\\
-B\left(\int\limits_{\Omega}\left|\nabla v_1\right| dx\right)\triangle v_1 &= \lambda_2\gamma\left(x\right)g\left(u_0\right)+\mu_2\eta\left(x\right)\tau\left(v_0\right)-\frac{v_k-u_{k-1}}{\tau'}\\
&\geq \lambda_2\gamma\left(x\right)g\left(u_1\right)+\mu_2\eta\left(x\right)\tau\left(v_1\right)-\frac{v_k-v_{k-1}}{\tau'}\\
&= -B\left(\int\limits_{\Omega}\left|\nabla v_2\right|^2 dx\right)\triangle v_2.
\end{aligned}
$$

Then, $u_1 \geq u_2$, $v_1 \geq v_2$.
Similarly, $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ because

$$
\begin{aligned}
-A\left(\int\limits_{\Omega}\left|\nabla u_2\right|^2 dx\right)\triangle u_2 &= \lambda_1\alpha\left(x\right)f\left(v_1\right)+\mu_1\beta\left(x\right)h\left(u_1\right)-\frac{u_k-u_{k-1}}{\tau'}\\
&\geq \lambda_1\alpha\left(x\right)f\left(\underline{v}\right)+\mu_1\beta\left(x\right)h\left(\underline{u}\right)-\frac{u_k-u_{k-1}}{\tau'}\\
&\geq -A\left(\int\limits_{\Omega}\left|\nabla \underline{u}\right|^2 dx\right)\triangle \underline{u},\\
-B\left(\int\limits_{\Omega}\left|\nabla v_2\right|^2 dx\right)\triangle v_2 &= \lambda_2\gamma\left(x\right)g\left(u_1\right)+\mu_2\eta\left(x\right)\tau\left(v_1\right)-\frac{v_k-v_{k-1}}{\tau'}\\
&\geq \lambda_2\gamma\left(x\right)g\left(\underline{u}\right)+\mu_2\eta\left(x\right)\tau\left(\underline{v}\right)-\frac{v_k-v_{k-1}}{\tau'}\\
&\geq -B\left(\int\limits_{\Omega}\left|\nabla \underline{v}\right|^2 dx\right)\triangle \underline{v}.
\end{aligned}
$$

Repeating this argument, we get a bounded monotone sequence $\{(u_n, v_n)\} \subset \left(H_0^1\left(\Omega\right)\times H_0^1\left(\Omega\right)\right)$ satisfying

$$\overline{u} = u_0 \geq u_1 \geq u_2 \geq ... \geq u_n \geq ... \geq \underline{u} > 0 \tag{4.18}$$

and

$$\overline{v} = v_0 \geq v_1 \geq v_2 \geq ... \geq v_n \geq ... \geq \underline{v} > 0. \tag{4.19}$$

Using the continuity of the functions f, h, g, τ and the definition of the sequences $\{u_n\}, \{v_n\}$,there exist constants $C_i > 0$, $i = 1, ..., 4$ independent of n such that

$$|f(v_{n-1})| \leq C_1, \quad |h(u_{n-1})| \leq C_2, \ |g(u_{n-1})| \leq C_3 \tag{4.20}$$

and

$$|\tau(u_{n-1})| \leq C_4 \text{ for all } n.$$

Multiplying the first equation of (4.17) by u_n, integrating, using the Holder inequality and Sobolev embedding, we can show that

$$\begin{aligned} a_1 \int_\Omega |\nabla u_n|^2 dx &\leq A\left(\int_\Omega |\nabla u_n|^2 dx\right) \int_\Omega |\nabla u_n|^2 dx \\ &= \lambda_1 \int_\Omega \alpha(x) f(v_{n-1}) u_n dx + \mu_1 \int_\Omega \beta(x) h(u_{n-1}) u_n dx - \int_\Omega \frac{u_k - u_{k-1}}{\tau'} u_n dx \\ &\leq \lambda_1 \|\alpha\|_\infty \int_\Omega |f(v_{n-1})| |u_n| dx + \mu_1 \|\beta\|_\infty \int_\Omega |h(u_{n-1})| |u_n| dx - \int_\Omega \frac{u_k - u_{k-1}}{\tau'} |u_n| dx \\ &\leq C_1 \lambda_1 \int_\Omega |u_n| dx + C_2 \mu_1 \int_\Omega |u_n| dx - \int_\Omega \frac{u_k - u_{k-1}}{\tau'} |u_n| dx \\ &\leq C_5 \|u_n\|_{H_0^1(\Omega)}, \end{aligned}$$

or

$$\|u_n\|_{H_0^1(\Omega)} \leq C_5, \ \forall n, \tag{4.21}$$

where $C_5 > 0$ is a constant independent of n. Similarly, there exists $C_6 > 0$ independent of n such that

$$\|v_n\|_{H_0^1(\Omega)} \leq C_6, \quad \forall n. \tag{4.22}$$

From (4.21) and (4.22), we infer that $\{(u_n, v_n)\}$ has a subsequence which weakly converges in $H_0^1(\Omega)$ to a limit (u, v) with the properties $u \geq \underline{u} > 0$ and $v \geq \underline{v} > 0$. Being monotone and also by using a standard regularity argument, $\{(u_n, v_n)\}$ converges itself to (u, v).
Now, passing the limit in (4.17), we deduce that (u, v) is a positive solution of system (4.4).
The proof of theorem is completed. ■

Conclusions

In this thesis, our result is an extension for our previous study in ([13, 16, 44]) which studied the stationary case, this idea is new for evolutionary case of this kind of problem, This thesis deals with the existence of positively solution and its asymptotic behavior for parabolic system of $(p(x), q(x))$-Laplacian system of partial differential equations using a sub and super solution according to some given boundary conditions, which is familiar in physics, since it appears clearly natural in inflation cosmology and super symmetric filed theories, quantum mechanics, and nuclear physics (see [10, 40]). This sort of problem has many applications in several branches of physics such as nuclear physics, optics, and geophysics (see [11, 16]). In future work, we will try to extend this study for the hyperbolic case of the presented problem, but by using the semigroup theory.

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