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Résumé

Dans cette thèse, nous présentons une méthode d'analyse d'homotopieoptimale pour obtenir une solution approximative pour des équations aux dérivées partielles d'ordre fractionnaire.

Cette méthode a été appliquée pour obtenir une solution numérique de l'équation différentielle partiellehyperbolique fractionnaire. Une autre méthode appelée méthode asymptotique d'homotopie optimale a été appliquée pour obtenir les solutions analytiques approchées de deux problèmes oscillatoires non linéaires d'ordre fortement fractionnaire.

En conséquence, ces méthodes nous permettent de contrôler la région convergente de la solution en série. Quelques exemples numériques sont présentés pour prouver l'éfficacité de la méthode.

Les mots clés : Équation différentielle fractionnaire, Dérivé fractionnaire de Caputo, Solution en série, Méthode d'analyse d'homotopie.

Abstract

In this thesis, we present an optimal homotopy analysis method to obtain approximate solution for partial differential equation of fractional order. This method was applied to obtain a numerical solution of time-fractional hyperbolic partial differential equation.

Another method called the optimal homotopy asymptotic method was applied to get the approximate analytic solutions for two strongly fractional-order nonlinear benchmark oscillatory problems.

As a result, these methods allow us to control the convergent region of the series solution. Some numerical examples are presented to prove the accuracy of the method.

Keywords: Fractional differential equation , Caputo fractional derivative, Series solution, Homotopy analysis method

ملخص

في هذه الأطروحة نقدم طريقة الهوموتوبي التحليلية لإيجاد حل تقريبي للمعادلة التفاضلية الجزئية ذات الرتب الكسرية.

هذه الطريقة تم تطبيقها لإيجاد حل عددي للمعادلة التفاضلية الجزئية الزائدية الكسرية, كما تم تطبيق طريقة أخرى تدعى طريقة الهوموتوبي المقاربة المثالية لإيجاد حلول تحليلية تقريبية لمعادلتي تذبذب غير الخطيتين ذات الترتيب الكسري.

كنتيجة هذه الطرق تسمح لنا بالتحكم في تقارب سلسلة الحلول, بعض الأمثلة العددية تم تقديمها لإثبات نجاعة الطريقة.

الكلمات المفتاحية::المعادلة التفاضلية الكسرية, المشتق الكسري ل Caputo, سلسلة الحلول, طريقة الهوموتوبي التحليلية.

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CONTENTS

Table of symboles and abreviations

$\Gamma(z)$	The Gamma Function.
$B(z, w)$	The Beta Function.
$E_\alpha(z)$	The one -parameter representation of Mittag -Leffler Function.
$E_{\alpha,\beta}(z)$	The two -parameter representation of Mittag -Leffler Function.
$Erfc(z)$	The Error Function.
I^α	The Riemann-Liouville Fractional Integral of the order α.
${}^{RL}D^\alpha$	The Riemann-Liouville Fractional Derivative of the order α.
${}_CD^\alpha$	Caputo Fractional Derivative of the order α.
$\mathcal{L}\{f(t); s\}$	The Laplace transform.
$\mathcal{F}\{f(t); w\}$	The Fourier transform.
VIM	Variational iteration method.
FDM	Finite difference method.
FEM	Finite element method.
FVM	Finite volume method.
BEM	Boundary element method.
$GDTM$	Generalized differential transform method.
ADM	Adomian decomposition method.
$FoHP$	The fractional-order hyperbolic problem.
HAM	Homotopy Analysis Method.
HPM	Homotopy Perturbation Method.
$ODEs$	Ordinary differential equations.
$OHAM$	The optimal homotopy asymptotic method.
$SHAM$	Spectral homotopy analysis method.
$LHAM$	The linearization-based approach of HAM.

List of figures and tables

General Introduction

The concept of fractional calculus linked back to 1695, when **Liebniz** (the inventor of both integer derivative $d^n f(x)/dx^n$ and integral $\int f(x)dx$) wrote a letter to **Hôpital** asking him about the meaning of the semi-derivative. **Euler** was the first one who was interrested in this topic, he noticed that the semi-derivative has a meaning, then **Fourier** suggested the idea of using the equality

$$\frac{d^p f(x)}{dx^p} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lambda^p d\lambda \int_{-\infty}^{+\infty} f(t)\cos(\lambda x - t\lambda + p\frac{\pi}{2})dt$$

in order to define the derivative for non-integer order. This topic attracted the interest of other well-known mathematicians as **Laplace**, **Liouville**, **Riemann**, **Grûnwald**, **Letnikov** .... In the last decade, fractional-order partial differential equation (FoPDEs) played a vital role in many fields such as physics, biology, chemistry, ... Actually, obtaining exact solutions for such equations in its nonlinearity case is extremely tough according to the view of many mathematicians. Thus, proposing numerical schemes for obtaining approximate solutions for such equations is a very urgent need. To meet this need, several iterative approaches have been recently suggested, such as variational iteration method (VIM) [74], finite difference method (FDM) [65], generalized differential transform method (GDTM) [5, 7, 20], adomian decomposition method (ADM) [68], homotopy perturbation method (HPM) [75], and homotopy analysis method (HAM) [29, 82].The fractional-order hyperbolic problem (FoHP) is one of the most important nonlinear problems in mathematical physics. This type is simply generated by replacing the second-order derivative of its classical equation by another one of fractional-order, say α, where $1 < \alpha \leq 2$ [5, 18]. This replacement in such orders provides much more degrees of freedom, letting researchers to model several real-life phenomena more comfortably than before. Over the past few years, lots of researchers paid their attentions to explore the solutions of the FoHPs and discuss their stabilities. The nonlocal problems for the degenerate FoHP were studied by Kilbas et al in [41]. Some conditions were set up by Kirane and Laskri in [42] to guarantee the nonexistence of solutions for some types of FoHPs. For obtaining the solutions of some FoHPs, difference approaches were proposed in [9].

The nonlinear fractional-order oscillators are typically considered to be a significant exemplar of such equations. The strongly nonlinear oscillator, which is one of the major types of these oscillators, could be dealt with by means of three main schemes. Constructing new or using some special existent functions that relies on the nature of nonlinearity is the first scheme. On the other hand, the second scheme could be represented by appropriately rescaling the displacement, and then inserting a small parameter into motion equation. Whereas, the third scheme can be delineated by introducing a further small parameter, and then transporting motion equation into a linear oscillator perturbation [12, 39, 44].In general terms, the nonlinear fractional-order oscillators have been examined and explored by many researchers. In particular, Shen et al. studied the primary resonance of fractional-order van der Pol oscillator analytically and numerically while using the averaging method [87], and then used the incremental harmonic balance method to analyze some dynamical properties of fractional-order nonlinear oscillator [88]. The dynamical response of the fractional-order stochastic Duffing equation was explored by Xu et al in [91]. Some novel dynamical features of fractional-order Duffing oscillator had been studied by Chen et al in [14–16]. They proposed a new powerful bifurcation control approach that is based on the $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller [16]. However, obtaining accurate solutions of most of these nonlinear equations is considered to be an extremely difficult mission for lots of researchers.

The homotopy analysis method (HAM) was proposed in 1992 by **Liao**, who used the basic ideas of the homotopy in topology. Several nonlinear fractional-order engineering and mechanics problems have been, very recently, solved using HAM such as theKdV–Burgers–Kuramoto and BBM–Burgers problems, Riccati DEs, the Klein–Gordon equation, the heat-like PDEs according with Neumann boundary conditions, diffusion-wave equations and others [43]. Such method has been also engaged in chemistry field by Hariharan [28]. He, actually, has obtained numerical solutions of Schrödinger equation and viscid equations of gas dynamics [72]. The role of this technique is to obtain out the best parameters of the convergence control by minimizing the square residual error as much as possible.

The HAM contains an auxiliary parameter called the convergence control parameter h, a nonzero auxiliary function H and a linear operator L, based on these parameters this method provides a convenient way to guarantee the convergence of series solution. The optimal homotopy analysis method (OHAM) was recently proposed and developed by Marinca et al as a generalization of the classical HAM [24, 32, 59, 61, 62, 85]. Several solutions of significant nonlinear problems within lots of studies were then, consequently, constructed based on using this method (see [11, 24, 27, 30, 35–38, 80, 85]). In view of many of these studies, it was demonstrated that this method is a reliable, straightforward, and effective tool for offering accurate analytical approximate solutions to lots of strongly nonlinear problems [11, 61, 80]. Besides, it was revealed that its key characteristic is its ability to optimally control the convergence of approximate series solutions [11, 61, 80].

The main objective of this thesis is to solve fractional partial differential equation by using an optimal homotopy analysis method.

Our thesis is organized as follows: the first chapter introduced the basic idea of fractional calculus: some special functions, fractional integral, fractional derivatives, Laplace transforms and Fourier transforms of fractional derivatives. In the second chapter, we describe the homotopy analysis method, an analytic approach to have an approximate solution for nonlinear differential equations, some definitions and examples are given to present thevapproach clearly. Third chapter present the so called Linearization based approach of HAM, a new modification of the classical homotopy analysis method based on employing Taylor series solution, to accelerate the convergence of the solution for such nonlinear hyperbolic partial differential equations of fractional order, we give some examples to prove the effeciency of the new method. The last chapter, contains one of the recent approximate methods namely the optimal asymptotic homotopy method to give an approximate solutions for two strongly fractional-order nonlinear benchmark osciliatory problems, some numerical results are given to demonstrate the accurate of this method.

CHAPTER 1

REVIEW OF FRACTIONAL CALCULUS

In this chapter we introduce the basic ideas of fractional calculs, specifically some special functions, the fractional integral and the fractional derivative (Riemann-Liouville fractional derivative and Caputo fractional Derivative) and their properties.

1.1 Special functions

1.1.1 Gamma function

One of the basic functions of the fractional calculus which generalizes the fact $n!$ is called Euler's gamma function.

Definition 1.1 *[51] The gamma function $\Gamma(z)$ is defined by the integral*

$$\Gamma(z) = \int_0^\infty \exp(-t)t^{z-1}dt \tag{1.1}$$

Some properties of gamma function [51] The gamma function satisfies the following functional equation:

$$\Gamma(z+1) = z\Gamma(z) \tag{1.2}$$

Which can be easily proved by integrating by parts.
Obviousily $\Gamma(1) = 1$ and using (1.2) we obtain

$$\Gamma(n+1) = n\Gamma(n) = n.(n-1)! = n!$$

The derivative of the gamma function can be expressed as follows

$$\frac{d^n}{dz^n}\Gamma(z) = \int_0^\infty t^{z-1}\exp(-t)(\ln t)^n dt, z > 0$$

1.1.2 Beta function

Definition 1.2 *[51] The beta function is usually defined by*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt \tag{1.3}$$

It is connected with the gamma function by the relation

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{1.4}$$

The integral (1.3) makes sense when $\operatorname{Re} z = 0$ or $\operatorname{Re} w = 0$ $(z \neq 0, w \neq 0)$.

Properties of Beta functions [51]

- $$B(z,w) = B(w,z)$$
- $$B(z+1,w) = B(z,w)\frac{z}{w+z}$$
- $$B(z,w) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2z-1}(\cos\theta)^{2w-1}d\theta, z > 0, w > 0$$

1.1.3 Mittag-Leffler Function

The Mittag-Leffler function is a direct generalization of the expenential function $\exp(z)$ which plays a very important role in the fractional calculus theory [51].

Definition 1.3 *The one-parameter representation of Mittag-Leffler function is denoted by*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \alpha > 0 \tag{1.5}$$

Definition 1.4 *The two-parameter function of the Mittag-Leffler type is defined by the series expantion*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha > 0, \beta > 0 \tag{1.6}$$

It follows from the definition (1.4) that

- $$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$
- $$E_{\frac{1}{2},1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\frac{k}{2}+1)} = e^{z^2} Erfc(-z)$$
  where $Erfc(z)$ is the error function complement defined by
  $$Erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt$$
- $$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^k+1}{(k+1)!} = \frac{e^z - 1}{z}$$
- $$E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh(z)$$
- $$E_{2,2}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{z(2k+1)!} = \frac{\sinh(z)}{z}$$
- $$E_{2,1}(-z^2) = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos(z)$$
- $$E_{2,2}(-z^2) = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \frac{\sin(z)}{z}$$
- $$\frac{1}{\Gamma(\nu)} \int_0^z (z-t)^{\nu-1} E_{\alpha,\beta}(\lambda t^\alpha)^\beta dt = z^{\beta+\nu+1} E_{\alpha+\beta+\nu}(\lambda z^\alpha), \nu > 0$$

1.2 Fractional integral

Definition 1.5 *[51] Let $\varphi(t) \in L_1(a,b)$. The left-sided and right-sided fractional integrals of the order α are defined as:*

$$(I^{\alpha}_{a+}\varphi)(x) \stackrel{def}{=} \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt, \quad x > a, \tag{1.7}$$

$$(I^{\alpha}_{b-}\varphi)(x) \stackrel{def}{=} \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)}{(t-x)^{1-\alpha}}dt, \quad x < a, \tag{1.8}$$

where $\alpha > 0$. The accepted names for the integrals (1.19) and (1.20) are the Riemann-Liouville fractionals integrals.

Properties: Fractional integration has the properties:

$$I^{\alpha}_{a+}I^{\beta}_{a+}\varphi = I^{\alpha+\beta}_{a^+}\varphi, \quad I^{\alpha}_{b-}I^{\beta}_{b-}\varphi = I^{\alpha+\beta}_{b^-}\varphi, \ \alpha > 0, \beta > 0 \tag{1.9}$$

$$\begin{aligned} I^{\alpha}_{a+}[c_1\varphi_1(x) + c_2\varphi_2(x)] &= c_1 I^{\alpha}_{a+}\varphi_1(x) + c_2 I^{\alpha}_{a+}\varphi_2(x) \\ I^{\alpha}_{b-}[c_1\varphi_1(x) + c_2\varphi_2(x)] &= c_1 I^{\alpha}_{b-}\varphi_1(x) + c_2 I^{\alpha}_{b-}\varphi_2(x) \end{aligned} \tag{1.10}$$

Example 1.1 *[52] We calcule the Riemann-Liouville fractional integral of the order α of the function $\varphi(x) = x^{\beta}$ for $x > 0$ and $\beta > -1$*

We have

$$\begin{aligned} I^{\alpha}x^{\beta} &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\beta}dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (1-\frac{t}{x})^{\alpha-1}x^{\alpha-1}t^{\beta}dt \end{aligned}$$

by substitution $u = \dfrac{t}{x}$, we get

$$\begin{aligned}
I^{\alpha}x^{\beta} &= \frac{1}{\Gamma(\alpha)}\int_0^x (1-u)^{\alpha-1}x^{\alpha-1}(ux)^{\beta}x du \\
&= \frac{1}{\Gamma(\alpha)}\int_0^x (1-u)^{\alpha-1}x^{\alpha+\beta}(u)^{\beta}du \\
&= \frac{1}{\Gamma(\alpha)}\int_0^x x^{\alpha+\beta}B(\beta+1,\alpha)du \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}x^{\alpha+\beta}
\end{aligned}$$

1.3 Riemann-Liouville Fractional Derivative

Definition 1.6 *[51] For functions $\varphi(x)$ given in the interval $[a,b]$, we define the fractional derivative of order α, $0<\alpha<1$ left-handed and right-handed respectively:*

$$^{RL}D^{\alpha}_{a+}\varphi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{\varphi(t)}{(x-t)^{\alpha}}dt \tag{1.11}$$

$$^{RL}D^{\alpha}_{b-}\varphi(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{\varphi(t)}{(t-x)^{\alpha}}dt \tag{1.12}$$

Fractional derivatives (1.23) and (1.24) are usually named Riemann-Liouville derivatives .
For $\alpha>0$ we defined the fractional derivatives as:

$$\begin{aligned}
^{RL}D^{\alpha}_{a+}\varphi(x) &= (\frac{d}{dx})^n I^{n-\alpha}_{a^+}\varphi(x) \\
&= \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_a^x \frac{\varphi(t)}{(x-t)^{\alpha-n+1}}dt, n-1<\alpha<n, x>a
\end{aligned} \tag{1.13}$$

$$\begin{aligned}
^{RL}D^{\alpha}_{b-}\varphi(x) &= (-\frac{d}{dx})^n I^{n-\alpha}_{b^-}\varphi(x) \\
&= \frac{1}{\Gamma(n-\alpha)}(-\frac{d}{dx})^n\int_x^b \frac{\varphi(t)}{(t-x)^{\alpha-n+1}}dt, n-1<\alpha<n, x<b
\end{aligned} \tag{1.14}$$

The special case of the fractional derivative when $\alpha=\frac{1}{2}$ is called the semi-derivative.
Properties: The Riemann-Liouville derivatives have the following properties:

1.
$$^{RL}D^{\alpha}_{a+}I^{\alpha}_{a+}\varphi(x) = \varphi(x) \text{ and } ^{RL}D^{\alpha}_{b-}I^{\alpha}_{b-}\varphi(x) = \varphi(x) \quad \alpha > 0$$

2. If the fractional Derivatives $D^{m}_{a^+}$ and $D^{m+\alpha}_{a^+}$ ($D^{m}_{b^-}$ and $D^{m+\alpha}_{b^-}$ respectively) exist then
$$D^{m}D^{\alpha}_{a+}\varphi(x) = D^{m+\alpha}_{a^+}\varphi(x) \text{ and } D^{m}D^{\alpha}_{b-}\varphi(x) = D^{m+\alpha}_{b^-}\varphi(x), m \in \mathbb{N}$$

3. If $\varphi(x) \in L_1(a,b)$ and $\varphi_{n-\alpha}(x) \in AC^n[a,b]$, then the equality
$$I^{\alpha}_{a+}D^{\alpha}_{a+}\varphi(x) = \varphi(x) - \sum_{j=1}^{n}\frac{\varphi^{(n-j)}_{n-\alpha}(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j},$$
holds almost everywhere on $[a,b]$.

4. Let $\alpha > 0$ and $\beta > 0$ be such that $n-1 < \alpha \leq n, m-1 < \beta \leq m$ $(n,m \in \mathbb{N})$ and $\alpha + \beta < n$, and let $\varphi \in L_1(a,b)$ and $\varphi_{m-\alpha} \in AC^m([a,b])$. Then we have the following index rule
$$^{RL}D^{\alpha}_{a+} \, ^{RL}D^{\beta}_{a+}\varphi(x) = (^{RL}D^{\alpha+\beta}_{a^+}\varphi(x)) - \sum_{j=1}^{m}(D^{\beta-j}_{a^+}\varphi(a^+))\frac{(x-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)}$$

Example 1.2 *[52] Let $n-1 < \alpha < n$, $n \in \mathbb{N}$ and $\nu > -1$ the Riemann-liouville derivative of the function $\varphi(x) = (x-a)^{\nu}$ is:*

$$^{RL}D^{\alpha}(x-a)^{\nu} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^x \frac{(\tau-a)^{\nu}}{(x-\tau)^{\alpha-n+1}}d\tau \tag{1.15}$$

substituting into the formula (1.31) $\tau = a + s(x-a)$, we get

$$\begin{aligned} ^{RL}D^{\alpha}(x-a)^{\nu} &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}(x-a)^{n+\nu-\alpha}\int_0^1(1-s)^{n-\alpha-1}s^{\nu}ds \\ &= \frac{\Gamma(n+\nu-\alpha+1)B(n-\alpha,\nu+1)}{\Gamma(n-\alpha)\Gamma(\nu-\alpha+1)}(x-a)^{\nu-\alpha} \\ &= \frac{\Gamma(n+\nu-\alpha+1)\Gamma(n-\alpha)\Gamma(\nu+1)}{\Gamma(n-\alpha)\Gamma(\nu-\alpha+1)\Gamma(n+\nu-\alpha+1)}(x-a)^{\nu-\alpha} \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}(x-a)^{\nu-\alpha} \end{aligned}$$

Remark 1.1 *The Riemann-Liouville fractional derivative of a constant, in general is not equal to zero*

$$^{RL}D^{\alpha}c = \frac{c}{\Gamma(1-\alpha)}(x-a)^{-\alpha}$$

1.4 Caputo Fractional Derivative

Definition 1.7 *[56] The left and right Caputo derivatives with order $\alpha > 0$ of the given function $\varphi(t)$,in the interval $[a, b]$ are defined as:*

$$
\begin{aligned}
{}_C D^{\alpha}_{a^+}\varphi(x) &= D^{-(m-\alpha)}_{a^+}[\varphi^m(t)] \\
&= \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{\varphi^m(t)}{(x-t)^{\alpha-m-1}}dt
\end{aligned}
\tag{1.16}
$$

and

$$
{}_C D^{\alpha}_{b^-}\varphi(x) = \frac{(-1)^m}{\Gamma(m-\alpha)}\int_x^b \frac{\varphi^m(t)}{(t-x)^{\alpha-m-1}}dt \tag{1.17}
$$

respectively, where m is a positive integer satisfying $m-1 < \alpha \leq m$.

The Riemann–Liouville derivative and Caputo derivative of $\varphi(t)$ have the following relation

$$
{}^{RL}D^{\alpha}_{a^+}\varphi(x) = {}_C D^{\alpha}_{a^+}\varphi(x) + \sum_{k=0}^{m-1}\frac{\varphi^k(a)(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \tag{1.18}
$$

where $m-1 < \alpha < m$, m is a positive integer, $\varphi \in C^{m-1}[a, x]$ and $\varphi^{(m)}$ is integrable on $[a, x]$. In fact, (1.34) can be obtained by repeatedly performing integration by parts.
Furthermore, if $\varphi \in C^m[a, x]$, then from (1.32) or the Taylor series expansion, we have

$$
{}^{RL}D^{\alpha}_{a^+}[\varphi(x) - \phi(x)] = {}_C D^{\alpha}_{a^+}\varphi(x) \tag{1.19}
$$

where $\phi(x) = \sum_{k=0}^{m-1}\frac{\varphi^k(a)(x-a)^k}{\Gamma(k+1)}$, On the other hand, it is easy to find that

$$
{}^{RL}D^{\alpha}_{a^+}\varphi(x) = {}_C D^{\alpha}_{a^+}\varphi(x) \tag{1.20}
$$

The main advantage of Caputo's approach is that the initial conditions of fractional differential equations with Caputo derivatives accept the same form as for integer-order differential equations.
Properties:

1. If $\varphi_1, \varphi_2 \in C_\mu, \mu > -1, c_1, c_2 \in \mathbb{R}, {}_C D^\alpha\varphi_1(x)$ and ${}_C D^\alpha\varphi_2(x)$ exist then:

$$
D^\alpha[c_1\varphi_1(x) + c_2\varphi_2(x)] = c_1\, {}_C D^\alpha\varphi_1(x) + c_2\, {}_C D^\alpha\varphi_2(x)
$$

2. If $\varphi \in C_\mu, \mu > -1, {}_C D^\alpha\varphi(x)$ exist then:

$$
{}_C D^\alpha I^\alpha\varphi(x) = \varphi(x)
$$

3.
$$I^{\alpha}{}_{C}D^{\alpha}\varphi(x) = \varphi(x) - \sum_{k=0}^{n-1} \frac{\varphi^{k}(0)}{k!} x^{k}$$

4.
$$_{C}D^{\alpha}{}_{C}D^{\beta}\varphi(x) ={}_{C} D^{\alpha+\beta}\varphi(x)$$

Example 1.3 *[41] Let $n-1 < \alpha < n$, $n \in \mathbb{N}$ and $\nu > -1$ the Caputo fractional derivative of the function $\varphi(x) = (x-a)^{\nu}$ by using (1.34) is:*

$$\begin{aligned}
{}_{C}D^{\alpha}_{a+}(x-a)^{\nu} &= {}^{RL}D^{\alpha}_{a+}(x-a)^{\nu} - \sum_{k=0}^{m-1} \frac{\frac{d^k}{dx^k}(a-a)^{\nu}(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \\
&= {}^{RL}D^{\alpha}_{a+}(x-a)^{\nu} - \sum_{k=0}^{m-1} \frac{\frac{d^k}{dx^k}(0)^{\nu}(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \\
&= {}^{RL}D^{\alpha}_{a+}(x-a)^{\nu} \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}(x-a)^{\nu-\alpha}
\end{aligned}$$

Remark 1.2 *The Caputo fractional derivative of a constant is equal to zero*

$$_{C}D^{\alpha}k = 0$$

1.5 Laplace transforms of fractional derivatives

The Laplace transform of the equation $f(t)$ is defined as:

$$F(s) = \mathcal{L}\{f(t); s\} = \int_{0}^{\infty} e^{-st} f(t)dt \tag{1.21}$$

For the existance of (1.41) the function $f(t)$ must be of exponential order α, which means that there exist positive constants M and T such that

$$|f(t)| \leq Me^{\alpha T}, \quad \text{for all } t < T$$

The original function $f(t)$ can be restored from the Laplace transform $F(s)$ with the help of the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\{G(s); t\} = \frac{1}{2\pi\imath} \int_{c-\imath\infty}^{c+\imath\infty} e^{st}F(s)ds, \quad c = \mathrm{Re}(s) < c_0 \tag{1.22}$$

where c_0 lies in the right half plane of the absolute convergence of the Laplace integral.
The direct evolution of the inverse Laplace transform using (1.42) is often complicated; however, sometime it gives useful information on the behavior of the unknown original $f(t)$ which we look for.
Next, we present two important properties that will be useful in obtaining the Laplace transform of the fractional derivative operators.
The Laplace transform of the convolution:

$$f(t) * g(t) = \int_0^t f(t-s)g(s)ds = \int_0^t f(s)g(t-s)ds \tag{1.23}$$

is given as

$$\mathcal{L}\{f(t) * g(t); s\} = F(s)G(s) \tag{1.24}$$

where $F(s)$ and $G(s)$ are Laplace transforms of $f(t)$ and $g(t)$, respectively, and $f(t)$ and $g(t)$ are equal to zero for $t < 0$.
Another useful property which we need is the formula for the Laplace transform of the derivative of an integer n of the function $f(t)$.

$$\mathcal{L}\{f^n(t); s\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^k(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{n-k-1}(0) \tag{1.25}$$

which can be obtained from the definition of the Laplace transform (1.41) by integrating by parts under the assumption that the corresponding integrals exist.
The Laplace transform of some basic functions:

$$\begin{aligned}
\mathcal{L}(t^\nu) &= \frac{\Gamma(\nu+1)}{s^{\nu+1}}, \quad \nu > -1 \\
\mathcal{L}(e^{at}) &= \frac{1}{s-a} \\
\mathcal{L}(\sin(kt)) &= \frac{k}{s^2+k^2} \\
\mathcal{L}(\cos(kt)) &= \frac{s}{s^2+k^2} \\
\mathcal{L}(\sinh(kt)) &= \frac{k}{s^2-k^2} \\
\mathcal{L}(\cosh(kt)) &= \frac{s}{s^2-k^2}
\end{aligned} \tag{1.26}$$

Now, let us start with the Laplace transform of the fractional integral $I^\alpha f(t)$ can be rewritten as:

$$I^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} f(s) d s=\frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t). \tag{1.27}$$

It is easy to calculate that.

$$G(s)=\mathcal{L}\{t^{\alpha-1}; s\}=\Gamma(\alpha) s^{-\alpha}. \tag{1.28}$$

Hence

$$\mathcal{L}\{I^{\alpha} f(t); s\}=\frac{1}{\Gamma(\alpha)} \mathcal{L}\{t^{\alpha-1} * f(t); s\}=s^{-\alpha} F(s). \tag{1.29}$$

1.5.1 Laplace transform of the Riemann-Liouville fractional derivative

Now, we define the Laplace transform of the Riemann–Liouville derivative operator with order α,$m-1 \leq \alpha<m$ [56, 64] The optimal homotopy analysis method applied on
nonlinear time-fractional hyperbolic partial differential equations. Let

$$g(t)=D_{0,t}^{-(m-\alpha)} f(t). \tag{1.30}$$

Then

$${}^{RL}D_{0,t}^{\alpha} f(t)=g^{(m)}(t). \tag{1.31}$$

Applying (1.46) gives

$$\mathcal{L}\{{}^{RL}D_{0,t}^{\alpha} f(t); s\}=\mathcal{L}\{g^{(m)}(t); s\}=s^{m} \mathcal{L}\{g(t); s\}-\sum_{k=0}^{m-1} s^{k} g^{m-k-1}(0). \tag{1.32}$$

By (1.47) one has

$$\mathcal{L}\{g(t); s\}=\mathcal{L}\{D_{0,t}^{-(m-\alpha)} f(t); s\}=s^{-(m-\alpha)} \mathcal{L}\{f(t); s\}. \tag{1.33}$$

Combining (1.48)–(1.51) gives the Laplace transform of the Riemann–Liouville derivative as:

$$\mathcal{L}\{{}^{RL}D_{0,t}^{\alpha} f(t); s\}=s^{m} \mathcal{L}\{f(t); s\}-\sum_{k=0}^{m-1} s^{k}[{}^{RL}D_{0,t}^{\alpha-k-1} f(t)], \quad m-1 \leq \alpha<m \tag{1.34}$$

1.5.2 Laplace transform of the Caputo fractional derivative

To establish the Laplace transform formula for the Caputo fractional derivative [48, 56], we write The α-th order Caputo derivative of $f(t)$ as:

$$ {}_{C}D_{0.t}^{\alpha} f(t) = D_{0,t}^{-(m-\alpha)} g(t), \quad g(t) = f^{(m)}(t). \tag{1.35}$$

Using (1.46) and (1.49) gives

$$\begin{aligned} \mathcal{L}\{{}_{C}D_{0,t}^{\alpha} f(t); s\} &= \mathcal{L}\{D_{0,t}^{-(m-\alpha)} g(t); s\} = s^{-(m-\alpha)} \mathcal{L}\{g(t); s\} \\ &= s^{-(m-\alpha)} \left[s^{m} \mathcal{L}\{f(t); s\} - \sum_{k=0}^{m-1} s^{m-k-1} f^{(k)}(0) \right] \\ &= s^{\alpha} \mathcal{L}\{f(t); s\} - \sum_{k=0}^{m-1} s^{m-k-1} f^{(k)}(0). \end{aligned} \tag{1.36}$$

Therfore, the Laplace transform of the Caputo derivative operator reads us:

$$\mathcal{L}\{{}_{C}D_{0,t}^{\alpha} f(t); s\} = s^{\alpha} \mathcal{L}\{f(t); s\} - \sum_{k=0}^{m-1} s^{m-k-1} f^{(k)}(0), \quad m-1 < \alpha \leq m.\text{”} \tag{1.37}$$

1.6 Fourier transform of fractional derivatives

The expenential Fourier transform of a continuous function $f(t)$ absolutly integrable in $(-\infty, \infty)$ is defined by

$$\mathcal{F}\{f(t); w\} = \int_{-\infty}^{\infty} e^{\imath w t} f(t) dt \tag{1.38}$$

and the original $f(t)$ can be restored from the Fourier transform $\mathcal{F}\{f(t); w\}$ with the help of the inverse Fourier transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}\{f(t); w\} e^{-\imath w t} dw \tag{1.39}$$

The Fourier transform of the convolution

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \tag{1.40}$$

of the two functions $f(t)$ and $g(t)$ which are defined in $(-\infty, \infty)$ is equal to the product of their Fourier transforms:

$$\mathcal{F}\{f(t) * g(t); w\} = \mathcal{F}\{f(t); w\}\mathcal{F}\{g(t); w\} \tag{1.41}$$

We will use the formula (1.61) for the evaluation of the Fourier transform of the fractional derivatives [56, 57].
Another useful property of the Fourier transform which is frequently used in solving applied problems, is the Fourier transform of the derivatives of $f(t)$. Namely if $f(t), f'(t), \ldots, f^{(n)}(t)$ vanich for $t \to \pm\infty$, then the Fourier transform of the $n-$th derivative of $f(t)$ is

$$\mathcal{F}\{f^{(n)}(t); w\} = (-\imath w)^n \mathcal{F}\{f(t); w\}. \tag{1.42}$$

1.6.1 Fourier transform of the fractional integrals

Now, we investigate the Fourier transform of the fractional integral $D_{a,t}^{-\alpha}$ with the lower terminal $a = -\infty$ and $0 < \alpha < 1$. Let

$$h_+(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0 \\ 0, & t \leq 0 \end{cases}. \tag{1.43}$$

Then

$$D_{-\infty,t}^{-\alpha} f(t) = h_+(t) * f(t). \tag{1.44}$$

It is easy to calculate that

$$\mathcal{F}\{h_+(t); w\} = (\imath w)^{-\alpha}. \tag{1.45}$$

Therefore

$$\begin{aligned} \mathcal{F}\{D_{-\infty,t}^{-\alpha} f(t); w\} &= \mathcal{F}\{h_+(t) * f(t); w\} \\ &= \mathcal{F}\{h_+(t); w\}\mathcal{F}\{f(t); w\} \\ &= (\imath w)^{-\alpha}\mathcal{F}\{f(t); w\}. \end{aligned} \tag{1.46}$$

For the right fractional integral operator $D_{t,\infty}^{-\alpha}$ one has

$$D_{t,\infty}^{-\alpha} f(t) = h_+(-t) * f(t). \tag{1.47}$$

Note that

$$\mathcal{F}\{h_+(-t);w\} = (-\imath w)^{-\alpha}. \tag{1.48}$$

Hence

$$\begin{aligned}\mathcal{F}\{D_{t,\infty}^{-\alpha}f(t);w\} &= \mathcal{F}\{h_+(-t)*f(t);w\} \\ &= \mathcal{F}\{h_+(-t);w\}\mathcal{F}\{f(t);w\} \\ &= (-\imath w)^{-\alpha}\mathcal{F}\{f(t);w\}.\end{aligned} \tag{1.49}$$

1.6.2 Fourier transform of Fractional Derivatives

Next, we discuss the Fourier transform for the fractional derivatives. Suppose that $m-1<\alpha<m$, $f(t)$ is sufficiently smooth and $f^{(k)}(-\infty)(k=0,1,\ldots,m-1)$ are bounded. Then from (1.34), we see that the left Riemann–Liouville derivative and the left Caputo derivative have the same form:

$$\left.\begin{array}{r} {}^{RL}D_{-\infty,t}^{\alpha}f(t) \\ {}_{C}D_{-\infty,t}^{\alpha}f(t)\end{array}\right\} = D_{-\infty,t}^{-(m-\alpha)}f^{(m)}(t), \quad m-1<\alpha<m \tag{1.50}$$

One can similarly obtain

$$\left.\begin{array}{r} {}^{RL}D_{t,\infty}^{\alpha}f(t) \\ {}_{C}D_{t,\infty}^{\alpha}f(t)\end{array}\right\} = (-1)^m D_{-\infty,t}^{-(m-\alpha)}f^{(m)}(t), \quad m-1<\alpha<m \tag{1.51}$$

Now, let us turn to the evaluation of the Fourier transform of (1.67). From (1.66)and (1.68) one has

$$\begin{aligned}\mathcal{F}\{{}^{RL}D_{-\infty,t}^{\alpha}f(t);w\} &= \mathcal{F}\{{}^{RL}D-(m-_{-\infty,t}^{\alpha)}f^{(m)}(t);w\} \\ &= (\imath w)^{-(m-\alpha)}\mathcal{F}\{f^{(m)}(t);w\} = (\imath w)^{-(m-\alpha)}(\imath w)^m\mathcal{F}\{f(t);w\} \\ &= (\imath w)^{\alpha}\mathcal{F}\{f(t);w\}\end{aligned} \tag{1.52}$$

We can similarly obtain

$$\mathcal{F}\{{}^{RL}D_{t,\infty}^{\alpha}f(t);w\} = (-\imath w)^{\alpha}\mathcal{F}\{f(t);w\} \tag{1.53}$$

CHAPTER 2

HOMOTOPY ANALYSIS METHOD (HAM)

2.1 Introduction

In 1992, the homotopy analysis method was proposed by Shijun Liao, it is an analytical approach that provides us a convenient way to guarantee the convergence of series solution of nonlinear problems by introducing an auxiliary parameter h, called the convergence-control parameter. In 2003, Liao described systematically the basic ideas of the HAM and some applications mostly related to nonlinear ODEs.

Then, the HAM has been successfully applied by many researchers to solve a lot of nonlinear problems in science, finance and engineering. The main advantage of this method is its flexibility to select the auxiliary linear operator, the initial approximation, the auxiliary function and the auxiliary control parameter [49–54].

2.2 Description of the Homotopy Analysis Method

The homotopy analysis method is based on the concept of the homotopy, a fundamental concept in topology and differential geometry. Shortly speaking, a homotopy describes a kind of continuous variation or deformation in mathematics. Essentially, a homotopy defines a connection between different things in mathematics, which contain same characteristics in some aspects.

Definition 2.1 *A homotopy between two continuous functions $f(x)$ and $g(x)$ from a topological space X to a topological space Y is formally defined to be a continuous function $\mathcal{H} : X \times [0,1] \to Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that, if $x \in X$ then $\mathcal{H}(x;0) = f(x)$ and $\mathcal{H}(x;1) = g(x)$.*

The concept of homotopy defined above for functions can be easily expanded to equations.
For example, let us consider the general nonlinear equation:

$$N(u(x,t)) = 0, \tag{2.1}$$

where N is a nonlinear operator and $u(x,t)$ is unknown function of the independent variables x and t.

2.2.1 Zero-order deformation equation

We construct the homotopy

$$(1-q)L[\phi(x,t,q) - u_0(x,t)] = qhH(x,t)\{N(\phi(x,t))\}, \tag{2.2}$$

where L is an auxiliary linear operator with the property $L[0] = 0$, $q \in [0,1]$ is the embedding parameter in topology (called the homotopy parameter), $\phi(x,t;q)$ is the solution of (2.2) for $q \in [0,1]$, $u_0(x,t)$ is an initial guess, $h \neq 0$ is the so-called "convergence-control parameter", and $H(x,t)$ is an auxiliary function that is non-zero almost everywhere, respectively. Note that, in the frame of the homotopy, we have great freedom to choose the auxiliary linear operator L, the initial guess $u_0(x,t)$, the auxiliary function $H(x,t)$ and the value of the convergence-control parameter h.
When $q = 0$, due to the property $L[0] = 0$, we have from (2.2) the solution

$$\phi(x,t;0) = u_0(x,t), \tag{2.3}$$

When $q = 1$, since $h \neq 0$ and $H(x,t) \neq 0$, equation (2.2) is equivalent to the original nonlinear equation (2.1) so that we have

$$\phi(x,t;1) = u(x,t), \tag{2.4}$$

where $u(x,t)$ is the solution of the original equation (2.1). Thus, as the homotopy parameter q increases from 0 to 1, the solution $\phi(x,t;q)$ of (2.2) varies continuously from the initial guess $u_0(x,t)$ to the solution $u(x,t)$ of the original equation (2.1). As a result, equation (2.2) is called the zeroth-order deformation equation the base of HAM wich builds a connection (i.e. a continuous mapping/deformation) between a given nonlinear problem and a relatively
much simpler linear ones.
It should be noted that we have great freedom and large flexibility in the frame of the HAM to construct the so-called zeroth-order deformation equation using the concept homotopy in topology. Especially, the convergence-control parameter h plays an important role in the frame of the HAM.
Expanding $\phi(x;t;q)$ in Maclaurin series with respect to q i,e

$$\phi(x,t,q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m \tag{2.5}$$

converges at $q = 1$. Then, we have the approximation series

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \tag{2.6}$$

2.2.2 High-order deformation equation

Substituting the series (2.5) into the zeroth-order deformation equation (2.2) and differentiate m-times with respect to q then dividing them by $m!$ and finally setting $q = 0$, we get the following m^{th}-order deformation equation:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t)\{R_{m-1}(x,t)\} \tag{2.7}$$

where

$$R_k(x,t) = \frac{1}{k!} \frac{\partial^k}{\partial q^k} [N(u(x,t))]|_{q=0} \tag{2.8}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{2.9}$$

The so-called m$^{th}-$order deformation equation (2.7) is a linear which can be easily solved by using symbolic computation software such as Matlab.

2.3 Convergence of the Homotopy Analysis Method

Liao in [51, 52] proved that if the series solution $\sum_{k=0}^{\infty} u_k(x,t)$ given in (2.6) is convergent then it is an exact solution of the nonlinear problem (2.1)

Theorem 2.1 *[77] Let the solution components $u_0, u_1, u_2, \ldots$ be defined as given in (2.6). The series solution $\sum_{k=0}^{\infty} u_k(x,t)$ converges if $\exists 0 < \gamma < 1$ such that $\|u_{k+1}\| \leq \gamma \|u_k\|, \forall k \geq k_0$, for some $k_0 \in \mathbb{N}$.*

Theorem 2.2 *[77] Assume that the series solution* $\sum_{k=0}^{\infty} u_k(x,t)$ *defined in (2.6), is convergent to the solution* $u(x,t)$*. If the truncated series* $\sum_{k=0}^{m} u_k(x,t)$ *is used as an approximation to the solution* $u(x,t)$ *of the problem (2.1), then the maximum absolute truncated error is estimated as,*

$$\left\| u(x,t) - \sum_{k=0}^{m} u_k(x,t) \right\| \leq \frac{1}{1-\gamma} \gamma^{m+1} \left\| u_0(x,t) \right\|. \tag{2.10}$$

2.4 The valid region of h and the h-curves

The convergence control parameter h plays a key role in the HAM. The value of the auxiliary parameter h can be freely chosen to increase the convergence rate of the solution series, the freedom of selecting h is subject to the so-called valid region of h which can be obtained by the so-called h-curves. These curves have been successfully handled in many nonlinear problems. It has been found by the HAM researchers that there often exists such an effective region that certain values of h obtained from some physical quantities result in a convergent homotopy series solution. Even though such a region can always be found, with less computational effort as compared to the squared residual, by plotting the curves of these unknown quantities versus h, it is easy to discover the valid region of h which corresponds to the line segment nearly parallel to the horizontal axis. However, such kind of h-curves can not tell us the best convergence-control parameter h, which corresponds to the fastest convergent series.

2.5 Some methods based on the Homotopy Analysis Method

2.5.1 Homotopy perturbation method

In 1998 **Jihuan He** published the so-called "homotopy perturbation method". Like the early HAM, the "homotopy perturbation method" is based on constructing a homotopy equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = -qN[\phi(x,t,q)] = 0, x \in \Omega, t > 0, q \in [0,1] \tag{2.11}$$

which is exactly the same as the zeroth-order deformation equation (2.2). Like the HAM, the solution $\phi(x,t;q)$ is also expanded into Maclaurin series

$$\phi(x,t;q) = u_0(x,t) + \sum_{n=0}^{\infty} u_n(x,t) q^n \tag{2.12}$$

and the approximation is gained by setting $q = 1$, say,

$$u(x,t) = u_0(x,t) + \sum_{n=0}^{\infty} u_n(x,t) \tag{2.13}$$

Obviously, (2.18) and (2.19) are exactly the same as (2.5) and (2.6), respectively.The only difference between the "homotopy perturbation method" and the early HAM is that the embedding parameter $q \in [0, 1]$ is regarded as a "small parameter" so that the governing equation of $u_n(x,t)$ is gained by substituting the series (2.18) into (2.17) and equating the coefficients of the like-power of q.
However, in 2007 **Hayat** and **Sajid** [31] proved that, substituting the Maclaurin series the series (2.18) into (2.17),then differentiating n times with respect to the embedding parameter q, after that dividing by $n!$ and finally equating the coefficients of the like-power of q and setting $q = 0$, one obtains

$$L[u_n(x) - \chi_n u_{n-1}(x)] = -\frac{1}{k!}\frac{\partial^k}{\partial q^k}[N(u(x,t))]|_{q=0} \tag{2.14}$$

which is the same as the high-order deformation equation (2.7) exactly!
Therefore, **Sajid** and **Hayat** [83] pointed out that "nothing is new in Dr. He's approach, except the new name the homotopy perturbation method".
Unfortunately, like the early HAM, the so-called "homotopy perturbation method" can not guarantee the convergence of approximations, so that it is valid only for weakly nonlinear problems with small physical parameters, as reported by many researchers.
Abbasbandy [1], **Liang** and **Jeffrey** [46], **Turkyilmazoglu** [89] and others proved in their works that the so-called "homotopy perturbation method" is exactly the same as the early HAM. Thus, as a special case of the modified HAM when $h = -1$, the so-called "homotopy perturbation method" can not give anything new indeed. Besides, they also reveal the importance of the convergence-control parameter h in theory. The use of the convergence-control parameter h is a milestone of the HAM: it is the convergence-control parameter h which provides us a convenient way to guarantee the convergence of series solution so that the HAM becomes independent of small/large physical parameters in essence. In fact, it is the convergence-control parameter h which differs the HAM from all other analytic approximation methods.

2.5.2 Optimal homotopy asymptotic method

In 2007, **Yabushita** et al [92] first used the minimum of squared residual of governing equations to determine optimal convergence-control parameters in the frame of the HAM which suggested to

use in 2008 by **Akyildiz** and **Vajravelu** [4]. Then **Marinca** and **Herişanu** [59] suggested the so-called “optimal homotopy asymptotic method” .
Conside rthe following partial differential equation:

$$\begin{aligned} L(u(x,t)) + N(u(x,t)) &= 0, \qquad x \in \Omega \\ B(u, \frac{\partial u}{\partial t}) = 0, &\qquad x \in \Gamma \end{aligned} \tag{2.15}$$

where L is a linear operator, x, t denote independent variable, $u(x,t)$ is unknown function, $N(u(x,t))$ is a nonlinear operator, B is a boundary operator and Γ is the boundary of the domain Ω .
We first construct the homotopy

$$\begin{aligned} &(1-q)L[\phi(x,t;q)] = H(q)N(\phi(x,t;q)) \\ &B(\phi(x,t;q), \frac{\partial \phi(x,t;q)}{\partial t}) = 0, \qquad x \in \Gamma \end{aligned} \tag{2.16}$$

where $q \in [0,1]$ is an embedding parameter, $H(q)$ is a nonzero function for $q \neq 0$ and $H(0) = 0$, $\phi(x,t;q)$ is an unknown function, respectively.
Obviously, when $q = 0$ and $q = 1$ it holds

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t) \tag{2.17}$$

respectively. Therefore, when q increase from 0 to 1, the solution $\phi(x,t)$ varies from $u_0(x,t)$ to the solution $u(x,t)$. The zeroth-order problem is obtained from (2.21) when $q = 0$,

$$L(u_0(x,t)) = 0, \qquad B(u_0, \frac{\partial u_0}{\partial t}) \tag{2.18}$$

The auxiliary function $H(q)$ is chosen in the form

$$H(q) = qC_1 + q^2C_2 + q^3C_3 + \cdots, \tag{2.19}$$

where $C_1, C_2, C_3, \ldots$ are constants which can be determined later.To get an approximate solution, $\phi(x,t;q,C_i)$ is expanded in a Taylors series about q as

$$\phi(x,t;q,C_i) = u_0(x,t) + \sum_{k=1}^{+\infty} u_k(x,t;C_i)q^k, \qquad i = 1,2,3,\ldots \tag{2.20}$$

Substituting equation (2.26) into equation (2.22) and equating the coefficients of like powers of q, the first and second-order problems are given as

$$L(u_0(x,t)) = C_1 N_0(u_0(x,t)), \qquad B(u_1, \frac{\partial u_1}{\partial t}) = 0 \tag{2.21}$$

$$\begin{aligned} L(u_2(x,t)) - L(u_1(x,t)) &= C_2 N_0(u_0(x,t)) + C_1[L(u_1(x,t)) + N_1(u_0(x,t), u_1(x,t))] \\ B(u_2, \frac{\partial u_2}{\partial t}) &= 0 \end{aligned} \tag{2.22}$$

and the general governing equations for $u_k(x,t)$ are given as

$$\begin{aligned} L(u_k(x,t)) \; & -L(u_{k-1}(x,t)) = C_k N_0(u_0(x,t)) \\ & + \sum_{i=1}^{k-1} C_i[L(u_{k-i}(x,t)) + N_{k-i}(u_0(x,t), u_1(x,t), \ldots, u_{k-1}(x,t))] \\ & k = 2, 3, \ldots, \qquad B(u_k, \frac{\partial u_k}{\partial t}) = 0 \end{aligned} \tag{2.23}$$

where $N_i; i > 0$, are the coefficients of q^i in the nonlinear operator N:

$$N(u(x,t)) = N_0(u_0(x,t) + qN_1(u_0(x,t), u_1(x,t)) + q^2 N_2(u_0(x,t), u_1(x,t), u_2(x,t)) + \ldots \tag{2.24}$$

It should be emphasized that the u_k for $k > 0$ are governed by the linear equations (2.24), (2.27), (2.28) and (2.29) with the linear boundary conditions that come from the original problem, which can be easily solved. The convergence of the series(2.26) depends upon the auxiliary constants $C_1, C_2, \ldots$. If it is convergent at $q = 1$, one has

$$u(x,t;C_i) = \sum_{k=1}^{\infty} u_k(x,t;C_i) \tag{2.25}$$

Generally speaking, the solution of equation (2.21)can be determined approximately in the form

$$\tilde{u}^{(m)} = u_0(x,t) + \sum_{k=1}^{m} u_k(x,t;C_i) \tag{2.26}$$

We note that the last coefficient C_k can be a function of x, t. Substituting equation (2.32) into equation (2.22) results in the following residual:

$$R(x,t;C_i) = L(\tilde{u}^{(m)}(x,t;C_i)) + N(\tilde{u}^{(m)}(x,t;C_i)), i = 1, 2, \ldots \tag{2.27}$$

If $R(x,t;C_i) = 0$ then $\tilde{u}^{(m)}(x,t;C_i)$ happens to be the exact solution. Generally such a case will not arise for nonlinear problems, but we can minimize the functional

$$J(C_i) = \int_0^t \int_\Omega R^2(x,t;C_i)\,dx\,dt \tag{2.28}$$

where R is the residual. The unknown constants $C_i (i = 1, 2, \ldots, m)$ can be optimally identified from the conditions

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \cdots = \frac{\partial J}{\partial C_m} = 0 \tag{2.29}$$

The disadvantage of the OHAM is the requirement to solve a set of coupled nonlinear algebraic equation for the unknown convergence-control parameters $C_1, C_2, C_3, \ldots, C_m$ which will be obtained from relation (2.35). It is clear that for the low order of m, the nonlinear algebraic system can be solved with some ease but if m is large it becomes more difficult to solve.

2.5.3 Spectral Homotopy Analysis Method

The approach that use spectral methods falls under the aegis of the Spectral Analysis Method (SHAM). Motsa et al [69] was one of the first authors to consider the so-called "spectral homotopy analysis method" by using the Chebyshev pseudospectral method to solve the linear high-order deformation equations and choosing the auxiliary linear operator L in terms of the Chebyshev collocation matrices.

In theory, any a continuous function in a bounded interval can be best approximated by Chebyshev polynomial. So, the SHAM provides us larger freedom to choose the auxiliary linear operator L and initial guess in the frame of the HAM. It is valuable to expand the SHAM for nonlinear partial differential equations. Besides, it is easy to employ the optimal convergence-control parameter in the frame of the SHAM. Thus, the SHAM has great potential to solve more complicated nonlinear problems, although further modifications in theory and more applications are needed.

Chebyshev polynomial is just one of special functions. There are many other special functions such as Hermite polynomial, Legendre polynomial, Airy function, Bessel function, Riemann zeta function and so on. Since the HAM provides us extremely large freedom to choose auxiliary linear operator L and initial guess, it should be possible to develop a "generalized spectral HAM" which can use proper special functions for some nonlinear problems.

2.5.4 Generalized boundary element method

In essence, the HAM replaces a nonlinear problem by means of an infinite number of linear sub-problems, since the high-order deformation equation is always linear and governed by the auxiliary linear operator L. If the initial guess and the auxiliary linear operator L are so properly chosen that the analytic solution of the highorder deformation equation can be gained, then we obtain the analytic homotopy approximation exactly, whose convergence is guaranteed by choosing proper value of the convergence-control parameter. However, obviously, the linear high-order deformation equation can be solved by means of different numerical techniques, such as the finite difference method (FDM), the finite element method (FEM), the finite volume method (FVM), the boundary element method (BEM), and so on. So, in theory, it is very easy to combine the HAM with advanced numerical techniques. Since the numerical techniques are valid for differential equations defined in rather complicated domain, the combination of the HAM with numerical techniques can greatly enlarge the application fields of the HAM.

For example, based on the HAM, Liao [47] proposed the so-called "generalized boundary element method". The traditional BEM is often valid for a linear differential equation $L_0(u) = 0$, whose solution can be expressed by integration of a fundamental solution on the boundary. When the traditional BEM is applied to solve a nonlinear differential equation

$$L_0(u) + N_0(u) = 0$$

where $L_0(u)$ and $N_0(u)$ denote the linear and nonlinear parts of the governing equation, one often rewrites $L_0(u) = -N_0(u)$ and uses iteration approach by regarding the right-hand side term as the known ones. Unfortunately, this approach has strong restrictions on the linear operator L_0, and thus does not work if the fundamental solution of L_0 is unknown, or if the highest order of derivative of L_0 is lower than that of the governing equation, or if the linear operator L_0 does not exist at all, and so on. However, the HAM provides us extremely large freedom to choose the auxiliary linear operator L. So, in the frame of the HAM, we can always choose such a proper auxiliary linear operator L that the linear high-order deformation equation can be solved by means of the traditional BEM. Combining the HAM with the traditional BEM in this way, many nonlinear problems can be solved by means of the so-called generalized BEM. For example, by means of the generalized BEM, in 2005 Wu and Liao successfully obtained the convergent results of driven cavity viscous flows at Reynolds number up to $R_e = 10000$, governed by the exact Navier-Stokes equation. Note that, one often obtains convergent numerical result of driven cavity flow with only $R_e = 1000$ by means of traditional BEM.

2.6 Application of HAM for solving fractional differential equations

In recent years, the fractional differential equations have gained importance, due to their numerous applications in many fields of physics and engineering. Song and zhang, Cang and his co-authors used the HAM to solve nonlinear differential equation for the first time then many researchers used the same analytic approach in their works [5, 29, 76, 79], we consider the following nonlinear fractional differential equation to illustrate its basic ideas:

$$D^{\alpha}u(x,t) + N(u(x,t)) = 0 \tag{2.30}$$

where $m-1 < \alpha \leq m$, N is a nonlinear operator, D^{α} is the Caputo fractional derivative and $u(x,t)$ is unknown function of the independent variables x and t.
The so-called zero-order deformation equation can be defined as:

$$1-q)L[\phi(x,t;q) - \phi_0(x,t)] = qhH(x,t)\{D^{\alpha}u(x,t) + N(u(x,t))\} \tag{2.31}$$

where $0 \leq q \leq 1$ is the embedding parameter, $h \neq 0$ is a non zero auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, ϕ_0 the initial guess of $u(x,t)$ and L is an auxiliary linear operator that may be defined as $L = \dfrac{d^m}{dt^m}$ or $L = \dfrac{d^{\alpha}}{dt^{\alpha}}$.
When $q = 0$ the equation (2.37) becomes

$$L[\phi(x,t;0) - \phi_0(x,t)] = 0, \tag{2.32}$$

and when $q = 1$ the zero-order deformation equation reduces to the original equation (2.36). Thus as q varies from 0 to 1, the solution of (2.37) varies from the initial guess to the solution $u(x,t)$. Expanding
$\phi(x,t;q)$ in Taylor series with respect to q, one has

$$\phi(x,t;q) = \phi_0(x,t) + \sum_{n=0}^{\infty} \phi_n(x,t)q^n, \tag{2.33}$$

where

$$\phi_n(x,t) = \frac{\partial^n \phi(x,t;q)}{\partial q^n}|_{q=0}. \tag{2.34}$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameter h are so properly chosen, the series (2.39) converges at $q = 1$, one has

$$u(x,t) = \phi_0(x,t) + \sum_{n=0}^{\infty} \phi_n(x,t). \tag{2.35}$$

Differentiating Eq. (2.37) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t)\{R_{m-1}(x,t)\}, \tag{2.36}$$

where

$$R_k(x,t) = \frac{1}{k!}\frac{\partial^k}{\partial q^k}[D^{\alpha}u(x,t) + N(u(x,t))]|_{q=0}, \tag{2.37}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}. \tag{2.38}$$

The so-called $m^{th}-$order deformation equation (2.42) is a linear which can be easily solved by using symbolic computation software such as Matlab.

2.7 Numerical Examples

Example 2.1 *Consider the following nonlinear differential equation*

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad x \in \mathbb{R}, t \in [0,1]. \tag{2.39}$$

Subject to the initial condition

$$u(x,0) = 2x. \tag{2.40}$$

Which has the following exact solution:

$$u(x,t) = \frac{2x}{1+2t}. \tag{2.41}$$

Following the homotopy analysis approach, if we select the auxiliary linear operator as $L[u(x,t)] = \frac{\partial u(x,t)}{\partial t}$*, we can construct the homotopy*

$$(1-q)L[\phi(x,t,q) - \phi_0(x,t)] = qhH(x,t)\{\frac{\partial \phi(x,t)}{\partial t} + \phi(x,t)\frac{\partial \phi(x,t)}{\partial x} - \frac{\partial^2 \phi(x,t)}{\partial x^2}\}. \tag{2.42}$$

Taking $H(x,t) = 1$ *and substituting (2.5) into the homotopy (2.48) and equating the terms with identical powers of* q*, we obtain the following deformation equations:*

$$\begin{cases} \frac{\partial \phi_1(x,t)}{\partial t} & = & h(\frac{\partial \phi_0(x,t)}{\partial t} + \phi_0(x,t)\frac{\partial \phi_0(x,t)}{\partial x} - \frac{\partial^2 \phi_0(x,t)}{\partial x^2}) \\ \frac{\partial \phi_2(x,t)}{\partial t} & = & h(\frac{\partial \phi_1(x,t)}{\partial t} + N_1(\phi_0, \phi_1)) \\ \vdots & & \\ \frac{\partial \phi_k(x,t)}{\partial t} & = & h(\frac{\partial \phi_{k-1}(x,t)}{\partial t} + N_{k-1}(\phi_0, \phi_1, \ldots, \phi_{k-1})), \end{cases} \tag{2.43}$$

where

$$N_{k-1} = \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial q^{k-1}}[(\phi_0 + \ldots + q^{k-1}\phi_{k-1})\frac{\partial}{\partial x}(\phi_0 + \ldots + q^{k-1}\phi_{k-1}) - \frac{\partial^2}{\partial x^2}(\phi_0 + \ldots + q^{k-1}\phi_{k-1})]. \tag{2.44}$$

To solve the above equations, we select the initial guess as $\phi_0(x,t) = 2x$*, the first few components of the homotopy analysis solution for Equation(2.45) can be recursively derived as:*

$$\begin{cases} \phi_1(x,t) &= 2hx(1+2t) \\ \phi_2(x,t) &= 2hx(1+h)(1+2t) + 8h^2x(t+t^2) \\ \phi_3(x,t) &= 2hx(1+h)^2(1+2t) + 8h^2x(1+h)(t+t^2) + 8h^2x(1+h)(t+t^2) \\ & +32h^3x(\frac{t^2}{2}+\frac{t^3}{3}) + 4h^3x(t+2t^2+\frac{4t^3}{3}) \\ \vdots \end{cases} \tag{2.45}$$

and so on, in the same manner, the rest of components can be obtained.
The approximate solution can be given by using the formula (2.6). In figure 2.1, we plot the exact solutions, the approximate solutions obtained using the Homotopy Analysis Method (HAM) for problem (2.45)when $x = 1, N = 6$ *and* $h = -1$.

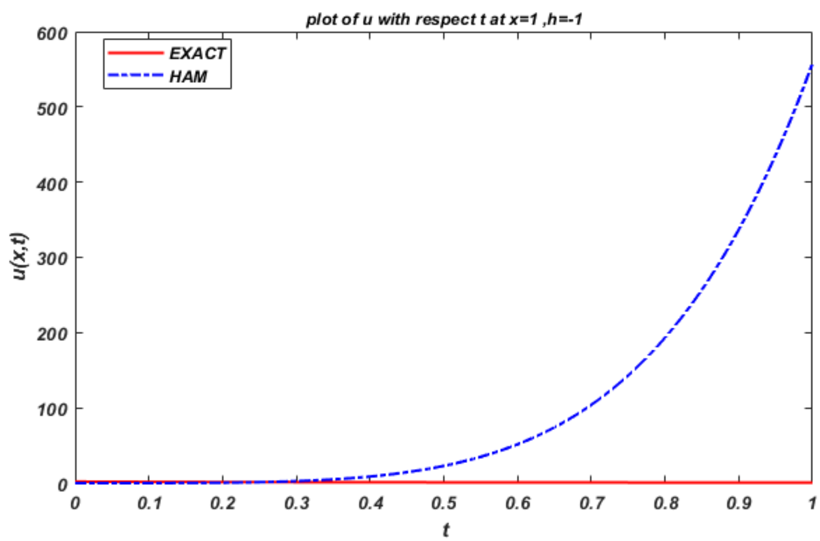

Figure 2.1: plot of approximate solution $u(x,t) = \sum_{k=0}^{N} \phi_k(x,t)$ and the exact solution of the equation (2.45) for $x = 1, h = -1$ and $N = 6$.

Example 2.2 *Consider the non-linear fractional differential equation:*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u(x,t)}{\partial x} = 0 \tag{2.46}$$

subject to the initial condition

$$u(x,0) = x. \tag{2.47}$$

The exact solution of this equation, when $\alpha = 1$ is:

$$u(x,t) = \frac{x}{1+t}. \tag{2.48}$$

To solve the equation (2.52) according to the HAM, we select the linear operator as $L(u(x,t)) = \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}$*, then, we can construct the homotopy*

$$(1-q)L[\phi(x,t,q) - \phi_0(x,t)] = qhH(x,t)\{\frac{\partial \phi(x,t)}{\partial t} + \phi(x,t)\frac{\partial \phi(x,t)}{\partial x}\} \tag{2.49}$$

Taking $H(x,t) = 1$ and substituting (2.5) into (2.55) then equating the terms of identical powers of q, we obtain the following deformation equations

$$\begin{cases} \dfrac{\partial^\alpha \phi_1(x,t)}{\partial t^\alpha} &= h(\dfrac{\partial^\alpha \phi_0(x,t)}{\partial t^\alpha} + \phi_0(x,t)\dfrac{\partial \phi_0(x,t)}{\partial x}) \\ \dfrac{\partial^\alpha \phi_2(x,t)}{\partial t^\alpha} &= h(\dfrac{\partial^\alpha \phi_1(x,t)}{\partial t^\alpha} + N_1(\phi_0, \phi_1)) \\ \vdots & \\ \dfrac{\partial^\alpha \phi_k(x,t)}{\partial t^\alpha} &= h(\dfrac{\partial^\alpha \phi_{k-1}(x,t)}{\partial t^\alpha} + N_{k-1}(\phi_0, \phi_1, \dots, \phi_{k-1})) \end{cases} \tag{2.50}$$

where

$$N_{k-1} = \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial q^{k-1}}[(\phi_0 + \ldots + q^{k-1}\phi_{k-1})\frac{\partial}{\partial x}(\phi_0 + \ldots + q^{k-1}\phi_{k-1})] \tag{2.51}$$

To solve the above equations, we select the initial guess as $\phi_0(x,t) = x$, the first few components of the homotopy analysis solution for Equation(2.52) can be recursively derived as:

$$\begin{cases} \phi_1(x,t) &= hx(1 + \dfrac{t^\alpha}{\Gamma(\alpha+1)}) \\ \phi_2(x,t) &= hx(1+h)(1 + \dfrac{t^\alpha}{\Gamma(\alpha+1)}) + 2h^2x(\dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}) \\ \phi_3(x,t) &= (1+h)[hx(1+h)(1 + \dfrac{t^\alpha}{\Gamma(\alpha+1)}) + 2h^2x(\dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)})] \\ & +2hx(hx(1+h)(\dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{t^\alpha}{\Gamma(\alpha+1)}) + 2h^2x(\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)})) \\ & +h^3x(\dfrac{t^\alpha}{\Gamma(\alpha+1)} + 2\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}) \\ \vdots & \end{cases} \tag{2.52}$$

and so on, in the same manner, the rest of components can be obtained.

The approximate solution can be given by using the formula (2.6). In figure 2.2, we plot the exact solutions, the approximate solutions obtained using the Homotopy Analysis Method (HAM) for *problem (2.52)when $x = 1, N = 5$ and $h = -2$.*

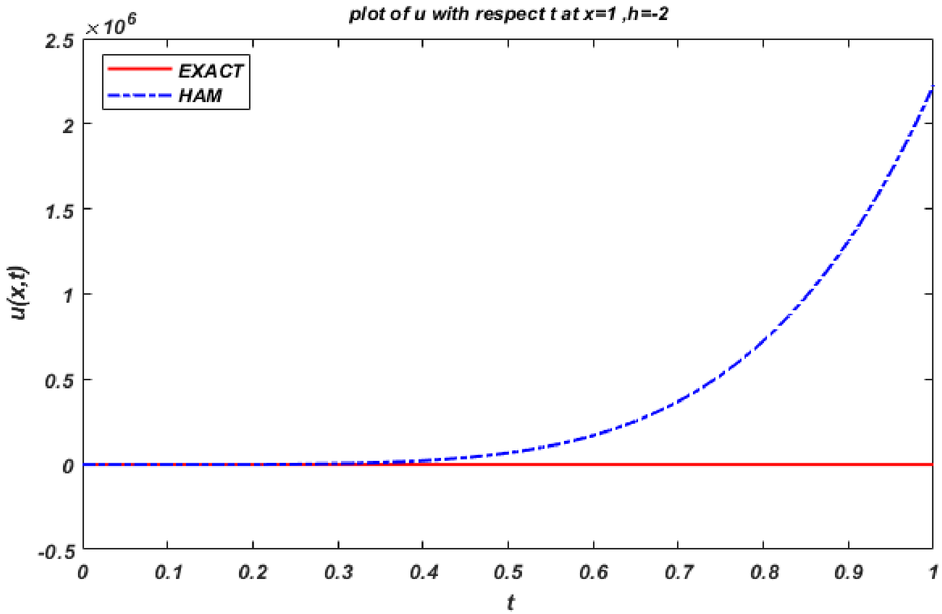

Figure 2.2: plot of approximate solution $u(x,t) = \sum_{k=0}^{N} \phi_k(x,t)$ and the exact solution of the equation (2.52) for $x = 1, h = -2$ and $N = 5$.

CHAPTER 3

THE OPTIMAL HOMOTOPY ANALYSIS METHOD APPLIED TO NONLINEAR TIME-FRACTIONAL HYPERBOLIC PDES

3.1 Introduction

In most recent times, a novel approach that relies on a stochastic arithmetic has been introduced by [72]. It has been shown that the optimization with using this approach is serving us in obtaining the optimal values for each of iteration, the parameters of convergence control, and the approximate solution deduced by the HAM [72]. Besides, Odibat [73, 76] has showed a novel procedure, established based on employing Taylor series linearization method, for designing an optimal auxiliary linear operator with its corresponding optimal initial guessing when implementing the so-called the linearization-based approach of HAM or simply LHAM. He has revealed that such two optimum contributors will accelerate the convergence of series solutions for the nonlinear fractional-order DEs (FoDEs). That is, a powerful improvement of OHAM called LHAM has been introduced in [73, 76] for solving the nonlinear FoDEs. In addition in this chapter we presente the proposed method, and we offer a further extension, enabling one to implement this new scheme on the time-fractional hyperbolic PDE which has the following form:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \gamma\frac{\partial^2}{\partial x^2}u(x,t) + N[u(x,t)], \qquad t > 0, \tag{3.1}$$

subject to the following initial and boundary conditions

$$\begin{cases} u(x,0) = f(x), \quad \frac{\partial}{\partial t}u(x,0) = g(x), \\ u(x,t) \to 0, \text{ as } |x| \to \infty, \quad t > 0, \end{cases} . \tag{3.2}$$

where $1 < \alpha \leq 2$, $\gamma \in \mathbb{R}$, N is a the nonlinear term given in the problem.
This work was published in Numerical Methods for Partial Differential Equations titled The optimal homotopy analysis method applied on nonlinear time-fractional hyperbolic partial differential equations. [10]

3.2 The HAM

In this part, for the purpose of dealing with the nonlinear problem given in (3.1), a trustworthy approach of HAM is introduced by means of deriving its basic ideas from the references [8, 50–52, 54, 55]. In accordance with how a suitable homotopy is identified, one can establish the following one:

$$(1-q)L[\Phi(x,t;q)-\phi_0(x,t)] = qhH(\frac{\partial^\alpha}{\partial t^\alpha}\Phi(x,t;q) - \gamma\frac{\partial^2}{\partial x^2}\Phi(x,t;q) - N[\Phi(x,t;q)]) \tag{3.3}$$

where $q \in [0,1]$ is the embedding parameter, h is a non zero auxiliary parameter, $\phi_0(x,t)$ is an initial guess, $H(x,t) \neq 0$ is an auxiliary function and L is an auxiliary linear operator with the following property

$$L[u(x,t)] = 0, \text{ when } u(x,t) = 0. \tag{3.4}$$

Obviously, on can note that when $q = 0$ then $\Phi(x,t;0) = \phi_0(x,t)$ whereas when $q = 1$, then $\Phi(x,t;1) = u(x,t)$. Thus, , the solution $\Phi(x,t;q)$ will varies from the initial guess $\phi_0(x,t)$ to the solution $u(x,t)$ corresponding to the increasing of q from 0 up to 1. However the solution of (3.3) can be expressed as a power series in q as follows:

$$\Phi(x,t;q) = \phi_0(x,t) + \sum_{k=1}^{\infty} q^k \phi_k(x,t). \tag{3.5}$$

If the right-hand side of (3.5) converges at $q = 1$, then, the so-called homotopy series solution will be obtained in the following form:

$$u(x,t) = \phi(x,t;1) = \phi_0(x,t) + \sum_{m=1}^{\infty} \phi_m(x,t). \tag{3.6}$$

Differentiating (3.3) m times with respect to q, then setting $q = 0$, and finally dividing them by $m!$, yields the following so-called higher order deformation equations:

$$L[\phi_m(x,t) - \chi_m\phi_{m-1}(x,t)] = hH\mathcal{R}[\phi_{m-1}(x,t)], \qquad m \geq 1, \tag{3.7}$$

where

$$\mathcal{R}[\phi_{m-1}(x,t)] = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}(\frac{\partial^\alpha}{\partial t^\alpha}\Phi(x,t;q) - \gamma\frac{\partial^2}{\partial x^2}\Phi(x,t;q) - N[\Phi(x,t;q)])|_{q=0}, \tag{3.8}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}. \tag{3.9}$$

One further principal, the HAM success is based on the suitable choice for each of the initial guess $\phi_0(x,t)$, the convergence control parameter h, the nonzero function $H(x,t)$ and also the linear operator L. In other words, such method allows one to adapt and also control the series solution convergence of the nonlinear problem, opening minds to the need of finding optimal values for all these assumptions of the problem.

3.3 The proposed design method

As a result of using the OHAM, there are several evidences showing how the optimal selections of both auxiliary the linear operator and the initial guessing control the convergence region for the series solutions and adjust the rate of such convergence for the nonlinear problem (see [73, 76]). This, indeed, motivated us to construct an appropriate auxiliary linear operator on the basis of using the Taylor series linearization of such problem. This, surely, would accelerate the convergence of series solution as reported in the same references above cited. However, the basic idea of the proposed approach begins first with a linearization of the nonlinear problem given in (3.1). That is, we firstly assume G as a function of three variables in the form $G(D^\alpha u, u_{xx}, u) = D^\alpha u - \gamma u_{xx} - N[u]$. As $\frac{\partial G}{\partial D^\alpha u}|_{t=0} = 1$ and $\frac{\partial G}{\partial u_{xx}}|_{t=0} = -\gamma$, the Taylor series linearization of G at $t=0$ can be derived as:

$$G(D^\alpha u, u_{xx}, u) \approx D^\alpha u - \gamma u_{xx} + \frac{\partial G}{\partial u}|_{t=0} u \tag{3.10}$$

Immediately, we can establish the following optimal linear operator:

$$L(u(x,t)) = \frac{\partial^\alpha}{\partial t^\alpha}u(x,t) - \gamma\frac{\partial^2}{\partial x^2}u(x,t;) - C(x)u(x,t), \tag{3.11}$$

where $1 < \alpha \leq 2$, and where $C(x)$ can be obtained according to the following relation:

$$C(x) = \frac{\partial N}{\partial u}|_{t=0}, \tag{3.12}$$

Thus, the LHAM suggests the following homotopy

$$\begin{aligned}&(1-q)[\frac{\partial^\alpha}{\partial t^\alpha}-qhH(\gamma\frac{\partial^2}{\partial x^2}+C(x))][\Phi(x,t;q)-\phi_0(x,t)]\\&=qhH(\frac{\partial^\alpha}{\partial t^\alpha}\Phi(x,t;q)-\gamma\frac{\partial^2}{\partial x^2}\Phi(x,t;q)-N[\Phi(x,t;q)]).\end{aligned} \tag{3.13}$$

Hence the solution $u(x,t)=\sum_{k=0}^{\infty}\phi_k(x,t)$ of (3.1) can be readily obtained so that the components functions $\phi_k(x,t)$, $k\geq 1$ satisfy the following deformation equations

$$\begin{aligned}&\frac{\partial^\alpha}{\partial t^\alpha}(\phi_{k+1}(x,t)-\chi_k\phi_k(x,t))\\&=h(\frac{\partial^\alpha}{\partial t^\alpha}\phi_k-\gamma\frac{\partial^2}{\partial x^2}\phi_k-N_k[\phi_0,\phi_1,\ldots,\phi_k]+(\gamma\frac{\partial^2}{\partial x^2}+C(x))(\chi_k\phi_k-\chi_{k-1}\phi_{k-1})),\end{aligned} \tag{3.14}$$

where

$$N_k[\phi_0,\ldots,\phi_k]=\frac{1}{k!}[\frac{\partial^k}{\partial q^k}(N[\phi_0+q\phi_1+q^2\phi_2+\ldots+q^k\phi_k])]|_{q=0}.$$

Taking $H=1$ in (3.13) and using $u(x,t)=\sum_{k=0}^{\infty}\phi_k(x,t)$ yield the following states

$$\begin{cases}\frac{\partial^\alpha}{\partial t^\alpha}\phi_1 &= h(\frac{\partial^\alpha}{\partial t^\alpha}\phi_0-\gamma\frac{\partial^2}{\partial x^2}\phi_0-N[\phi_0]),\\ \frac{\partial^\alpha}{\partial t^\alpha}\phi_2 &= (1+h)\frac{\partial^\alpha}{\partial t^\alpha}\phi_1-h(\gamma\frac{\partial^2}{\partial x^2}\phi_1+N_1[\phi_0,\phi_1]-(\gamma\frac{\partial^2}{\partial x^2}+C(x))\phi_1),\\ \frac{\partial^\alpha}{\partial t^\alpha}\phi_{k+1} &= (1+h)\frac{\partial^\alpha}{\partial t^\alpha}\phi_k+h(\gamma\frac{\partial^2}{\partial x^2}+C(x))(\phi_k-\phi_{k-1})\\ & \quad -h(\gamma\frac{\partial^2}{\partial x^2}\phi_k+N_k[\phi_0,\phi_1,\ldots,\phi_k]),\qquad k\geq 2\end{cases} \tag{3.15}$$

One might observe that the initial approximation ϕ_0 could be chosen to be as $\phi_0=(g(x)t+f(x))$, while for all $k\geq 1$, the other component functions $\phi_k(x,t)$ are obtained by solving (3.14). Besides, the truncated series $\sum_{k=0}^{M}\phi_k(x,t)$, is used to be as an approximate solution for problem (3.1) which its solution has the form $u(x,t)=\sum_{k=0}^{\infty}\phi_k(x,t)$. However, the following examples will show the achievement of such new approach which will be performed by obtaining an optimal auxiliary linear operator, and then designing a suitable linearization-based homotopy for some given problems.

3.4 Numerical results

This section examines the effectiveness of the proposed LHAM by providing several numerical comparisons performed between the results of using the homotopy constructed in (3.13), and the results of using the standard HAM. These comparisons are carried out by considering two nonlinear problems.

Example 3.1 *Consider the nonlinear time-fractional hyperbolic PDE given in (3.1) with* $\gamma = 0$ *and* $N[u(x,t)] = \frac{\partial}{\partial x}(u(x,t)\frac{\partial u(x,t)}{\partial x})$. *That is,*

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \frac{\partial}{\partial x}(u(x,t)\frac{\partial u(x,t)}{\partial x}), \qquad ,t>0, \tag{3.16}$$

associated with the following initial conditions:

$$u(x,0) = x^2, \quad u_t(x,0) = -2x^2. \tag{3.17}$$

where $1 < \alpha \leq 2$ *and* $x \in \mathbb{R}$. *The exact solution to problem (3.16), when* $\alpha = 2$, *is given by*

$$u(x,t) = (\frac{x}{(t+1)})^2 \tag{3.18}$$

In regarding to the standard HAM, usually it has been performed by first selecting of the auxiliary linear operator, which could be as $L(u(x,t)) = \frac{\partial^\alpha}{\partial t^\alpha}u(x,t)$, *then constructing a suitable homotopy, which is also could be as in the following form:*

$$(1-q)\frac{\partial^\alpha}{\partial t^\alpha}[\Phi(x,t;q) - \phi_0(x,t)] = qhH[\frac{\partial^\alpha}{\partial t^\alpha}\Phi(x,t;q) - \frac{\partial}{\partial x}(\Phi(x,t;q)\frac{\partial \Phi(x,t;q)}{\partial x})]. \tag{3.19}$$

Taking $H(x,t) = 1$, *and substituting* $\Phi(x,t) = \sum_{k=0}^{\infty}\phi_k(x,t)$ *into (3.19) yield the following deformation equation:*

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha}\phi_1 & = & h(\frac{\partial^\alpha}{\partial t^\alpha}\phi_0 - \frac{\partial}{\partial x}(\phi_0\frac{\partial \phi_0}{\partial x})), \\ \frac{\partial^\alpha}{\partial t^\alpha}\phi_{k+1} & = & \frac{\partial^\alpha}{\partial t^\alpha}\phi_k + h(\frac{\partial^\alpha}{\partial t^\alpha}\phi_k - N_k[\phi_0,\phi_1,\ldots,\phi_k]), \quad k \geq 1, \end{cases} \tag{3.20}$$

where

$$\begin{gathered} N_k[\phi_0,\phi_1,\ldots,\phi_k] \\ = \frac{1}{k!}[\frac{\partial^k}{\partial q^k}(\frac{\partial}{\partial x}((\phi_0 + q\phi_1 + q^2\phi_2 + q^3\phi_3 + \ldots)\frac{\partial(\phi_0 + q\phi_1 + q^2\phi_2 + q^3\phi_3 + \ldots)}{\partial x}))]|_{q=0}. \end{gathered} \tag{3.21}$$

Due to $\phi_0(x,t) = u(x,0) + u_t(x,0)t$, *one can selects the initial approximation to be as* $\phi_0(x,t) = x^2(1-2t)$, *which recursively yields the following first few components of the homotopy analysis solution:*

$$\begin{cases} \phi_1 & = & -6hx^2(\frac{t^\alpha}{\Gamma(\alpha+1)} - 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 8\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}), \\ \phi_2 & = & -6h(1+h)x^2(\frac{t^\alpha}{\Gamma(\alpha+1)} - 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 8\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}) \\ & & +72h^2x^2(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2(\alpha+3)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ & & +8(\alpha+3)\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - 16(\alpha+3)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)}). \\ \vdots & & \end{cases} \tag{3.22}$$

On the other side, the optimal linear operator could be designed according to the LHAM as follows:

$$L(u(x,t)) = \frac{\partial^\alpha}{\partial t^\alpha}u(x,t) - 2qhHu(x,t). \tag{3.23}$$

Now, as $\phi_0(x,t) = x^2(1-2t)$, the following homotopy could be constructed:

$$(1-q)(\frac{\partial^\alpha}{\partial t^\alpha} - 2qhH)[\Phi(x,t;q) - \phi_0(x,t)] = qhH[\frac{\partial^\alpha}{\partial t^\alpha}\Phi(x,t;q) - \frac{\partial}{\partial x}(\Phi(x,t;q)\frac{\partial \Phi(x,t;q)}{\partial x})]. \tag{3.24}$$

Setting $H = 1$ and using $\Phi(x,t;q) = \sum_{k=0}^{\infty} q^k \phi_k(x,t)$ yield the following deformation equations:

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha}\phi_1 &= h(\frac{\partial^\alpha}{\partial t^\alpha}\phi_0 - \frac{\partial}{\partial x}(\phi_0\frac{\partial \phi_0}{\partial x})), \\ \frac{\partial^\alpha}{\partial t^\alpha}\phi_2 &= (1+h)\frac{\partial^\alpha}{\partial t^\alpha}\phi_1 - h(N_1(\phi_0,\phi_1) - 2\phi_1), \\ \frac{\partial^\alpha}{\partial t^\alpha}\phi_{k+1} &= (1+h)\frac{\partial^\alpha}{\partial t^\alpha}\phi_k - h(N_k[\phi_0,\phi_1,\ldots,\phi_k] - 2(\phi_k - \phi_{k-1})), \quad k \geq 2, \end{cases} \tag{3.25}$$

Consequently, the following states of the homotopy analysis solution can be recursively derived from solving the above equations:

$$\begin{cases} \phi_1 &= -6hx^2(\frac{t^\alpha}{\Gamma(\alpha+1)} - 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 8\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}), \\ \phi_2 &= -6h(1+h)x^2(\frac{t^\alpha}{\Gamma(\alpha+1)} - 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 8\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}) \\ & +72h^2x^2(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2(\alpha+3)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ & +8(\alpha+3)\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - 16(\alpha+3)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)}) \\ & -12h^2x^2(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 4\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 8\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)}). \\ \vdots \end{cases} \tag{3.26}$$

In furtherance of obtaining very satisfactory numerical results through using LHAM, we intend to explore the effect of the convergent control parameter h on $u(x,0)$ through the interval $[-3,1.5]$.

Actually, several h-curves have been numerically plot in Figure 1 according to different values of x. All these results may allow one to approximately determine the convergence interval of h that, undoubtedly, necessary to find the LHAM's solution, and may also lead to choose h itself. In this example, we choose such parameter to be equal -0.5. Its, indeed, a proper choice during implementing LHAM due to enable us to successfully gain a series solution convergence of problem (3.16). For more insight, Figure 2 and Table 1 show some graphical comparisons between the exact solution and the other two approximate solutions of problem (3.16) obtained by implementing the standard HAM and the LHAM according to different values of x, when $\alpha = 2$, $M = 4$, and $h = -0.5$.

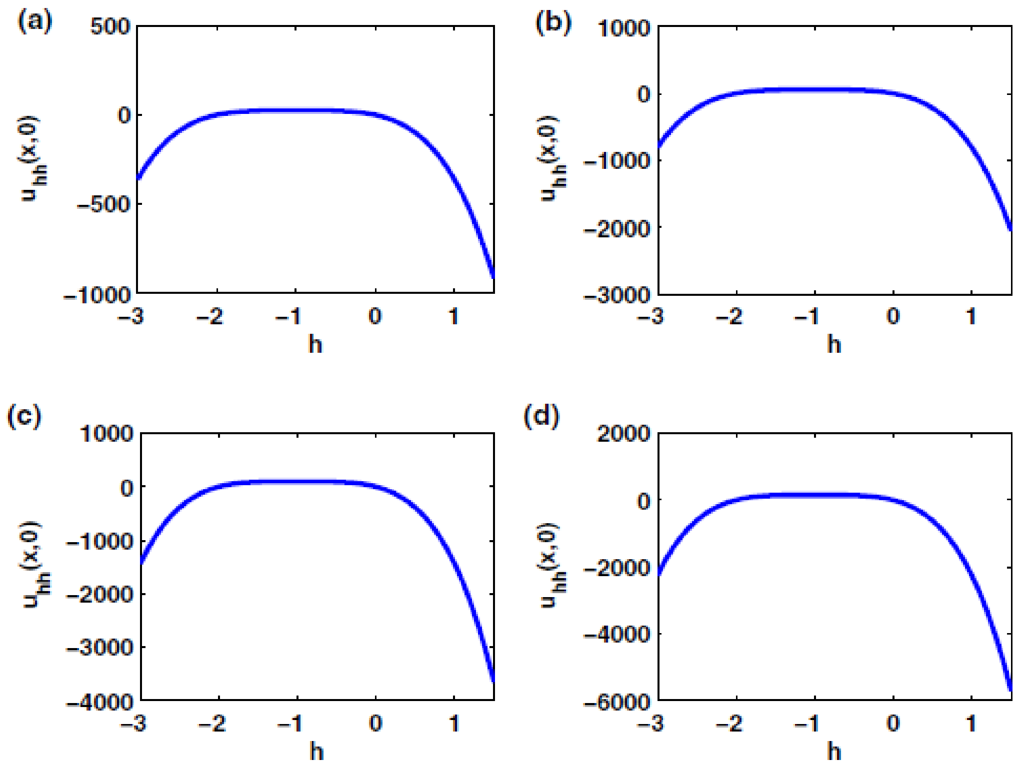

Figure 3.1:Plots of several h-curves according different values of x

For more insight, Figure 3.2 shows some graphical comparisons between the exact solution and the other two approximate solutions of problem (3.16) obtained by implementing the standard HAM and the LHAM according to different values of x, when $\alpha = 2$, $M = 4$, and $h = -0.5$.

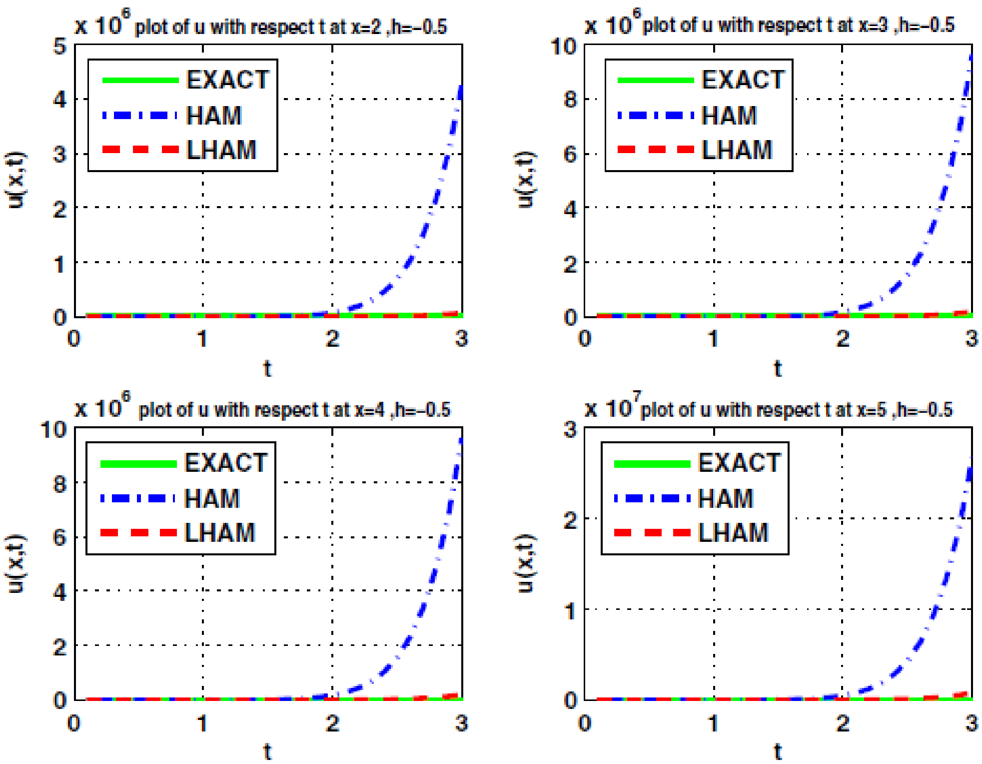

Figure 3.2: Plots of approximate solutions using HAM and LHAM versus the exact solution of (3.16) for $\alpha = 2$ and $h = -0.5$

TABLE1 The numerical solution of problem (3.16) for $\alpha = 2$ and $h = -0.5$

	$x = 2$			$x = 3$			$x = 4$			$x = 5$		
t	HAM	LHAM	Exact	HAM	LHAM	Exact	HAM	LHAM	Exact	HAM	LHAM	Exact
0.1	3.2989	3.2989	3.3058	7.4226	7.4225	7.4380	7.4226	7.4225	7.4380	20.6183	20.6181	20.6612
0.2	2.7508	2.7503	2.7778	6.1894	6.1882	6.2500	6.1894	6.1882	6.2500	17.1927	17.1895	17.3611
0.3	2.3051	2.3022	2.3669	5.1864	5.1801	5.3254	5.1864	5.1801	5.3254	14.4067	14.3905	14.7929
0.4	1.9328	1.9182	2.0408	4.3488	4.3173	4.5918	4.3488	4.3173	4.5918	12.0799	12.0040	12.7551
0.5	1.6596	1.5746	1.7778	3.7342	3.5492	4.0000	3.7342	3.5492	4.0000	10.3727	9.9168	11.1111
0.6	1.7192	1.2599	1.5625	3.8683	2.8585	3.5156	3.8683	2.8585	3.5156	10.7453	8.1511	9.7656
0.7	3.0414	0.9800	1.3841	6.8431	2.2745	3.1142	6.8431	2.2745	3.1142	19.0085	6.9358	9.6505
0.8	8.5085	0.7710	1.2346	19.1441	1.9090	2.7778	13.1441	1.9090	2.7778	53.1780	6.8527	7.7160
0.9	25.7556	0.7329	1.1080	57.9501	2.0398	2.4931	57.9501	2.0398	2.4931	160.9726	9.1390	6.9252
1.0	72.7723	1.1040	1.0000	163.7376	3.2901	2.2500	163.7376	3.2901	2.2500	454.8266	16.3053	6.2500

Example 3.2 *Consider the following time-fractional Klein-Gordon type equation:*

$$\frac{\partial^\alpha}{\partial x^\alpha}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - \frac{3}{4}u(x,t) + \frac{3}{2}u^3(x,t), \quad 1 < \alpha \leq 2, \tag{3.27}$$

associated with the following initial conditions:

$$u(x,0) = -\sec(x), \quad u_t(x,0) = \frac{1}{2}\sec(x)\tan(x). \tag{3.28}$$

Note that the exact solution of the above problem for $\alpha = 2$ is of the form:

$$u(x,t) = -\sec(x + \frac{1}{2}t). \tag{3.29}$$

Throughout using the standard HAM, we select the auxiliary linear operator as $L(u(x,t)) = \frac{\partial^\alpha}{\partial t^\alpha}u(x,t)$, and then we establish the following homotopy:

$$(1-q)\frac{\partial^\alpha}{\partial t^\alpha}[\Phi(x,t;q)-\phi_0(x,t)] = qhH[\frac{\partial^\alpha}{\partial t^\alpha}\Phi(x,t;q)-\frac{\partial^2}{\partial x^2}\Phi(x,t;q)+\frac{3}{4}\Phi(x,t;q)-\frac{3}{2}\Phi^3(x,t;q)]. \tag{3.30}$$

Taking $H(x,t) = 1$, and substituting $\Phi(x,t) = \sum_{k=0}^{\infty}\phi_k(x,t)$ into (3.24) yield the following deformation equation:

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha}\phi_1 & = & h(\frac{\partial^\alpha}{\partial t^\alpha}\phi_0 - \frac{\partial^2}{\partial x^2}\phi_0 + \frac{3}{4}\phi_0 - \frac{3}{2}\phi_0^3), \\ \frac{\partial^\alpha}{\partial t^\alpha}\phi_{k+1} & = & \frac{\partial^\alpha}{\partial t^\alpha}\phi_k + h(\frac{\partial^\alpha}{\partial t^\alpha}\phi_k - \frac{\partial^2}{\partial x^2}\phi_k + \frac{3}{4}\phi_k - \frac{3}{2}N_k[\phi_0,\phi_1,\ldots,\phi_k]), k \geq 1, \end{cases} \tag{3.31}$$

where

$$N_k[\phi_0,\phi_1,\ldots,\phi_k] = \frac{1}{k!}[\frac{\partial^k}{\partial q^k}(\phi_0 + q\phi_1 + \cdots + q^k\phi_k)^3]|_{q=0}.$$

Now, because of $\phi_0(x,t) = u(x,0) + u_t(x,0)t$, one can selects $\phi_0(x,t) = \frac{1}{2}t\sec(x)\tan(x) - \sec(x)$, *which leads us to the following few parts of solution:*

$$
\begin{cases}
\phi_1 = -\frac{9}{8}h\tan^3(x)\sec^3(x)\frac{t^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{9}{4}h\tan^2(x)\sec^3(x)\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\
\quad -\frac{1}{2}h\tan(x)\sec(x)(\tan^2(x) + \frac{19}{2}\sec^2(x) - \frac{3}{4})\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\
\quad +h\sec(x)(\tan^2(x) + \frac{5}{2}\sec^2(x) - \frac{3}{4})\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\
\phi_2 = h(1+h)(-\frac{9}{8}\tan^3(x)\sec^3(x)\frac{t^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{9}{4}\tan^2(x)\sec^3(x)\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\
\quad -\frac{1}{2}\tan(x)\sec(x)(\tan^2(x) + \frac{19}{2}\sec^2(x) - \frac{3}{4})\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \sec(x)(\tan^2(x) \\
\quad +\frac{5}{2}\sec^2(x) - \frac{3}{4})\frac{t^{\alpha}}{\Gamma(\alpha+1)}) + \frac{27}{8}h^2\sec^3(x)\tan(x)(3\tan^4(x) + 2\sec^4(x) \\
\quad +\tan^2(x)(9\sec^2(x) - \frac{1}{4}))\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{9}{4}h^2\sec^3(x)(9\tan^4(x) + 2\sec^4(x) \\
\quad +\tan^2(x)(19\sec^2(x) - \frac{3}{4}))\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{1}{2}h^2\sec(x)\tan(x)(\tan^4(x) \\
\quad +61\sec^4(x) + 36\sec^3(x) + 2\tan^2(x)(27\sec^4(x) + 9\sec(x)) - \frac{3}{2}\tan^2(x) \\
\quad -\frac{87}{8}\sec^2(x) + \frac{19}{2})\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - h^2\sec(x)(\tan^4(x) + \frac{19}{2}\sec^4(x) \\
\quad +\frac{1}{2}\tan^2(x)(63\sec^2(x) - 3) - \frac{21}{8}\sec^2(x) + \frac{9}{16})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
\quad +\frac{81}{16}h^2(\alpha+5)(\alpha+4)\sec^5(x)\tan^5(x)\frac{t^{2\alpha+5}}{\Gamma(2\alpha+6)} - \frac{81}{4}h^2(\alpha+4)(\frac{1}{8}(\alpha+11)) \\
\quad \sec^5(x)\tan^4(x)\frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)} + \frac{9}{4}h^2\sec^3(x)\tan^3(x)(\frac{9}{2}\sec^2(x)(\alpha+5) \\
\quad +\frac{1}{4}(\alpha+2)(\alpha+3)\tan^2(x) - \frac{3}{16}(\alpha+2)(\alpha+3)))\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{9}{4}h^2\sec^3(x)\tan^2(x) \\
\quad (\frac{1}{2}\sec^2(x)(\frac{5}{2}(\alpha+1)(\alpha+2) + 9) + \frac{1}{2}(\alpha+2)(\alpha+3)\tan^2(x) \\
\quad -\frac{3}{8}(\alpha+2)(\alpha+3))\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{9}{2}h^2\sec^3(x)\tan(x)(\frac{1}{2}(2\alpha+3)\tan^2(x) \\
\quad -\frac{3}{8}(2\alpha+3) + \frac{5}{2}(\alpha+1)\sec^2(x))\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{9}{2}h^2\sec^3(x)(\tan^2(x) \\
\quad +\frac{5}{2}\sec^2(x) - \frac{3}{4})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \\
\vdots
\end{cases}
\tag{3.32}
$$

From the LHAM point of view, the optimal linear operator can be designed as:

$$
L(u(x,t)) = \frac{\partial^\alpha}{\partial t^\alpha}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) - \frac{3}{2}(\frac{1}{4} - 3\sec^2(x))u(x,t). \tag{3.33}
$$

As $\phi_0(x,t) = \frac{1}{2}t\sec(x)\tan(x) - \sec(x)$, *the following homotopy might be constructed:*

$$\begin{aligned}&(1-q)(\frac{\partial^\alpha}{\partial t^\alpha} - (\tfrac{\partial^2}{\partial x^2} + \tfrac{3}{2}(\tfrac{1}{4} - 3\sec^2(x)))qhH)[\Phi(x,t;q) - \phi_0(x,t)]\\ &= qhH[\frac{\partial^\alpha}{\partial t^\alpha}\Phi(x,t;q) - \frac{\partial^2}{\partial x^2}\Phi(x,t;q) + \tfrac{3}{4}\Phi(x,t;q) - \tfrac{3}{2}\Phi^3(x,t;q)].\end{aligned} \tag{3.34}$$

Taking $H = 1$ *and using* $\Phi(x,t;q) = \sum_{k=0}^{\infty} q^k\phi_k(x,t)$ *yield the following deformation equations:*

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha}\phi_1 &= h(\frac{\partial^\alpha}{\partial t^\alpha}\phi_0 - \frac{\partial^2}{\partial x^2}\phi_0 + \frac{3}{4}\phi_0 - \frac{3}{2}\phi_0^3), \\ \frac{\partial^\alpha}{\partial t^\alpha}\phi_{k+1} &= \frac{\partial^\alpha}{\partial t^\alpha}\phi_k + h(\frac{\partial^\alpha}{\partial t^\alpha}\phi_k - \frac{\partial^2}{\partial x^2}\phi_k + \frac{3}{4}\phi_k - \frac{3}{2}N_k[\phi_0,\phi_1,\ldots,\phi_k] \\ & \quad +(\frac{\partial^2}{\partial x^2} + \frac{3}{2}(\frac{1}{4} - 3\sec^2(x))(\phi_k - \phi_{k-1})), \qquad k \geq 1, \end{cases} \tag{3.35}$$

One can recursively obtain the first few components of the solution of the problem (3.21) by solving above equations. These components are of the form:

$$\begin{cases} \phi_1 &= -\frac{9}{8}h\tan^3(x)\sec^3(x)\frac{t^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{9}{4}h\tan^2(x)\sec^3(x)\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\ & -\frac{1}{2}h\tan(x)\sec(x)(\tan^2(x) + \frac{19}{2}\sec^2(x) - \frac{3}{4})\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ & +h\sec(x)(\tan^2(x) + \frac{5}{2}\sec^2(x) - \frac{3}{4})\frac{t^\alpha}{\Gamma(\alpha+1)} \\ \phi_2 &= -h(1+h)(\frac{9}{8}\tan^3(x)\sec^3(x)\frac{t^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{9}{4}\tan^2(x)\sec^3(x)\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\ & +\frac{1}{2}\tan(x)\sec(x)(\tan^2(x) + \frac{19}{2}\sec^2(x) - \frac{3}{4})\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \sec(x)(\tan^2(x) \\ & +\frac{5}{2}\sec^2(x) - \frac{3}{4})\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{3}{2}(\frac{3}{4} - 3\sec^2(x))h(\frac{9}{8}\tan^3(x)\sec^3(x)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \\ & -\frac{81}{4}h^2(\alpha+4)(\frac{1}{8}(\alpha+11))\sec^5(x)\tan^4(x)\frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)} + \frac{9}{4}h^2\sec^3(x)\tan^3(x) \\ & -(\frac{9}{2}\sec^2(x)(\alpha+3) + \frac{1}{4}(\alpha+2)(\alpha+3)\tan^2(x) - \frac{3}{16}(\alpha+2)(\alpha+3)))\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \\ & -\frac{9}{4}h^2\sec^3(x)\tan^2(x)(\frac{1}{2}\sec^2(x)(\frac{5}{2}(\alpha+)(\alpha+2)+9) + \frac{1}{2}(\alpha+2)(\alpha+3)\tan^2(x) \\ & -\frac{3}{8}(\alpha+2)(\alpha+3))\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{9}{2}h^2\sec^3(x)\tan(x)(\frac{1}{2}(2\alpha+3)\tan^2(x)\frac{3}{8}(2\alpha+3) \\ & +\frac{5}{2}(\alpha+1)\sec^2(x))\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\frac{9}{2}h^2\sec^3(x)(\tan^2(x) + \frac{5}{2}\sec^2(x) - \frac{3}{4})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & -\frac{9}{4}\tan^2(x)\sec^3(x)\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{1}{2}\tan(x)\sec(x)(\tan^2(x) + \frac{19}{2}\sec^2(x) - \frac{3}{4})\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ & -\sec(x)(\tan^2(x) + \frac{5}{2}\sec^2(x) - \frac{3}{4}))\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{81}{16}h^2(\alpha+5)(\alpha+4)\sec^5(x)\tan^5(x)\frac{t^{2\alpha+5}}{\Gamma(2\alpha+6)}. \\ \vdots & \end{cases} \tag{3.36}$$

In order to explore the effect of the convergent control parameter h on $u(x,0)$, we plot several h-curves, as shown in Figure 3.3, according to different values of x on $[-3,3]$. In view of these results, we may choose also $h=-0.5$ because it is a proper choice through implementing LHAM. Likewise Example 1, we have performed some numerical comparisons, as shown in Figure 3.4 and Table 2, between the exact solution and the other two approximate solutions of problem (3.27) obtained by implementing the standard HAM and the LHAM according to different values of x, when $\alpha=2, M=3$, and $h=-1$.

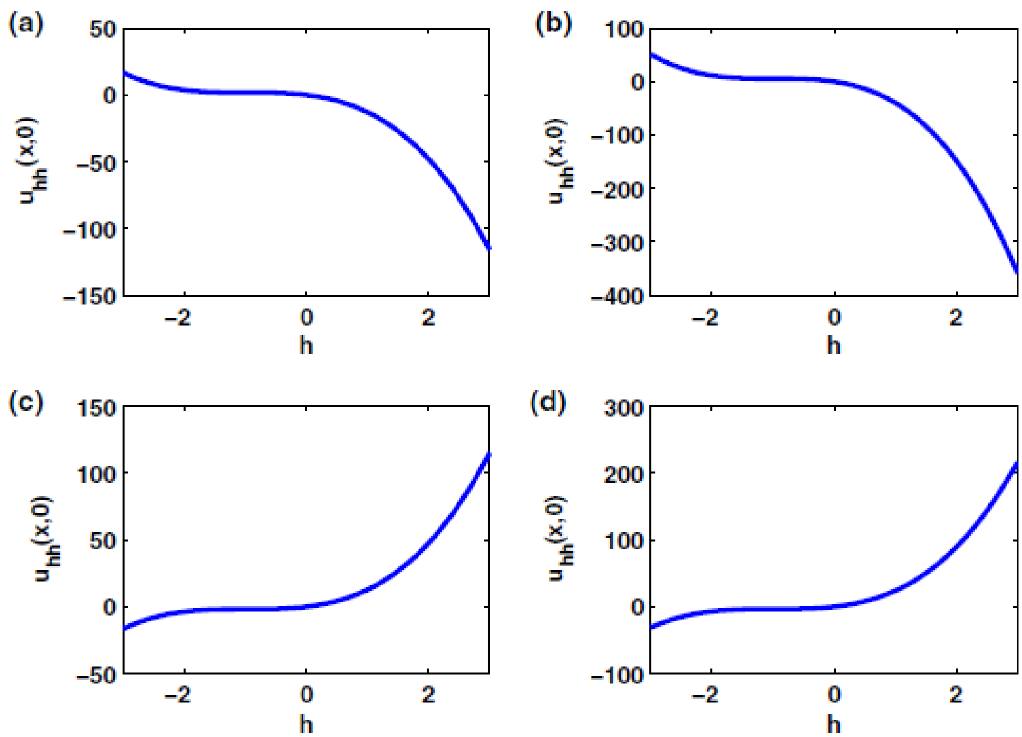

Figure 3.3: Plots of several h-curves according different values of x

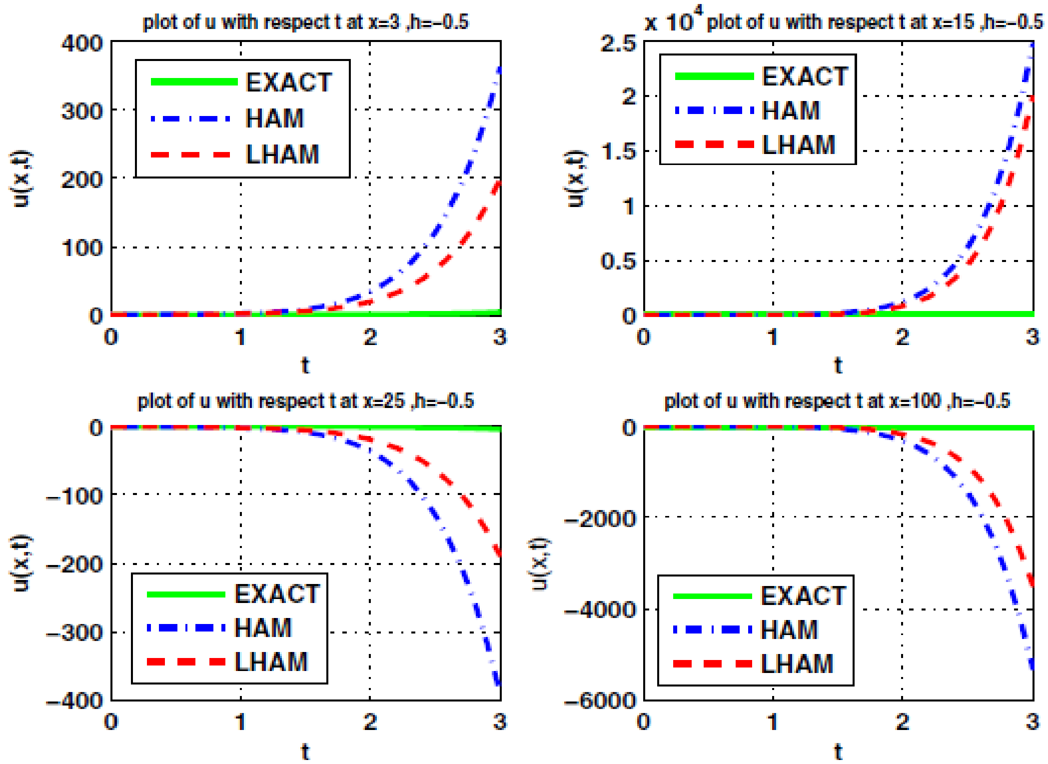

Figure 3.4: Plots of two approximate solutions using HAM and LHAM versus the exact solution of (3.27) for $\alpha = 2$ and $h = -1$.

Table 2 The numerical solutions of problem (3.27) for $\alpha = 2$ and $h = -0.5$.

	$x = 3$			$x = 15$			$x = 25$			$x = 100$		
t	HAM	LHAM	Exact	HAM	LHAM	Exact	HAM	LHAM	Exact	HAM	LHAM	Exact
0.0	1.0101	1.0101	1.0101	1.3163	1.3163	1.3163	−1.0089	−1.0089	−1.0089	−1.1597	−1.1597	−1.1597
0.1	1.0255	1.0255	1.0042	1.3991	1.3990	1.2638	−1.0237	−1.0237	−1.0034	−1.2094	−1.2094	−1.1280
0.2	1.0579	1.0576	1.0009	1.5434	1.5413	1.2183	−1.0554	−1.0552	−1.0005	−1.2943	−1.2935	−1.1006
0.3	1.1089	1.1077	1.0000	1.7703	1.7576	1.1788	−1.1056	−1.1045	−1.0001	−1.4228	−1.4179	−1.0772
0.4	1.1812	1.1774	1.0017	2.1206	2.0709	1.1445	−1.1770	−1.1730	−1.0023	−1.6100	−1.5914	−1.0574
0.5	1.2795	1.2690	1.0059	2.6729	2.5204	1.1149	−1.2742	−1.2633	−1.0069	−1.8828	−1.8274	−1.0408
0.6	1.4110	1.3866	1.0127	3.5725	3.1757	1.0894	−1.4046	−1.3790	−1.0142	−2.2882	−2.1477	−1.0273
0.7	1.5869	1.5358	1.0221	5.0747	4.1598	1.0677	−1.5800	−1.5257	−1.0241	−2.9051	−2.5874	−1.0166
0.8	1.8241	1.7250	1.0343	7.6059	5.6862	1.0494	−1.8175	−1.7118	−1.0368	−3.8607	−3.2029	−1.0086
0.9	2.1467	1.9665	1.0495	11.8502	8.1142	1.0342	−2.1428	−1.9488	−1.0525	−5.3526	−4.0834	−1.0033
1.0	2.5887	2.2771	1.0679	18.8652	12.0303	1.0220	−2.5913	−2.2534	−1.0714	−7.6771	−5.3668	−1.0005

Based on the numerical results illustrated in Figures 3.2 and 3.4 and Tables 1 and 2, one can obviously deduce that the approximate solution obtained using the LHAM is precisely better than the other one obtained using the standard HAM.

3.5 Discussion and conclusion

In this chapter, an optimal approach of HAM, called the LHAM, is presented to solve nonlinear time-fractional hyperbolic PDEs. The proposed approach employs the Taylor series approximation of the nonlinear PDEs in order to obtain an optimal auxiliary linear operator and its corresponding initial approximation. All illustrative examples show that the results obtained by using the LHAM outperforms the other results obtained by the standard HAM.

CHAPTER 4

THE OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR SOLVING TWO STRONGLY BENCHMARK OSCILLATORY PROBLEMS WITH FRACTIONAL ORDER

4.1 Introduction

The Optimal Homotopy Asymptotic Method (OHAM) is reliable, straightforward, and effective tool for offering accurate analytical approximate solutions to lots of strongly nonlinear problems [11, 61, 80]. Besides, it was revealed that its key characteristic is its ability to optimally control the convergence of approximate series solutions [11, 61, 80]. In this section we employs this method to provide approximate analytic solution for two strongly benchmark nonlinear oscillatory problems with fractional order through establishing an optimal auxiliary linear operator, an auxiliary function, and also an auxiliary control parameter. These two nonlinear oscillators are: The fractional-order Duffing-relativistic oscillator and the fractional-order stretched elastic wire oscillator (with a mass attached to its midpoint).
This work was published in Mathematics titled The Optimal Homotopy Asymptotic Method for Solving Two Strongly Fractional-Order Nonlinear Benchmark Oscillatory Problems [86].

4.2 The Homotopy Asymptotic Method

The HAM is a common analytical approach for solving both weakly and strongly nonlinear problems. In pursuance of this method, approximate series solutions are accurately obtained even if these problems have fractional-order derivatives [78]. In this part, a modified approach of HAM is presented for the purpose of handling some types of nonlinear FoDEs that have the following general form:

$$D^{\alpha}u(t) = \mathcal{N}(t, u(t)), \qquad 1 < \alpha \leq 2, \qquad t > 0, \tag{4.1}$$

subject to the initial conditions

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{4.2}$$

where N is a nonlinear operator, $u(t)$ is an unknown continuous function of the independent variable t, and D^{α} is the Caputo differential operator of order α that can be defined, as follows:

$$D^{\alpha}f(t) = J^{m-\alpha}D^{m}f(t), \tag{4.3}$$

where $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, and the real function $f \in C^{m}[0,T]$. Here, D^{m} is the traditional integer-order differential operator of order m, and J^{μ} is the Riemann-Liouville integral operator of order $\mu = m - \alpha > 0$, which can be defined by:

$$J^{\mu}f(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\eta)^{\mu-1}f(\eta)\,d\eta, \qquad t > 0. \tag{4.4}$$

For more insight regarding further properties that are associated with these two operators, Caputo and Riemann-Liouville operators, the reader may referred to [41]. However, in view of the HAM, the following homotopy can be established:

$$(1-q)L[\Phi(t;q) - u_0] = qhH(t)\big(D^{\alpha}\Phi(t;q) - \mathcal{N}(t, \Phi(t;q))\big), \tag{4.5}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non zero auxiliary parameter, u_0 is an initial guess, $h(t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, and $\phi(t;q)$ is an unknown function . Observe that homotopy (4.5) becomes simply $L[\Phi(t;0) - u_0] = 0$ when $q = 0$, whereas it returns back to its original nonlinear form given in (4.1) when $q = 1$. Therefore, as q differs from 0 up to 1, $\Phi(t;q)$ differs from the initial guess u_0 up to the exact solution $u(t) = \phi(t;1)$ that constructed for (4.1). Regardless, $\Phi(t;q)$ could be expanded with respect to q by Taylor series as follows:

$$\Phi(t;q) = u_0 + \sum_{m=1}^{\infty} q^{m}u_m(t). \tag{4.6}$$

Note that, whenever the series $u_0 + \sum_{m=1}^{\infty} q^{m}u_m(t)$ converges at $q = 1$, then the following homotopy series solution could be established:

$$u(t) = \Phi(t;1) = u_0 + \sum_{m=1}^{\infty} u_m(t), \tag{4.7}$$

which should satisfy (4.1). In the same vein, one can track the same procedure that was established in [11, 61, 80] for the purose of identifying each term of u_m's that given in series (4.6). Now, substituting series (4.6) in homotopy (4.5) and then equating the coefficients of the similar powers of the q yields the following m^{th}-order deformation equation:

$$L\left(u_m(t) - \chi_m u_{m-1}(t)\right) = hH(t)\mathcal{R}[u_{m-1}(t)], \qquad m \geq 1, \tag{4.8}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases} \tag{4.9}$$

and

$$\mathcal{R}[u_{m-1}(t)] = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\Big[D^{\alpha}\Phi(t;q) - \mathcal{N}(t,\Phi(t;q))\Big]_{q=0}. \tag{4.10}$$

In light of the previous considerations, a further modification has been proposed for the HAM that can be employed simply and directly for obtaining series solutions for nonlinear FoDEs. It has clearly appeared that the success of this modification relies on the favorable choice for each of the auxiliary parameter h, the auxiliary function $H(t)$, and the auxiliary linear operator L. For more details about the proper selection of the auxiliary function $H(t)$, and the auxiliary control parameter h, the reader may refer to the references [3, 70, 90].

4.3 An OHAM for fractional-order nonlinear oscillators

This section targets introducing the OHAM for the purpose of generally establishing approximate solutions for the strongly fractional-order nonlinear oscillatory problems that can be expressed by the following form [79]:

$$D^{\alpha}u(t) + f(u(t)) = 0, \tag{4.11}$$

subject to the following initial conditions:

$$u(0) = u_0, \qquad u'(0) = 0, \tag{4.12}$$

where D^{α} is the Caputo operator of order $1 < \alpha \leq 2$, f is a nonlinear function, and $u(t)$ is an unknown continuous function of the independent variable t. First of all, we set out to rewrite the nonlinear oscillator given in (4.11) to be in the following form:

$$F(D^{\alpha}u(t), u(t)) = 0, \tag{4.13}$$

where F is a nonlinear function. The idea of constructing our proposed algorithm relies initially on choosing an optimal auxiliary linear operator by taking into account that the nonlinear function F can be written by a Taylor series at $t = 0$. Therefore, making a linearization of the function F at $t = 0$ yields the following linear approximation:

$$F(D^{\alpha}u(t), u(t)) \cong F(D^{\alpha}u(0), u(0)) + \frac{\partial F}{\partial D^{\alpha}u}(D^{\alpha}u(0), u(0))D^{\alpha}u(t) + \frac{\partial F}{\partial u}(D^{\alpha}u(0), u(0))u(t). \tag{4.14}$$

Accordingly, solving straightforwardly the algebraic equation $F(D^{\alpha}u(0), u(0)) = 0$ for $D^{\alpha}u(0)$ leads us to design an optimal auxiliary linear operator L in the form:

$$L[u(t)] = D^{\alpha}u(t) + k(u_0)u(t), \tag{4.15}$$

where the constant $k(u_0)$, which only depends on u_0, can be computed according to the following formula:

$$k(u_0) = \frac{\frac{\partial F}{\partial u}(u_0^{\alpha}, u_0)}{\frac{\partial F}{\partial D^{\alpha}u}(u_0^{\alpha}, u_0)}, \tag{4.16}$$

where $u_0^{\alpha} = D^{\alpha}u(0)$. One should observe that the designed linear operator is an optimal operator in the sense that the approximation $L[u(t)] = D^{\alpha}u(t) + k(u_0)u(t)$ is the best linear approximation to the function $F(D^{\alpha}u(t), u(t))$ near $t = 0$ [79]. In a subsequent step, the optimal approach of HAM for the nonlinear fractional-order oscillator problem given in (4.11) can be established by employing the linear operator given in (4.15) as proposed in the following homotopy:

$$(1-q)\big[D^{\alpha} + qk(u_0)hH(t)\big]\Big[\Phi(t;q) - u_0\Big] = qhH(t)F\big(D^{\alpha}\Phi(t;q), \Phi(t;q)\big). \tag{4.17}$$

It is worth noting that the proposed approach divides the linear operator $L[u(t)]$ into two main parts, namely $D^{\alpha}[u(t)]$ and $k(u_0)[u(t)]$, and it furthermore embeds them into the homotopy as $(D^{\alpha} + qk(u_0)hH(t))[u(t)]$. Besides, it utilizes u_0 as an initial approximation to simplify computations. However, the last step that allows to successfully implement OHAM considers that the nonlinear fractional-order oscillatory problem given in (4.11) has an approximate solution of the form: $u(t) = u_0 + \sum_{m=1}^{\infty} u_m(t)$. This solution can be easily obtained where the solution components u_m's should satisfy the following m^{th}-order deformation equation :

$$D^{\alpha}(u_m(t) - \chi_m u_{m-1}(t)) + k(u_0)hH(t)(\chi_m u_{m-1}(t) - \chi_{m-1}u_{m-2}(t)) = hH(t)R[u_{m-1}(t)], \tag{4.18}$$

where

$$\mathcal{R}[u_{m-1}(t)] = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\Big[F\big(D^{\alpha}\Phi(t;q), \Phi(t;q)\big)\Big]_{q=0}, \tag{4.19}$$

and where χ_m is previously defined in (9) such that $m \geq 1$..

4.4 Test problems

This section employs the OHAM to provide approximate analytic solutions for two strongly fractional-order nonlinear benchmark oscillatory problems with , namely: The fractional-order Duffing-relativistic oscillator, and the fractional-order stretched elastic wire oscillator (with a mass attached to its mid-point). All theoretical findings in this section have been numerically performed using MATLAB software package.

Example 4.1 *Consider the following fractional-order Duffing-relativistic oscillator:*

$$D^{\alpha}u(t) + \delta u(t) + \gamma u^3(t) = 0 \tag{4.20a}$$

subject to the following initial conditions:

$$u(0) = A, \qquad u'(0) = 0 \tag{4.20b}$$

where $1 < \alpha \leq 2$, δ is a constant, and γ is a positive non-dimensional cœfficient of nonlinearity that does need to be small [44], If one selects the linear operator L to be as $L = D^{\alpha}$, then the standard homotopy will be established as:

$$(1-q)D^{\alpha}[\Phi(t;q) - A] = qhH(t)(D^{\alpha}\Phi(t;q) + \delta\Phi(t;q) + \gamma\Phi^3(t;q)) \tag{4.21}$$

Thus, taking $H(t) = 1$ makes all components of the standard HAM solution to be gained by collecting the terms with similar powers of q via the following equation:

$$\begin{aligned}(1-q)D^{\alpha}[u_0 + qu_1(t) + q^2u_2(t) + \cdots] &= qh(D^{\alpha}(u_0 + qu_1(t) + q^2u_2(t) + \cdots) && (4.22)\\ &\quad +\delta(u_0 + qu_1(t) + q^2u_2(t) + \cdots) && (4.20)\\ &\quad +\gamma(u_0 + qu_1(t) + q^2u_2(t) + \cdots)^3)\end{aligned}$$

The optimal linear operator then has the following form:

$$L[u(t)] = D^{\alpha}u(t) + (\delta + 3\gamma A^2)u(t) \tag{4.23}$$

and, morever, the optimal homotopy when $u_0 = A$, will be of the form:

$$(1-q)[D^{\alpha} + q(\delta + 3\gamma A^2)hH(t)][\Phi(t;q) - A] = qhH(t)(D^{\alpha}\Phi(t;q) + \delta\Phi(t;q) + \gamma\Phi^3(t;q)) \tag{4.24}$$

Consequently the OHAM's solution can be obtained as $u(t) = A + \sum_{m=1}^{\infty} u_m(t)$, in which all components $u_m(t)$ of that solution satisfy the following m^{th}-order deformation equation:

$$D^{\alpha}(u_m(t) - \chi_m u_{m-1}(t)) + (\delta + 3\gamma A^2)hH(t)(\chi_m u_{m-1}(t) - \chi_{m-1}u_{m-2}(t)) = hH(t)\mathcal{R}[u_{m-1}(t)] \tag{4.25}$$

where

$$\mathcal{R}[u_{m-1}(t)] = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[D^{\alpha}\Phi(t;q) + \delta\Phi(t;q) + \gamma\Phi^3(t;q)]_{q=0} \tag{4.26}$$

and where $m \geq 1$.

Again, taking $H(t) = 1$ makes, this time all components of the OHAM's solution to be obtained by collecting the terms with similar powers of q via other equations that could be expressed by:

$$\begin{aligned}(1-q)[D^{\alpha} + q(\delta + 3\gamma A^2)h][u_0 + qu_1(t) + q^2u_2(t) + \cdots] &= qh(D^{\alpha}(u_0 + qu_1(t) + q^2u_2(t) + \cdots) && (4.27)\\ &\quad +\delta(u_0 + qu_1(t) + q^2u_2(t) + \cdots)\\ &\quad +\gamma(u_0 + qu_1(t) + q^2u_2(t) + \cdots)^3),\end{aligned}$$

which, consequently, implies the following recursive states:

$$\begin{cases} D^\alpha u_1(t) &= h(\delta u_0 + \gamma u_0^3) \\ D^\alpha u_2(t) &= D^\alpha u_1(t) - (\delta + 3\gamma A^2)hu_1(t) + h(D^\alpha u_1(t) + \delta u_1(t) + 3\gamma u_0^2 u_1(t) \\ \vdots & \\ D^\alpha u_k(t) &= D^\alpha u_{k-1}(t) - ((\delta + 3\gamma A^2)h(u_{k-1}(t) - u_{k-2}(t)) + h\mathcal{R}[u_{k-1}(t)], \end{cases} \tag{4.28}$$

subject to the initial conditions:

$$u_0(0) = A, \quad u_m'(0) = 0, \tag{4.29}$$

where $m = 1, 2, \ldots$. *Applying the operator* J^α *on (4.28) implies:*

$$\begin{cases} u_1(t) &= \dfrac{hA(\delta + \gamma A^2)}{\Gamma(\alpha+1)} t^\alpha \\ u_2(t) &= \dfrac{hA(1+h)(\delta + \gamma A^2)}{\Gamma(\alpha+1)} t^\alpha. \\ \vdots & \end{cases} \tag{4.30}$$

In similar manner, we can obtain the rest of all the components using MATLAB software code. In addition, the series solutions expression can be then written in the form:

$$u(t) \simeq u_0 + \sum_{m=1}^{N} u_m(t) = u_0 + u_1(t) + u_2(t) + \cdots, \tag{4.31}$$

or

$$\begin{aligned} u(t) \quad \simeq \quad & A + Ah(2+h)(\delta + \gamma A^2)(\frac{t^\alpha}{\Gamma(\alpha+1)}) + Ah(1+h)^2(\delta + \gamma A^2)(\frac{t^\alpha}{\Gamma(\alpha+1)}) \\ & + h(\delta + 3\gamma A^2)Ah(1+h)(\delta + \gamma A^2)(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}) \\ & + 3\gamma A^3 h^2 (\delta + \gamma A^2)^2 (\frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2 \Gamma(3\alpha+1)}) + \cdots. \end{aligned} \tag{4.32}$$

In connection with the selection of the value of the parameter h *or the so-called the convergent-control parameter* h, *in Figure 4.1 we draw it corresponding* h*-curves according to different values of* A, δ,γ *and* α. *In view of such figure, we deduce the convergence interval that guarantees, in return, a convergence of the approximate solution* $u(t)$. *Here, such interval is deduced to be as* $[-3, 3]$ *so that the value of* h *can be chosen within this scope. For more simplification, one may choose the auxiliary function* $H(t)$ *to be, e.g., equal* 1. *However, Figure 4.2 shows approximate solutions* $u(t)$ *for problem (4.20) by using the OHAM for different values of* $A, \alpha, h, \delta,$ *and* γ. *For more effective practice, we perform some graphical comparisons between the OHAM and the HAM as shown in Figure 4.3. Obviously, these comparisons reveal that the approximate solutions obtained* *by such methods are very close to each other, confirms the efficient and robustness of the OHAM. The reader may refer to the references [3, 70, 71] to obtain a complete overview about the h-curves and how they can be utilized to determine the admissible values of the parameter h.*

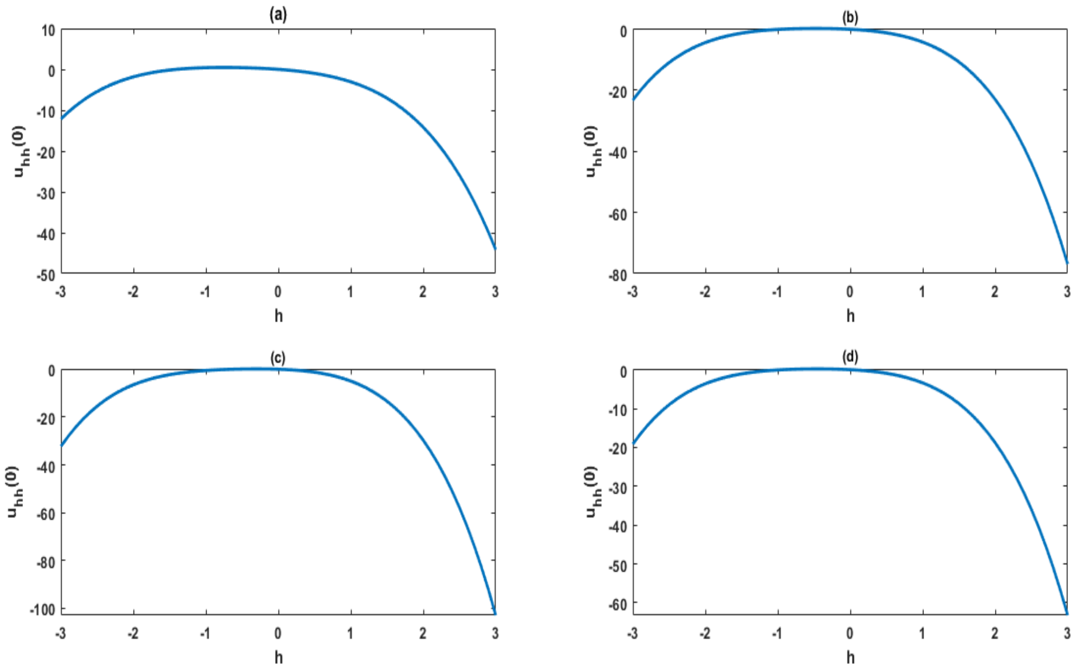

Figure 4.1- Plots of several h-curves according the values: (a) $A = 0.75, \gamma = \delta = 1$ *(b)* $A = 1.25, \gamma = 0.3, \delta = 0.5$ *(c)* $A = 2, \gamma = \delta = 1$ *(d)* $A = 1.5, \gamma = \delta = 1$.

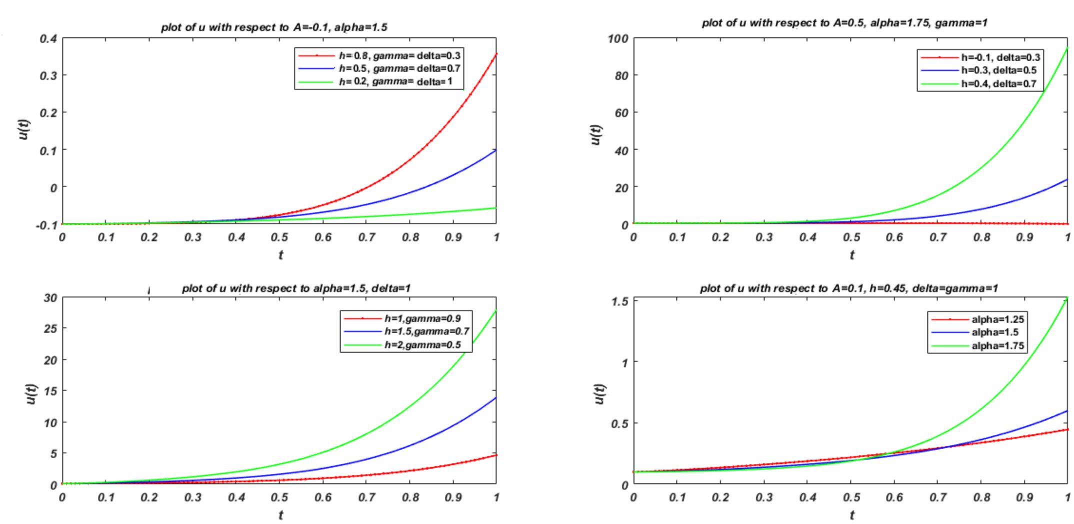

Figure 4.2- Plots of the OHAM's solutions $u(t)$ *according to different values of* A, δ, γ *and* α.

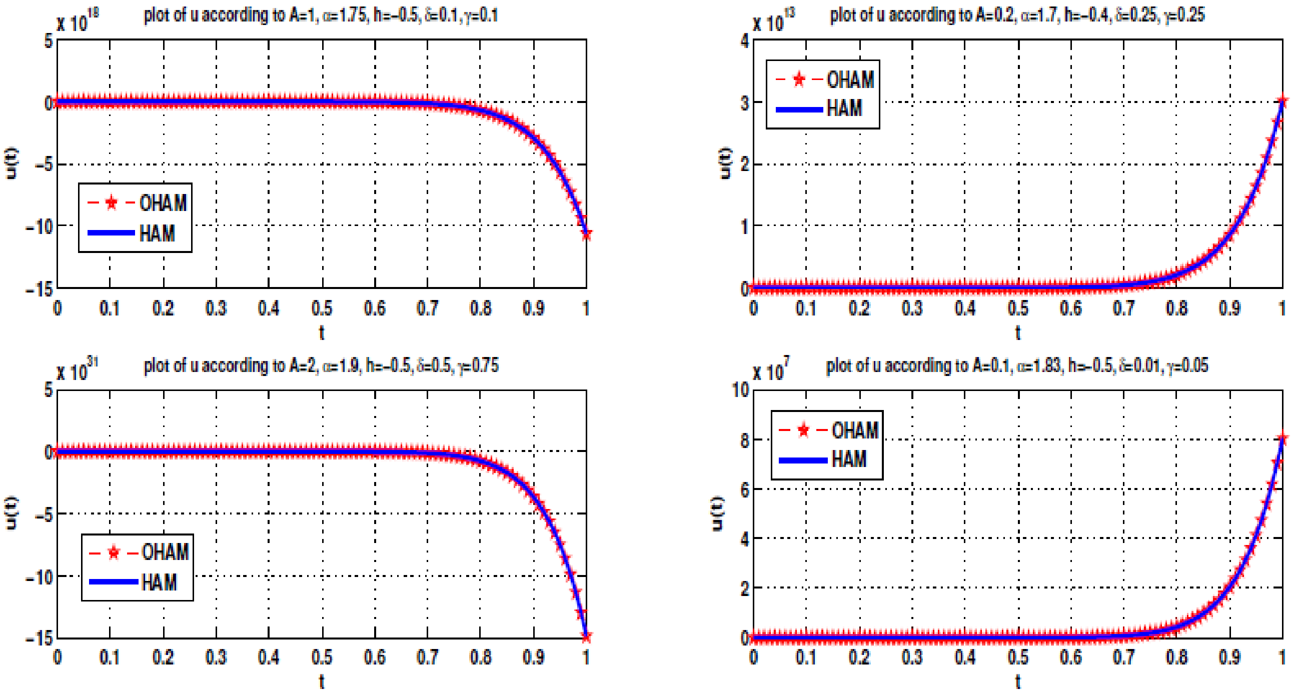

Figure 4.3- Plots of approximate solutions using OHAM and HAM for different values of $A, \gamma, \delta, \alpha$ and h.

Example 4.2 *Consider the following nonlinear fractional-order problem that represents the motion equation of the stretched elastic wire oscillator (with a mass attached to its midpoint):*

$$D^{\alpha}u(t) + u(t) - \frac{\lambda u(t)}{\sqrt{1+u^2(t)}} = 0, \tag{4.33a}$$

subject to the following initial conditions:

$$u(0) = A, \quad u'(0) = 0, \tag{4.33b}$$

where $0 < \lambda \leq 1$ and $1 < \alpha \leq 2$.
The optimal linear operator here is of the following form :

$$L[u(t)] = D^{\alpha}u(t) + (1 - \lambda(1+A^2)^{\frac{-1}{2}} + \lambda A^2((1+A^2)^{\frac{-3}{2}})u(t). \tag{4.34}$$

Furthermore, the optimal homotopy, when $u_0 = A$, is then of the form :

$$\begin{aligned}(1-q)(D^{\alpha} + q(1 - \lambda(1+A^2)^{\frac{-1}{2}} + \lambda A^2((1+A^2)^{\frac{-3}{2}})hH(t))[\Phi(t;q) - A] \\ = qhH(t)[D^{\alpha}\Phi(t;q) + \Phi(t;q) - \lambda\Phi(t;q)(1+\Phi^2(t;q))^{\frac{-1}{2}}].\end{aligned} \tag{4.35}$$

Consequently, the OHAM's solution can be formulated as $u(t) = A + \sum_{m=1}^{\infty} u_m(t)$, *in which* u_m*'s hold the following* m^{th}*-order deformation equation :*

$$\begin{aligned} D^{\alpha}(u_m(t) - \chi_m u_{m-1}(t)) + (1 - \lambda(1+A^2)^{\frac{-1}{2}} + \lambda A^2((1+A^2)^{\frac{-3}{2}}) \\ \times hH(t)(\chi_m u_{m-1}(t) - \chi_{m--1} u_{m-2}(t)) = hH(t)\mathcal{R}[u_{m-1}(t)], \end{aligned} \tag{4.36}$$

where

$$\mathcal{R}[u_{m-1}(t)] = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[D^{\alpha}\Phi(t;q) + \Phi(t;q) - \lambda\Phi(t;q)(1+\Phi^2(t;q))^{\frac{-1}{2}}]_{q=0}, \tag{4.37}$$

and where $m \geq 1$.
Now, taking $H(t) = 1$ *allows for one to gain all components of the OHAM's solution by collecting the terms with similar powers of* q *via the following equation:*

$$\begin{aligned} (1-q)[D^{\alpha} + q(1 - \lambda(1+A^2)^{\frac{-1}{2}} + \lambda A^2((1+A^2)^{\frac{-3}{2}})hH(t)][u_0 + qu_1(t) + q^2u_2(t) + \cdots] \\ = qh[D^{\alpha}(u_0 + qu_1(t) + q^2u_2(t) + \cdots) + (u_0 + qu_1(t) + q^2u_2(t) + \cdots) \\ \lambda(u_0 + qu_1(t) + q^2u_2(t) + \cdots)(1 + (u_0 + qu_1(t) + q^2u_2(t) + \cdots)^2)^{\frac{-1}{2}}. \end{aligned} \tag{4.38}$$

This leads us to establish the following recursive states :

$$\begin{cases} D^{\alpha}u_1(t) &= h(D^{\alpha}u_0(t) + u_0(t) - \lambda u_0(1+u_0^2)^{\frac{-1}{2}}, \\ D^{\alpha}u_2(t) &= (1+h)D^{\alpha}u_1(t) - hku_1(t) + h[u_1(t) - \lambda N_1(u_0, u_1(t))], \\ \vdots & \\ D^{\alpha}u_k(t) &= (1+h)D^{\alpha}u_{k-1}(t) - hk(u_{k-1}(t) - u_{k-2}(t)) + h[u_{k-1}(t) - \lambda N_{k-1}(u_0, u_1(t), \ldots, u_{k-1}(t))], \end{cases} \tag{4.39}$$

where $k = (1 - \lambda(1+A^2)^{\frac{-1}{2}} + \lambda A^2((1+A^2)^{\frac{-3}{2}})$, *and where*

$$\begin{aligned} N_{k-1}(u_0, u_1(t), \ldots, u_{k-1}(t)) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[-\lambda(u_0 + qu_1(t) + q^2u_2(t) + \cdots) \\ &\quad \times((1 + (u_0 + qu_1(t) + q^2u_2(t) + \cdots)^2)^{\frac{-1}{2}}]_{q=0}, \end{aligned} \tag{4.40}$$

subject to the following initial conditions:

$$u_0(0 = A, \quad u'_m(0) = 0 \tag{4.41}$$

where $m = 1, 2, \ldots$.
Now, applying j^{α} *on (4.39) yields:*

$$\begin{cases} u_1(t) &= Ah(1 - \dfrac{\lambda}{\sqrt{1+A^2}})\dfrac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ u_2(t) &= Ah(1+h)(1 - \dfrac{\lambda}{\sqrt{1+A^2}})\dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{2A^3h^2}{(1+A^2)^{\frac{3}{2}}}(1 - \dfrac{\lambda}{\sqrt{1+A^2}})\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ \vdots & \end{cases} \tag{4.42}$$

In a similar manner, the rest of other components can be obtained, and then the series solutions expression will be, as given before, in (4.31). That is :

$$\begin{aligned} u(t) &= A + Ah(2+h)(1-\frac{\lambda}{\sqrt{1+A^2}})\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad + \frac{2A^3h^2}{(1+A^2)^{\frac{3}{2}}}(1-\frac{\lambda}{\sqrt{1+A^2}})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots . \end{aligned} \tag{4.43}$$

Similarly to Example 4.1, Figure 4.4 illustrates several h-curves in accordance with different values of A, λ and α. Based on this figure, one may candidate the interval $[-2, 1.5]$ to be also the interval of convergence. The value of the parameter h can be, then, chosen from such interval. On the other hand, the auxiliary function $H(t)$ can be chosen once again 1. Taking the previous data into account when carrying out the OHAM via MATLAB software code generates the results shown in Figure 4.5 that represents the approximate solutions for problem (4.33) according to different values of A, α, h, and λ. For more insight,Figure 4.6 shows some graphical comparisons are performed between the OHAM and the HAM. Such comparisons reveal the influence and impact of the method under consideration.

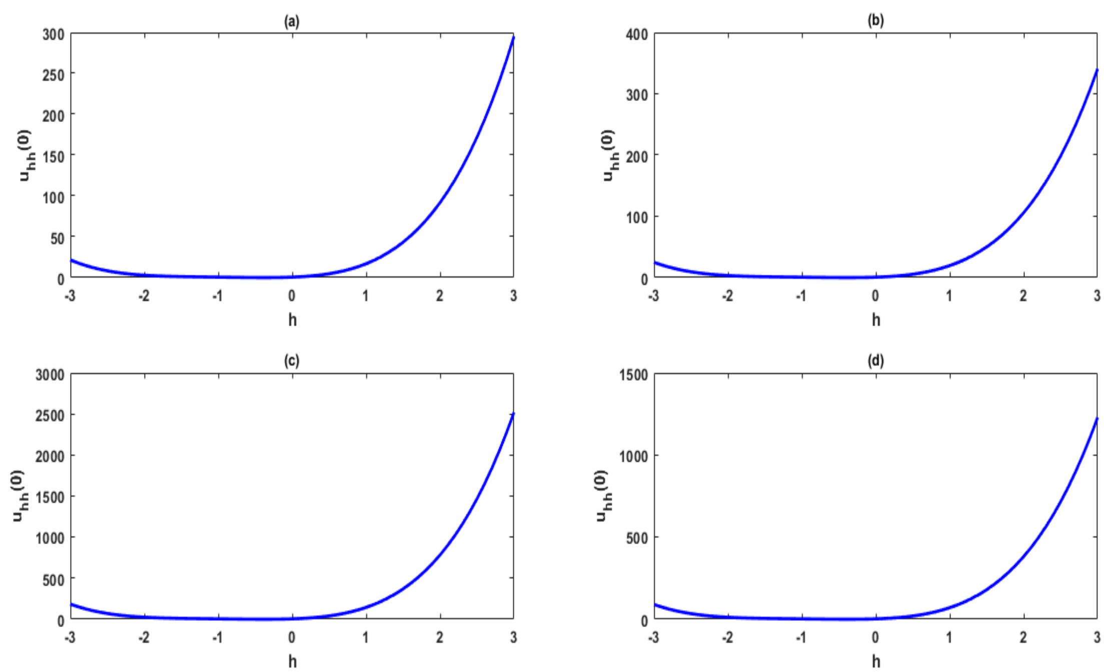

Figure 4.4 –Plots of several h-curves according the values: (a) $A = 0.75, \lambda = 0.5$, (b) $A = 1.25, \lambda = 0.5$, (c) $A = 2, \lambda = 0.9$, (d) $A = 1.5, \lambda = 0.95$.

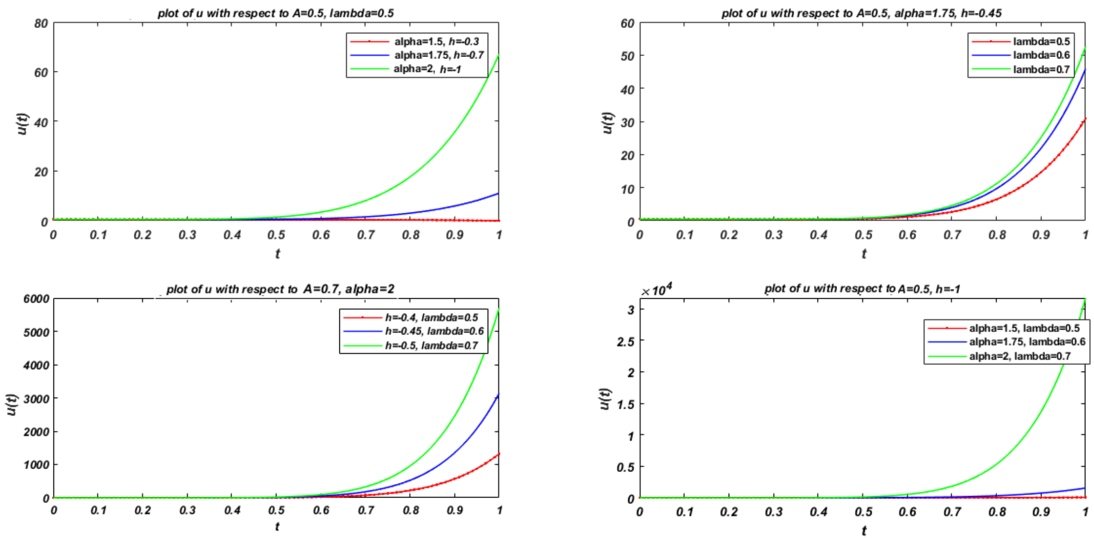

Figure 4.5- Plots of the OHAM's solutions $u(t)$ according to different values of A, λ, α and h.

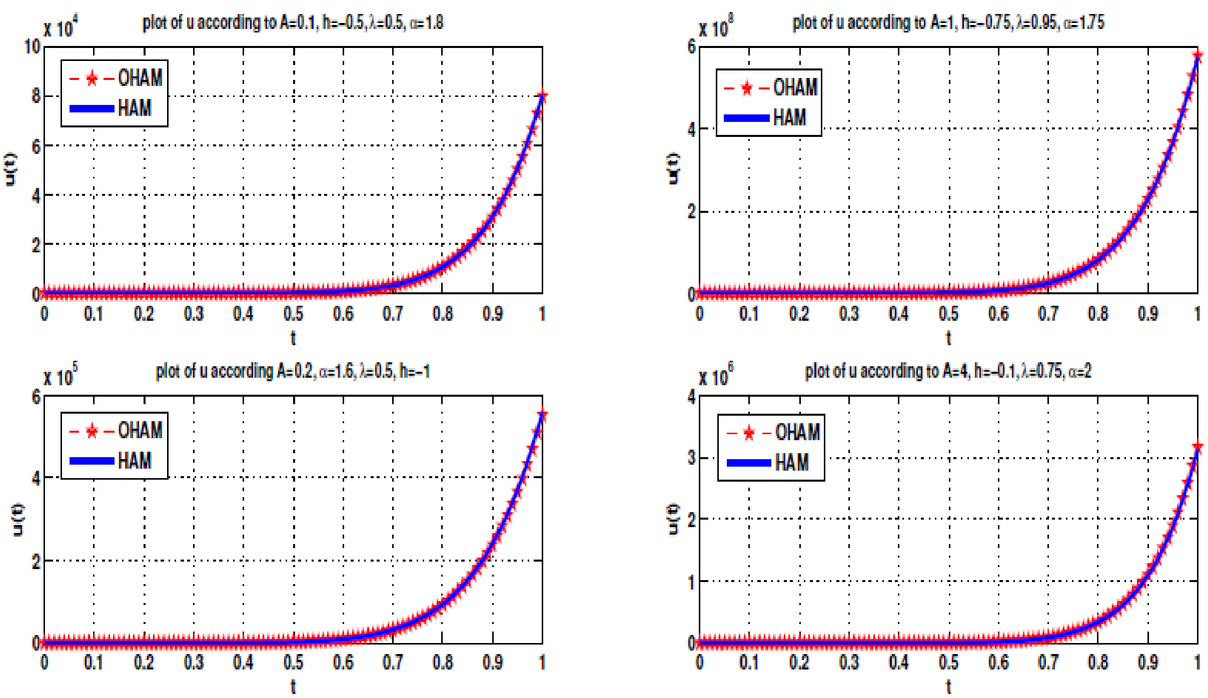

Figure 4.6- Plots of approximate solutions using OHAM and HAM for different values of A, λ, α and h.

4.5 Conclusion

In this chapter, a further modification for an Optimal Homotopy Asymptotic Method (OHAM) has been successfully implemented to solve two strongly fractional-order nonlinear benchmark oscillatory problems, namely: The fractional-order Duffing-relativistic oscillator, and the fractional-order stretched elastic wire oscillator (with a mass attached to its midpoint). Such modification has been performed by establishing an optimal auxiliary linear operator, an auxiliary function, and an auxiliary control parameter. The proposed scheme has shown its reliability in comparison with the approximate solutions that were obtained using HAM, and its efficiency in handling the considered problems.

General Conclusion

To conclude, we applied an optimal approach of the homotopy analysis method to obtain approximate solution of fractional partial differential equations. To get the desired results the basic ideas of fractional calculus and the homotopy analysis method are presented in the first and the second chapter, then in third and the last chapter we present an optimal homotopy analysis method to give an analytic approximate solution for nonlinear time-fractional hyperbolic PDEs and two strongly fractional-order nonlinear oscillators. After the presentation of our results we figured that:

- The homotopy anlysis method is an effective method to solve fractional partial differential equations.
- The efficiency of the proposed method based on the best selection of the auxiliary parameter h, the auxiliary function H, the linear operator L, and the initial approximation. All these parameters provides us to control the convergence of the series solution.
- The future research will focus on this method to find an analytic approximate solution for more complicated fractional differential problems.
- We are going to find an analytic approximate solutions for nonlinear differential problems with order $\alpha(t)$, where α is a known function of the independant variable t.

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