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Thème

# Sub super solutions method for elliptic systems involving (p1, ..., pm) Laplacian operator


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بسم الله الرحمن الرحيم

# شكر و تقدير

نحمد الله و نشكره شكرا جزيلا إذ هو خالقنا، و معيننا

فهو الأولى بالشكر في كل الأوقات و الظروف.

نحمد الله عز و جل و نثني عليه الخير كله الذي وفقنا

لإتمام هذا العمل، و نسأله ان يجعل هذا كله

خالصا لوجهه الكريم و أن ينفعنا به و ينتفع به من بعدنا.

نتقدم بجزيل الشكر و التقدير للاستاذ والدكتور المشرف "قفايفية رفيق"

، على كل ماقدمه لنا من توجيهات و معلومات قيمة ساهمت في اثراء

مضوع دراستنا كما نتقدم بجزيل الشكر الى اعضاء لجنة المناقشة

الموقرة"نابتي عبد الرزاق"و"زديري صنية".

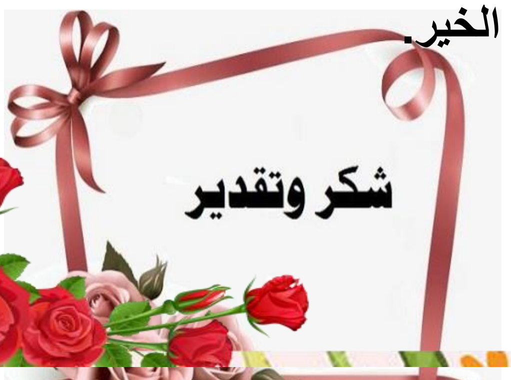
و لكل الأساتذة في تكويننا عبر مسيرتنا الدراسية

من الإبتدائية إلى الجامعة،

و إلى كل من قدم لنا يد المساعدة من قريبب أو من بعيد

فله منا خالص الإحترام و التقدير،

نسأل الله أن يجازي الجميع كل الخير.



# الإهداء

**نشكر الله العلي القدير الذي وفقنا في إنجاز هذا العمل المتواضع الذي كان نجاحنا بيديه وأهدي ثمرة جهدي هذا إلى :**

- **إلى من وضع المولى عز وجل الجنة تحت قدميها ووقرها في كتابه العزيز ،حملتني وهنا على وهن إلى والدتي الحبيبة.اطال الله في عمرها.**
- **-إلى طيب القلب الذي علمني بمثاليته وتواضع صفاته إلى والدي العزيز أطال الله في عمره**
- **- .إلى شموع البيت المنيرة إخوتي الأعزاء (رياض .شهيناز .سارة .نورالهدى.هندة .سلسبيل )**
- **إلى كل أساتذتي في جميع مراحل حياتي الدراسية الذين حملوا شعلة العلم**
- **إلى كل أصدقائي وزملائي في درب الحياة واللذين قاسموني مقاعد الدراسة في الجامعه. إلى طلبة السنة الثانية ماستر رياضيات بقسم الرياضيات واعلام الي -جامعة تبسة –دفعة 2021**

قتال نصر الدين

# الإهداء

أهدي ثمرة جهدي

إلى من حملتني كرها ووضعتني كرها، إلى أول من ابتسمتُ لها ونطقت اسمها، أمي الحبيبة الغالية

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إلى التي صبرت وكافحت معي كل هذه السنوات،الى عمتي التي درست وتربيت عندها في الصغر اسال الله لها الشفاء العاجل

إلى كل أساتذتي في جميع مراحل حياتي الدراسية الذين حملوا شعلة العلم

إلى كل أصدقائي وزملائي في درب الحياة

إلى طلبة السنة الثانية ماستر رياضيات بقسم الرياضيات واعلام الي -جامعة تبسة -دفعة2021

إلى من تسعهم ذاكرتي ولم تسعهم مذكرتي

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أهدي إليهم ثمرة جهدي

ابراهيم بوشوشة

# Abstract

In this memory we studied the weak positive solutions for a class of semilinear elliptic systems subject to homogeneous Dirichlet conditions on the bord. The technique used is the method sub- and Super solutions.

# ملخص

في هذه المذكرة نهتم بدراسة الحلول الضعيفة الموجبة لجمل معادلات اهليجية شبه خطية، تحت الشروط الحافوية المعدومة لديريشلي، التقنية المستخدمة هي طريقة الحلول تحتية و فوقية .

# Contents

# Introduction

Partial differential equations are of crucial importance in modelization and description of a wide variety of phenomena such as fluid dynamics, quantum physics, sound, heat, electrostatics, diffusion, gravitation, chemistry, biology, simulation of airplane, calculator charts and time prediction.

PDEs are equations involving functions of several variables and their derivatives and model multidimensional systems generalizing ODEs (ordinary differential equations), which deal with functions of a single variable and their derivatives.

The non-linearity is essential and depends on one phenomenon to another and is closely linked to its exact to its exact description.

Problems involving the $p$-Laplacian arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

Hai, Shivaji $[5]$ studied the existence of positive solution for the $p$-Laplacian system

$$\begin{cases} -\Delta_p u = \lambda f(v) \ \text{ in } \Omega, \\ \\ -\Delta_p v = \lambda g(u) \ \text{ in } \Omega, \\ \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

which $f(s), g(s)$ are the increasing functions in $[0, \infty)$ and satisfy

$$\lim_{s \to +\infty} \frac{f\left(M(g(s))^{\frac{1}{p-1}}\right)}{s^{p-1}} = 0, M > 0$$

the authors showed that the problem $(1)$ has at least one positive solution provided that $\lambda > 0$ is large enough.

In $[3]$ , the existence and nonexistence of positive weak solutions to the following quasilinear elliptic system:

$$\begin{cases} -\Delta_p u = \lambda u^{\alpha} v^{\gamma} \text{ in } \Omega, \\ \\ -\Delta_q u = \lambda u^{\delta} v^{\beta} \text{ in } \Omega, \\ \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

has been considered where the first eigenfunction of the operator $-\Delta_p$ has been used to construct the subsolution of problem (2) and the following results were obtained:

(i) If $\alpha, \beta \geq 0, \gamma, \delta > 0, \theta = (p-1-\alpha)(q-1-\beta) - \gamma\delta > 0$, then problem $(2)$ has a positive weak solution for each $\lambda > 0$.

(ii) If $\theta = 0$ and $p\gamma = q(p-1-\alpha)$, then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ problem $(2)$ has nonontrivial nonnegative weak solution.

In this Chapter 1 of this thesis reviews some useful preliminary notions as Sobolev spaces and $p$-Laplace operator.

In chapter 2 we present study the existence and multiplicity of positive weak solutions for a new class of $(p, q)$ Laplacian nonlinear elliptic system

$$\begin{cases} -\triangle_p u - |u|^{p-2} u = \lambda_1 a(x) f(v) + \mu_1 \alpha(x) h(u) \text{ in } \Omega, \\ \\ -\triangle_q v - |v|^{q-2} v = \lambda_2 b(x) g(u) + \mu_2 \beta(x) \gamma(v) \text{ in } \Omega, \\ \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

where

$$\triangle_s z = \operatorname{div}\left(|\nabla z|^{s-2} \nabla z\right), s > 1, \Omega \subset \mathbb{R}^N \ (N \geq 3)$$

is a bounded domain with smooth boundary $\partial\Omega, a(x), b(x), \alpha(x), \beta(x) \in C\left(\overline{\Omega}\right), \lambda_1, \lambda_2, \mu_1,$ and $\mu_2$ are nonnegative parameters.

The study of $(p, q)$ Laplacian nonlinear elliptic system is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. Many existence results have been obtained on this kind of problems, see for example.These problems arise in some physical models and are interesting in applications at combustion, mathematical biology, chemical reactions. Our approach is based on the method of sub- and supersolutions .

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K. AKROUT and R. Guefaifia [16] have studied the existence and nonexistence of positive weak solution for a generalized elliptic systems involving $(p_1, ..., p_m)$-Laplacian operator with zero Dirichlet boundary condition in bounded domain $\Omega \subset \mathbb{R}^n$ by using sub-super solutions method of the following form

$$\begin{cases} -\Delta_{p_i} u_i = \lambda_i f_i(u_1, ..., u_m) \text{ dans } \Omega, 1 \leq i \leq m \\ \\ u_i = 0 \text{ sur } \partial\Omega, \forall i, 1 \leq i \leq m \end{cases} \quad (4)$$

In Chapter 3 we study the existence and nonexistence of positive weak solution to the quasilinear elliptic system

$$\begin{cases} -\Delta_{p_i} u_i - |u_i|^{p_i - 2} u_i = \lambda_i f_i(u_1, ..., u_m) \text{ in } \Omega, 1 \leq i \leq m \\ \\ u_i = 0 \text{ on } \partial\Omega, \forall i, 1 \leq i \leq m \end{cases}$$

Our results are natural generalization and extension of previous studies.

# Chapter 1

# Preliminary

1- Continuous function spaces.

2- $L^p$ Space .

3- Sobolev Space.

4- Maximum principle.

5- Eigenvalue problems.

## 1.1 Continuous function spaces

We start this work by giving some useful notations and conventions

Let $x = (x_1, x_2, ..., x_n)$ denote the generic point of an open set $\Omega$ of $\mathbb{R}^n$. Let $u$ be a function defined from $\Omega$ to $\mathbb{R}$, we designate by $D^i u(x) = \frac{\partial u(x)}{\partial x_i}$ the partial derivative of $u$ with respect to $x_i \, (1 \leq i \leq n)$.Let's also define the gradient and the $p-$Laplacian from $u$, respectively as following

$$\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n})^T \text{ and } |\nabla u|^2 = \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2$$

$$\Delta_p u(x) = \text{div} \left( |\nabla u|^{p-2} \nabla u \right)(x), p \geq 2.$$

Note by $C(\Omega)$ the space of continuous functions from $\Omega$ to $\mathbb{R}$, $(C(\Omega), \mathbb{R}^m)$ the space of continuous functions from $\Omega$ to $\mathbb{R}^m$ and $C_b\left(\overline{\Omega}\right)$ the space of all continuous and bounded functions on $\overline{\Omega}$, , it is equipped with the norm $\|.\|_\infty$ ;

$$\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)|$$

For $k \geq 1$ integer, $C^k(\Omega)$ is the space of functions $u$ which are $k$ times derivable and whose derivation of order $k$ is continuous on $\Omega$.

$C_c^k(\Omega)$ is the set of functions of $C^k(\Omega)$ , whose support is compact and contained in $\Omega$.

We also define $C^k\left(\overline{\Omega}\right)$ , as the set of restrictions to $\overline{\Omega}$ of elements from $C^k(\mathbb{R}^n)$ or as being the set of functions of $C^k(\Omega)$ , ), such that for all $0 \leq j \leq k$, and for all $x_0 \in \partial\Omega$, the limit $\lim_{x \to x_0} D^j u(x)$ exists and depends only on $x_0$.

$C_0^\infty(\Omega)$ or $\mathfrak{D}(\Omega)$ , is the space of the infinitely differentiable functions, with compact supports called test function space.

## 1.2 $L^p$ Space

Let $\Omega$ be an open set of $\mathbb{R}^n$, equipped with the **Lebesgue measure** $dx$. We denote by $L^1(\Omega)$ the space of integrable functions on $\Omega$ with values in $\mathbb{R}$, it is provided with the norm

$$\|u\|_{L^1} = \int_\Omega |u(x)|\, dx$$

Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we define the space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R},\ f \text{ measurable and } \int_\Omega |f(x)|^p\, dx < +\infty \right\}$$

equipped with norm

$$\|u\|_{L^p} = \left( \int_\Omega |u(x)|^p\, dx \right)^{\frac{1}{p}}$$

We also define the space $L^\infty(\Omega)$ by

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R},\ f \text{ measurable}, \exists c > 0, \text{ so that } |f(x)| \leq c \text{ a.e. on } \Omega \}$$

it will be equipped with the essential-sup norm

$$\|u\|_{L^\infty} = ess \sup_{x \in \Omega} |u(x)| = \inf \{ c;\ |u(x)| \leq c \quad \text{a.e on } \Omega \}$$

We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $f\mathbf{1}_K \in L^p(\Omega)$ for any compact $K \subset \Omega$.

## 1.3 Sobolev space

### 1.3.1 Weak derivative

**Definition 1.1** *Let $\Omega$ be an open set of $\mathbb{R}$, and $1 \leq i \leq n$. A function $u \in L^1_{loc}(\Omega)$ has an $i^{th}$ weak derivative in $L^1_{loc}(\Omega)$ if there exists $f_i \in L^1_{loc}(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega u(x)\, \partial_i \varphi(x)\, dx = -\int_\Omega f_i(x)\, \varphi(x)\, dx$$

*This leads to say that the $i^{th}$ derivative within the meaning of distributions of $u$ belongs to $L^1_{loc}(\Omega)$ ,we write*

$$\partial_i u = \frac{\partial u}{\partial x_i} = f_i$$

### 1.3.2 $W^{1,p}(\Omega)$ space

Let $\Omega$ be a bounded or unbounded open set of $\mathbb{R}^n$, and $p \in \mathbb{R}$, $1 \leq p \leq +\infty$, the space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega);\ \text{such that } \partial_i u \in L^p(\Omega), 1 \leq i \leq n\}$$

where $\partial_i$ is the $i^{th}$ weak derivative of $u \in L^1_{loc}(\Omega)$.

**Theorem 1.1** [11] *There exists a constant $C$ ( depending only on $\Omega$ ) such that*

$$\|u\|_{L^\infty} \leq C \|u\|_{W^{1,p}},\quad \forall u \in W^{1,p}(\Omega), \forall\, 1 \leq p \leq +\infty$$

*In other words, $W^{1,p}(\Omega) \subset L^\infty(\Omega)$ with continuous injection for all $1 \leq p \leq +\infty$ .*

*Further, if $\Omega$ is **bounded** then*

$$\begin{aligned} &\textit{the injection } W^{1,p}(\Omega) \subset C\left(\overline{\Omega}\right) \textit{ is compact for all},\ 1 < p \leq +\infty \\ &\textit{the injection } W^{1,1}(\Omega) \subset L^q(\Omega) \textit{ is compact for all},\ 1 \leq q \leq +\infty, \end{aligned}$$

**Corollary 1.1** [16] *Suppose that $\Omega$ is an unbounded interval and $u \in W^{1,p}(\Omega)$ with $1 \leq p \leq +\infty$ .Then*

$$\lim_{\substack{|x| \to +\infty \\ x \in \Omega}} u(x) = 0$$

### 1.3.3 $W^{m,p}(\Omega)$ Space

Let $\Omega$ be an open set of $\mathbb{R}^n, m \geq 2$ integer number and p real number such that $1 \leq p \leq +\infty$, we define the space $W^{m,p}(\Omega)$as following

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega),\ \text{such that } \partial_i u \in L^p(\Omega),\ \forall \alpha, |\alpha| \leq m\}$$

where $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$ the length of $\alpha$ and $\partial_i u = \partial_1^{\alpha_1} ... \partial_n^{\alpha_n}$ is the weak derivative of a function $u \in L^1_{loc}(\Omega)$ in the sense of definition $(1.1)$.

The space $W^{m,p}(\Omega)$ is equiped with the norm

$$\|u\|_{W^{m,p}} = \|u\|_{L^p} + \sum_{0<|\alpha|\leq m} \|\partial_i u\|_{L^p}$$

For $p = 2$,The space $W^{m,2}(\Omega)$ is noted $H^m(\Omega)$.

### 1.3.4 $W_0^{1,p}(\Omega)$ Space

**Definition 1.2** *For* $1 \leq p < +\infty$ *-we define the space* $W_0^{1,p}(\Omega)$ *as being the closure of* $\mathcal{D}(\Omega)$ *in* $W^{1,p}(\Omega)$ *,and we write*

$$W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}}$$

## 1.4 Maximum principle

A large number of results of existence or uniqueness of solutions to boundary problems (elliptic or parabolic) can be established using the maximum principle. Here we give some variants of this result.

Let $\Omega$ be an open set of $\mathbb{R}^n$, $a(.) = (a_{ij}(.))_{1\leq i,j\leq n}$ a matrix, $b(.) = (b_i(.))_{1\leq i\leq n}$ a vector and $c$ a function. We consider the second-order symmetric operator $L$ defined by

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{i=1}^{n} b_i \partial_i u + cu \tag{1.1}$$

It is assumed that the square matrix $a$ satisfies the coercive (or elliptic) condition.

$$\exists\alpha > 0,\ \forall\xi \in \mathbb{R}^n,\ a(x)\,\xi.\xi = \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \geq \alpha\,|\xi|^2 \quad \text{a.e on } \Omega, \tag{1.2}$$

where $|\xi|$ -designates the Euclidean norm of $\xi$ in $\mathbb{R}^n$.

**Theorem 1.2 (Classical maximum principle)** [26] *Let $\Omega$ a bounded and connected open set, and $L$ as in (1.1). We suppose that $c \geq 0$, (1.2) is satisfied and $a_{ij}, b_i$, $c \in C\left(\overline{\Omega}\right)$. If $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ verifies $Lu \leq 0$ then we have*

$$\sup_{x\in\overline{\Omega}} u(x) \leq \sup_{\sigma\in\partial\Omega} u^+(\sigma) \quad \textit{where } u^+(\sigma) = \max(u(\sigma), 0)$$

**Theorem 1.3 (Hopf maximum principle)** [26] *Let $\Omega$ a bounded and connected open set, and $L$ as in (1.1). We suppose that $c \geq 0$, (1.2) is satisfied and $a_{ij}, b_i$, $c \in C\left(\overline{\Omega}\right)$. If $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ verifies $Lu \leq 0$ and if u reaches $a$ non negatif maximum in the interior of $\Omega$, then u is constant on $\Omega$.*

**Theorem 1.4 (Aleksandrov maximum principle)** [26] *Let $\Omega$ a bounded and connected open set, and $L$ as in (1.1). We suppose that $c \geq 0$, (1.2) is satisfied and $a_{ij}, b_i$, $c \in C\left(\overline{\Omega}\right)$and $f \in L^N(\Omega)$. There exists $C > 0$ depending on $N$, $\|b\|_{L^N(\Omega)}$ and the diameter of $\Omega$ such that; if $u \in W^{2,N}_{loc}(\Omega) \cap C\left(\overline{\Omega}\right)$ verifies $Lu \leq f$ then we have*

$$\sup_{x\in\overline{\Omega}} u(x) \leq \sup_{\sigma\in\partial\Omega} u(\sigma) + C\|f\|_{L^N(\Omega)}$$

**Lemma 1.1** *[Boundary Point Lemma]* [15] *suppose that $u$ is continuous in $\Omega$, $Lu \geq 0$ (resp. $Lu \leq 0$)in $\Omega$, and $u$ attains its maximum (resp; minimum)in point $p \in \partial\Omega$. then, all Directional derivativevers the exterior of$u$ at point $p$ are positive (resp; negative).*

## 1.5 Eigenvalue problems

**Definition 1.3** *We say that $u \in W^{1,p}_0(\Omega)$, $u \neq 0$, is an eigenfunction of the operator $-\triangle_p u$ if:*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla\varphi dx = \lambda \int_\Omega |u|^{p-2} u.\varphi dx \tag{1.3}$$

*for all $\varphi \in C^\infty_0(\Omega)$. The corresponding real number $\lambda$ is called eigenvalue.*

*Let $\lambda_1$ defined by*

$$\lambda_1 = \inf_{u\in W^{1,p}_0(\Omega), u\neq 0} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \tag{1.4}$$

*equivalent to*

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^p \, dx ; \int_\Omega |u|^p \, dx = 1, u \in W_0^{1,p}(\Omega), u \neq 0 \right\}$$

*$\lambda_1$ is the first eigenvalue of the $p-$Laplacien operator with null Dirichlet conditions at the edge.*

**Lemma 1.2** [27] *$\lambda_1$ is isolated, i.e : there exists $\delta > 0$ such that in the interval $(\lambda_1, \lambda_1 + \delta)$, there is no other eigenvalues of $(1.3)$.*

**Lemma 1.3** [27] *a) let $p \geq 2$, then for all $\xi_1, \xi_2 \in \mathbb{R}^n$*

$$|\xi_2|^p \geq |\xi_1|^p + p |\xi_1|^{p-2} \langle \xi_1, \xi_2 - \xi_1 \rangle + C(p) |\xi_1 - \xi_2|^p,$$

*b)let $p < 2$, then for all $\xi_1, \xi_2 \in \mathbb{R}^n$*

$$|\xi_2|^p \geq |\xi_1|^p + p |\xi_1|^{p-2} \langle \xi_1, \xi_2 - \xi_1 \rangle + C(p) \frac{|\xi_1 - \xi_2|^p}{(|\xi_2| + |\xi_1|)^{2-p}},$$

*where$C(p)$ is a component dependent only on $p$.*

**Lemma 1.4** [27] *The first eigenvalue $\lambda_1$ i.e : if $u.v$ are two eigenfunctions associated with $\lambda_1$ , then, there exists $k \in \mathbb{R}$ such that $u = kv$.*

**Lemma 1.5** [27] *Let $u$ be an eigenfunction associated with the eigenvalue $\lambda_1$, then u does not change sign on $\Omega$, Further if $u \in C^{1,\alpha}$,then $u(x) \neq 0, \forall x \in \overline{\Omega}$.*

**Proof** By the lemma $(1.4)$, we can assume that $u, v$ are positive on $\Omega$, and by taking

$$\begin{aligned} \varphi_1 &= \frac{(u^p - v^p)}{u^{p-1}}, \\ \varphi_2 &= \frac{(v^p - u^p)}{v^{p-1}}, \end{aligned}$$

two test functions in the weak formulation $(1.3)$, one obtains

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \left( \frac{u^p - v^p}{u^{p-1}} \right) dx &= \lambda \int_\Omega |u|^{p-2} u \left( \frac{u^p - v^p}{u^{p-1}} \right) dx \\ \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \left( \frac{v^p - u^p}{v^{p-1}} \right) dx &= \lambda \int_\Omega |v|^{p-2} v \left( \frac{v^p - u^p}{v^{p-1}} \right) dx \end{aligned} \tag{1.5}$$

The addition of these two formulas gives

$$0=\int_{\Omega}|\nabla u|^{p-2}\nabla u\nabla\left(\frac{u^p-v^p}{u^{p-1}}\right)dx+\int_{\Omega}|\nabla v|^{p-2}\nabla v\nabla\left(\frac{v^p-u^p}{v^{p-1}}\right)dx \tag{1.6}$$

And using the identities :

$$\begin{aligned}\nabla\left(\frac{u^p-v^p}{u^{p-1}}\right) &= \nabla u-p\frac{v^{p-1}}{u^{p-1}}\nabla v+(p-1)\frac{v^p}{u^p}\nabla u,\\ \nabla\left(\frac{v^p-u^p}{v^{p-1}}\right) &= \nabla v-p\frac{u^{p-1}}{v^{p-1}}\nabla u+(p-1)\frac{u^p}{v^p}\nabla v,\end{aligned} \tag{1.7}$$

we get the first term of $(1.6)$

$$\begin{aligned}\int_{\Omega}|\nabla u|^{p-2}\nabla u\nabla\left(\frac{u^p-v^p}{u^{p-1}}\right)dx &= \int_{\Omega}|\nabla u|^p\,dx-p\int_{\Omega}\frac{v^{p-1}}{u^{p-1}}|\nabla u|^{p-2}\nabla v\nabla u dx \\ &\quad+\int_{\Omega}(p-1)\frac{v^p}{u^p}|\nabla u|^p\,dx \\ &= \int_{\Omega}|\nabla\ln u|^p u^p dx-p\int_{\Omega}v^p|\nabla\ln u|^{p-2}\langle\nabla\ln u,\nabla\ln v\rangle dx \\ &\quad+\int_{\Omega}(p-1)|\nabla\ln u|^p v^p dx\end{aligned} \tag{1.8}$$

We have an analogous expression for the second term of $(1.6)$, where formula $(1.6)$ then becomes

$$\begin{aligned}0 &= \int_{\Omega}(u^p-v^p)\left(|\nabla\ln u|^p-|\nabla\ln v|^p\right)dx \\ &\quad-p\int_{\Omega}v^p\left(|\nabla\ln u|^{p-2}\langle\nabla\ln u,\nabla\ln v-\nabla\ln u\rangle\right)dx \\ &\quad-p\int_{\Omega}u^p\left(|\nabla\ln v|^{p-2}\langle\nabla\ln v,\nabla\ln u-\nabla\ln v\rangle\right)dx\end{aligned} \tag{1.9}$$

Choisissans $\xi_1=\nabla\ln u$ and $\xi_2=\nabla\ln v$ and we use the lemma $(1.3)$ we will have, for $p\geq 2$

$$0\geq\int_{\Omega}C(p)\,|\nabla\ln u-\nabla\ln v|\,(u^p+v^p)\,dx \tag{1.10}$$

or

$$0 = |\nabla \ln u - \nabla \ln v| \tag{1.11}$$

then $u = kv$.

For $p < 2$ ,we use the second part of the lemma $(1.3)$ as above. ■

**Theorem 1.5** $(dominated\ convergence\ theorem, Lebesgue)$ [26] *Let $\{f_n\}_{n\geq 1}$ be a sequence of functions of $L^1(\Omega)$converging almost everywhere to a measurable function $f$. It is assumed that there exists $g \in L^1(\Omega)$ such that for all $n \geq 1$, we get*

$$|f_n| \leq g \quad \text{a.e on } \Omega$$

*then:* $f \in L^1(\Omega)$ *and*

$$\lim_{n\to+\infty} \|f_n - f\|_{L^1} = 0, \text{ and } \int_\Omega f(x)\,dx = \lim_{n\to+\infty} \int_\Omega f_n(x)\,dx$$

**Definition 1.4** [26] *Let $\omega$ be a part of a Banach space $X$ and $F : \omega \to \mathbb{R}$. if $u \in \omega$, we say that $F$ is **Gâteaux** differentiable ( G-differentiable) at $u$, if there exists $l \in X'$ such that in each direction $z \in X$ where $F(u + tz)$ exists for $t > 0$ small enough, the directional derivative $F'_z(u)$ exists and we have*

$$\lim_{t\to 0^+} \frac{F(u + tz) - F(u)}{t} = \langle l, z\rangle .$$

*We write $F'(u) = l$.*

**Theorem 1.6** [26] *Let $\Omega \subset \mathbb{R}^n$ an open set $n \geq 3$,, For $p \in (1, +\infty)$ we define a functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by*

$$J(u) = \int_\Omega |\nabla u|^p\,dx$$

*then $J$ is **Gâteaux** differentiable in $W_0^{1,p}(\Omega)$ and*

$$J'(u)(v) = p\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v dx, \forall v \in W_0^{1,p}(\Omega)$$

**Proof** We consider the function $\varphi : \mathbb{R}^n \to \mathbb{R}$, defined by: $\varphi(x) = |x|^p$ ,it is a class function $C^1$,and $\nabla\varphi = p|x|^{p-2}x$,

then for all $x, y \in \mathbb{R}^n$,

$$\lim_{t\to 0} \frac{\varphi(x+ty)-\varphi(x)}{t} = p|x|^{p-2}x.y$$

as a result

$$\lim_{t\to 0} \frac{|\nabla u(x)+t\nabla v(x)|^p - |\nabla u(x)|^p}{t} = p|\nabla u(x)|^{p-2}\nabla u(x).\nabla v(x)$$

by the finite increase theorem, for almost everything $x \in \Omega$ and for $t > 0$, there is a function $\theta$ with values in $]0,1[$ such that one can write:

$$\begin{aligned} & |\nabla u(x)+t\nabla v(x)|^p - |\nabla u(x)|^p - tp|\nabla u(x)|^{p-2}\nabla u(x).\nabla v(x) \\ = \; & tp|\nabla u(x)+\theta(t,x)t\nabla v(x)|^{p-2}(\nabla u(x)+\theta(t,x)t\nabla v(x)).\nabla v(x) \\ & -tp|\nabla u(x)|^{p-2}\nabla u(x).\nabla v(x) \end{aligned} \tag{1.12}$$

Dividing by $t$,we get for almost all $x$

$$\lim_{t\to 0} \frac{|\nabla(u+tv)(x)|^p - |\nabla u(x)|^p - tp|\nabla u(x)|^{p-2}\nabla u(x).\nabla v(x)}{t} = 0.$$

On the other hand, we can increase the second member of the equality $(1.12)$ divided by $t$ by

$$h(x) = 2|\nabla v(x)|(|\nabla u(x)|+|\nabla v(x)|)^{p-1}$$

Using HÖlder's inequality we obtain:

$$|h| \le C\|\nabla v\|_p \left(\|\nabla u\|_p^{p-1} + \|\nabla v\|_p^{p-1}\right).$$

We may now apply dominated convergence theorem and conclude that:

$$J'(u)(v) = p\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v dx, \forall v \in W_0^{1,p}(\Omega)$$

then $J$ is **Gâteaux** differentiable. ■

**Lemma 1.6** (*Comparison lemma*) [8] *Let* $u, v \in W_0^{1,p}(\Omega)$ *satisfying*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi dx \leq \int_\Omega |\nabla v|^{p-2} \nabla v.\nabla \varphi dx \tag{1.13}$$

*for all* $\varphi \in W_0^{1,p}(\Omega), \varphi \geq 0$,*then* $u \leq v$ *a.e in* $\Omega$.

**Proof** This proof is based on the arguments presented in [12] and [4]. We start by defining the function $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by the formula

$$J(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx \tag{1.14}$$

It is clear that the functional $J$ is Gâteaux differentiable and continuous and its derivative at $u \in W_0^{1,p}(\Omega)$ is the function $J'(u) \in W_0^{-1,p}(\Omega)$ i.e

$$J'(u)(\varphi) = \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi dx, \varphi \in W_0^{1,p}(\Omega). \tag{1.15}$$

$J'(u)$ is continuous and bounded. We will show that $J'(u)$ is strictly monotonic in $W_0^{1,p}(\Omega)$. Indeed, for all $u, v \in W_0^{1,p}(\Omega), u \neq v$ without loss of generality, we can suppose that

$$\int_\Omega |\nabla u|^p dx \geq \int_\Omega |\nabla v|^p dx$$

Using the Cauchy inequality we have

$$\nabla u.\nabla v \leq |\nabla u| |\nabla v| \leq \frac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2\right) \tag{1.16}$$

From formula (1.14) we deduce

$$\int_\Omega |\nabla u|^p dx - \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla v dx \geq \frac{1}{2} \int_\Omega |\nabla u|^{p-2} \left(|\nabla u|^2 - |\nabla v|^2\right) dx \tag{1.17}$$

$$\int_\Omega |\nabla v|^p dx - \int_\Omega |\nabla v|^{p-2} \nabla v.\nabla u dx \geq \frac{1}{2} \int_\Omega |\nabla v|^{p-2} \left(|\nabla v|^2 - |\nabla u|^2\right) dx \tag{1.18}$$

If $|\nabla u| \geq |\nabla v|$ , by using $(1.14)-(1.16)$ we get .

$$
\begin{aligned}
I_1(u) &= J'(u)(u) - J'(u)(v) - J'(v)(u) + J'(v)(v) \\
&= \left( \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} |\nabla u|^{p-2} \nabla u . \nabla v dx \right) \\
&- \left( \int_{\Omega} |\nabla v|^{p-2} \nabla v . \nabla u dx - \int_{\Omega} |\nabla v|^p \, dx \right) \\
&\geq \int_{\Omega} \tfrac{1}{2} |\nabla u|^{p-2} \left( |\nabla u|^2 - |\nabla v|^2 \right) dx \\
&- \tfrac{1}{2} \int_{\Omega} |\nabla u|^{p-2} \left( |\nabla u|^2 - |\nabla v|^2 \right) dx \\
&= \tfrac{1}{2} \int_{\Omega} \left( |\nabla u|^{p-2} - |\nabla v|^{p-2} \right) \left( |\nabla u|^2 - |\nabla v|^2 \right) dx \\
&\geq \tfrac{1}{2} \int_{\Omega} \left( |\nabla u|^{p-2} - |\nabla v|^{p-2} \right) \left( |\nabla u|^2 - |\nabla v|^2 \right) dx
\end{aligned} \tag{1.19}
$$

If $|\nabla v| \geq |\nabla u|$ , by changing the role of $u$ and $v$ in $(1.14)-(1.16)$ we have

$$\begin{aligned}
I_2(v) &= J'(v)(v) - J'(v)(u) - J'(u)(v) + J'(u)(u) \\
&= \left( \int_\Omega |\nabla v|^p \, dx - \int_\Omega |\nabla v|^{p-2} \nabla v . \nabla u dx \right) \\
&- \left( \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v dx - \int_\Omega |\nabla u|^p \, dx \right) \\
&\geq \tfrac{1}{2} \int_\Omega |\nabla v|^{p-2} \left( |\nabla v|^2 - |\nabla u|^2 \right) dx \\
&- \tfrac{1}{2} \int_\Omega |\nabla v|^{p-2} \left( |\nabla v|^2 - |\nabla u|^2 \right) dx \\
&= \tfrac{1}{2} \int_\Omega \left( |\nabla v|^{p-2} - |\nabla u|^{p-2} \right) \left( |\nabla v|^2 - |\nabla u|^2 \right) dx \\
&\geq \tfrac{1}{2} \int_\Omega \left( |\nabla v|^{p-2} - |\nabla u|^{p-2} \right) \left( |\nabla v|^2 - |\nabla u|^2 \right) dx
\end{aligned} \tag{1.20}$$

From $(1.17)$ and $(1.18)$, we have

$$(J'(u) - J'(v))(u - v) = I_1 = I_2 \geq 0, \forall u, v \in W_0^{1,p}(\Omega)$$

In addition, if $u \neq v$ and $(J'(u) - J'(v))(u - v) = 0$ , then we have

$$\int_\Omega \left( |\nabla u|^{p-2} - |\nabla v|^{p-2} \right) \left( |\nabla u|^2 - |\nabla v|^2 \right) dx = 0,$$

If $|\nabla u| = |\nabla v|$ in $\Omega$, we deduce that

$$\begin{aligned}
(J'(u) - J'(v))(u - v) &= J'(u)(u - v) - J'(v)(u - v) \\
&= \int_\Omega |\nabla u|^{p-2} |\nabla u - \nabla v|^2 \, dx = 0,
\end{aligned} \tag{1.21}$$

i.e. $u - v$ is a constant. Given $u = v = 0$ on $\partial\Omega$ we are getting $u = v$ , which is contrary with $u \neq v$. Then $(J'(u) - J'(v))(u - v) > 0$ and $J'(u)$ is strictly monotonic in $W_0^{-1,p}(\Omega)$. Let $u, v$ two functions such that $(1.15)$ is satisfied, let's take $\varphi = (u - v)^+$, the positive part of $u - v$ as a test function in $(1.15)$, we get

$$(J'(u) - J'(v))(\varphi) = \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla\varphi dx - \int_\Omega |\nabla v|^{p-2} \nabla v.\nabla\varphi dx \leq 0. \tag{1.22}$$

Relationships $(1.20)$ and $(1.21)$ imply that $u \leq v$. ■

# Chapter 2

# Existence and multiplicity of positive weak solutions for a new class of $(p, q)$ Laplacian nonlinear elliptic system

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1- Introduction.
2- Definitions and notations.
2- Existence result.
2- Example 1.
4- Example 2.

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## 2.1 Introduction

In this chapter, we study the existence and multiplicity of positive weak solutions for a new class of $(p,q)$ Laplacian nonlinear elliptic system

$$\begin{cases} -\triangle_p u - |u|^{p-2} u = \lambda_1 a(x) f(v) + \mu_1 \alpha(x) h(u) \text{ in } \Omega, \\ \\ -\triangle_q v - |v|^{q-2} v = \lambda_2 b(x) g(u) + \mu_2 \beta(x) \gamma(v) \text{ in } \Omega, \\ \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where

$$\triangle_s z = \operatorname{div}\left(|\nabla z|^{s-2} \nabla z\right), s > 1, \Omega \subset \mathbb{R}^N \ (N \geq 3)$$

is a bounded domain with smooth boundary $\partial\Omega, a(x), b(x), \alpha(x), \beta(x) \in C\left(\overline{\Omega}\right), \lambda_1, \lambda_2, \mu_1,$ and $\mu_2$ are nonnegative parameters.

## 2.2 Definitions and notations

First, we make the following assumptions:

$(H1)$ Let $a(x), b(x), \alpha(x), \beta(x) \in C\left(\overline{\Omega}\right)$ such that $a(x) \geq a_1 > 0, b(x) \geq b_1 > 0, \alpha(x) \geq \alpha_1 > 0, \beta(x) \geq \beta_1 > 0$

$(H2)$ $f,\ g,\ h,\ \gamma \in C^1([0,\infty))$ be monotone functions such that

$$\lim_{s \to +\infty} f(s) = \lim_{s \to +\infty} g(s) = \lim_{s \to +\infty} h(s) = \lim_{s \to +\infty} \gamma(s) = +\infty.$$

$(H3)$ $\lim_{s \to +\infty} \frac{f\left(M(g(s))^{\frac{1}{q-1}}\right)}{s^{p-1}} = 0, \ \forall M > 0.$

$(H4)$ $\lim_{s \to +\infty} \frac{h(s)}{s^{p-1}} = \lim_{s \to +\infty} \frac{\gamma(s)}{s^{q-1}} = 0.$

We give the following two definitions before we give our main result

**Definition 2.1** *Let* $(u,v) \in W^{1,p}(\Omega) \cap C\left(\overline{\Omega}\right) \times W^{1,q}(\Omega) \cap C\left(\overline{\Omega}\right), (u,v)$ *is said a weak solution* *of (2.1) if it satisfies*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \xi dx - \int_\Omega |u|^{p-2} u.\xi dx =$$
$$\lambda_1 \int_\Omega a(x) f(v) \xi dx + \mu_1 \int_\Omega \alpha(x) h(u) \xi dx \text{ in } \Omega,$$

$$\int_\Omega |\nabla v|^{q-2} \nabla v.\nabla \zeta dx - \int_\Omega |v|^{q-2} v.\zeta dx =$$
$$\lambda_2 \int_\Omega b(x) g(u) \zeta dx + \mu_2 \int_\Omega \beta(x) \gamma(v) \zeta dx \text{ in } \Omega$$

*for all* $(\xi,\zeta) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

**Definition 2.2** *A pair of nonnegative functions* $(\underline{u},\underline{v}), (\overline{u},\overline{v})$ *in* $W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ *are called a weak subsolution and supersolution of (2.1) if they satisfy* $(\underline{u},\underline{v}), (\overline{u},\overline{v}) = (0,0)$ *on* $\partial\Omega$

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u}.\nabla \xi dx - \int_\Omega |\underline{u}|^{p-2} \underline{u}.\xi dx \le$$
$$\lambda_1 \int_\Omega a(x) f(\underline{v}) \xi dx + \mu_1 \int_\Omega \alpha(x) h(\underline{u}) \xi dx \text{ in } \Omega,$$

$$\int_\Omega |\nabla \underline{v}|^{q-2} \nabla \underline{v}.\nabla \zeta dx - \int_\Omega |\underline{v}|^{q-2} \underline{v}.\zeta dx \le$$
$$\lambda_2 \int_\Omega b(x) g(\underline{u}) \zeta dx + \mu_2 \int_\Omega \beta(x) \gamma(\underline{v}) \zeta dx \text{ in } \Omega$$

*and*

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u}.\nabla \xi dx - \int_\Omega |\overline{u}|^{p-2} \overline{u}.\xi dx \ge$$
$$\lambda_1 \int_\Omega a(x) f(\overline{v}) \xi dx + \mu_1 \int_\Omega \alpha(x) h(\overline{u}) \xi dx \text{ in } \Omega,$$

$$\int_\Omega |\nabla \overline{v}|^{q-2} \nabla \overline{v}.\nabla \zeta dx - \int_\Omega |\overline{v}|^{q-2} \overline{v}.\zeta dx \ge$$
$$\lambda_2 \int_\Omega b(x) g(\overline{u}) \zeta dx + \mu_2 \int_\Omega \beta(x) \gamma(\overline{v}) \zeta dx \text{ in } \Omega$$

*for all* $(\xi,\zeta) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

*We shall establish the following result.*

## 2.3 Existence result

**Theorem 2.1** [14] *Assume that the conditions* $(H1)-(H4)$ *hold, then problem (2.1) has a positive weak solution for each provided* $\lambda_1 + \mu_1$ *and* $\lambda_2 + \mu_2$ *are large.*

**Proof** We shall establish Theorem 2.1 by constructing a positive weak subsolution. $(\underline{u}, \underline{v}) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ and a supersolution $(\overline{u}, \overline{v}) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ of (2.1) such that $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$. That is, $(\underline{u}, \underline{v}), (\overline{u}, \overline{v})$ satisfy $(\underline{u}, \underline{v}) = (0,0) = (\overline{u}, \overline{v})$ on $\partial\Omega$.

Let $\sigma_r$ the first eigenvalue of $-\triangle_r$ with Dirichlet boundary conditions and $\phi_r$ the corresponding eigenfunction with $\phi_r > 0$ in $\Omega$ and $\|\phi_r\| = 1$ for $r = p, q$. Let $m, \eta, \delta > 0$ be such that

$|\nabla\phi_r|^r - \sigma_r\phi_r^r \geq m$ on $\overline{\Omega_\delta} = \{x \in \Omega, d(x, \partial\Omega) \leq \delta\}$ and $\phi_r \geq \eta$ on $\Omega \backslash \overline{\Omega}_\delta$ for $r = p, q$.Taking $k_0 > 0$ such that $a_1 f(t), \alpha_1 h(t), b_1 g(t), \beta_1 \gamma(t) > -k_0$.

We shall verify that

$$\underline{u} = \left[\frac{(\lambda_1 + \mu_1) k_0}{m}\right]^{1/p-1} \left(\frac{p-1}{p}\right) \phi_p^{p/p-1}, \tag{2.1}$$

and

$$\underline{v} = \left[\frac{(\lambda_2 + \mu_2) k_0}{m}\right]^{1/q-1} \left(\frac{q-1}{q}\right) \phi_q^{q/q-1}, \tag{2.2}$$

is a subsolution of (2.1) for $\lambda_1+\mu_1$ and $\lambda_2+\mu_2$ are large. Let the test function $\xi(x) \in W_0^{1,p}(\Omega)$ with $\xi(x) \geq 0$. Thus, from $(H1)$ we have

$$\begin{aligned}
\int_\Omega |\nabla\underline{u}|^{p-2} \nabla\underline{u}.\nabla\xi dx - \int_\Omega |\underline{u}|^{p-2} \underline{u}.\xi dx \ &\leq \ \int_\Omega |\nabla\underline{u}|^{p-2} \nabla\underline{u}.\nabla\xi dx \\
&= \left(\frac{(\lambda_1 + \mu_1) k_0}{m}\right) \int_\Omega \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\} \xi dx \\
&= \left(\frac{(\lambda_1 + \mu_1) k_0}{m}\right) \int_{\Omega_\delta} \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\} \xi dx \\
&\quad + \left(\frac{(\lambda_1 + \mu_1) k_0}{m}\right) \int_{\Omega\backslash\overline{\Omega}_\delta} \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\} \xi dx.
\end{aligned}$$

Note that on $\overline{\Omega_\delta}$ we have $|\nabla\phi_r|^r - \sigma_r\phi_r^r \geq m$ for $r = p, q$.Also on $\Omega\backslash\overline{\Omega}_\delta$ $\phi_r \geq \eta$ for $r = p, q$. If $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ are large in the definition of $\underline{u}$, $\underline{v}$, so by $(H2)$

$$a_1 f(\underline{v}), \alpha_1 h(\underline{u}), b_1 g(\underline{u}), \beta_1\gamma(\underline{v}) \geq \frac{k_0}{m}\max\{\sigma_p, \sigma_q\}. \tag{2.3}$$

Hence

$$\begin{aligned}\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}.\nabla\xi dx - \int_\Omega |\underline{u}|^{p-2}\underline{u}.\xi dx &\leq \left(\frac{(\lambda_1+\mu_1)k_0}{m}\right)\int_{\Omega_\delta}\left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\}\xi dx \\ &\quad + \left(\frac{(\lambda_1+\mu_1)k_0}{m}\right)\int_{\Omega\backslash\overline{\Omega}_\delta}\left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\}\xi dx \\ &\leq -(\lambda_1+\mu_1)k_0\int_{\Omega_\delta}\xi dx + \left(\frac{(\lambda_1+\mu_1)k_0}{m}\right)\int_{\Omega\backslash\overline{\Omega}_\delta}\sigma_p\xi dx \\ &\leq \int_{\Omega_\delta}\left[\lambda_1 a(x) f(\underline{v}) + \mu_1\alpha(x) h(\underline{u})\ \right]\xi dx \\ &\quad + \int_{\Omega\backslash\overline{\Omega}_\delta}\left[\lambda_1 a(x) f(\underline{v}) + \mu_1\alpha(x) h(\underline{u})\ \right]\xi dx \\ &= \int_\Omega\left[\lambda_1 a(x) f(\underline{v}) + \mu_1\alpha(x) h(\underline{u})\ \right]\xi dx\end{aligned}$$

Similarly,

$$\int_\Omega |\nabla\underline{v}|^{q-2}\nabla\underline{v}.\nabla\zeta dx - \int_\Omega |\underline{v}|^{q-2}\underline{v}.\zeta dx \leq \int_\Omega\left[\lambda_2 b(x) g(\underline{u}) + \mu_2\beta(x)\gamma(\underline{v})\right]\zeta dx$$

Therefore $(\underline{u}, \underline{v})$ is subsolution of problem $(2.1)$.

Next,we construct a supersolution of $(2.1)$. Let $\omega_r$ be a unique positive solution of

$$\begin{cases} -\triangle_r\omega_r = 1 \text{ in } \Omega, \\ \omega_r = 0 \text{ on } \partial\Omega. \end{cases}$$

for $r = p, q$. We denote

$$\overline{u} = \frac{C}{\nu_p}\left(\frac{1}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\omega_p, \tag{2.4}$$

$$\overline{v}=\left[\left(\frac{\lambda_2\|b\|_\infty+\mu_2\|\beta\|_\infty}{1-\nu_q^{q-1}}\right)g\left(C\left(\frac{1}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)^{\frac{1}{q-1}}\right]\omega_q. \tag{2.5}$$

where $\nu_r=\|\omega_r\|_\infty$, $r=p,q$ and $C>0$ is a large number to be chosen later, We shall verify that $(\overline{u},\overline{v})$ is a supersolution of $(2.1)$ such that $(\overline{u},\overline{v})\geq(\underline{u},\underline{v})$ . By $(H3)-(H4)$ we can choose $C$ large enough so that

$$\begin{aligned}\left(\frac{C}{\nu_p}\right)^{p-1} &\geq \lambda_1\|a\|_\infty f\left(\left[\left(\frac{\lambda_2\|b\|_\infty+\mu_2\|\beta\|_\infty}{1-\nu_q^{q-1}}\right)g\left(C\left(\frac{1}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)^{\frac{1}{q-1}}\right]\omega_q\right) \\ &\quad+\mu_1\|\alpha\|_\infty h\left(\frac{1}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\omega_p\end{aligned} \tag{2.6}$$

Hence

$$\int_\Omega|\nabla\overline{u}|^{p-2}\nabla\overline{u}.\nabla\xi dx-\int_\Omega|\overline{u}|^{p-2}\overline{u}.\xi dx = \left(\frac{C}{\nu_p}\right)^{p-1}\int_\Omega\xi dx$$

Using $(2.6)$

$$\begin{aligned}&\int_\Omega|\nabla\overline{u}|^{p-2}\nabla\overline{u}.\nabla\xi dx-\int_\Omega|\overline{u}|^{p-2}\overline{u}.\xi dx \\ \geq\ &\lambda_1\|a\|_\infty f\left(\left[\left(\frac{\lambda_2\|b\|_\infty+\mu_2\|\beta\|_\infty}{1-\nu_q^{q-1}}\right)g\left(C\left(\frac{1}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)^{\frac{1}{q-1}}\right]\omega_q\right)\int_\Omega\xi dx \\ &+\mu_1\|\alpha\|_\infty\int_\Omega h\left(C\left(\frac{1}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)\xi dx \\ \geq\ &\int_\Omega\left[\lambda_1 a(x)f(\overline{v})+\mu_1\alpha(x)h(\overline{u})\right]\xi dx.\end{aligned} \tag{2.7}$$

Next

$$\int_\Omega |\nabla \overline{v}|^{q-2} \nabla \overline{v}.\nabla \zeta dx - \int_\Omega |\overline{v}|^{q-2} \overline{v}.\zeta dx \tag{2.8}$$

$$= \left\{ (\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty) g \left( C \left( \frac{1}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right\} \omega_q \int_\Omega \xi dx$$

$$\geq \left[ \lambda_2 \|b\|_\infty g \left( C \left( \frac{1}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) + \mu_2 \|\beta\|_\infty g \left( C \left( \frac{1}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right] \int_\Omega \xi dx$$

By $(H4)$ choose $C$ large so that

$$g \left( C \left( \frac{1}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \geq \gamma \left( \left[ \left( \frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1-\nu_q^{q-1}} \right) g \left( C \left( \frac{1}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right]^{\frac{1}{q-1}} \|\omega_q\|_\infty \right)$$

Then from $(2.7)$ we have

$$\int_\Omega |\nabla \overline{v}|^{q-2} \nabla \overline{v}.\nabla \zeta dx - \int_\Omega |\overline{v}|^{q-2} \overline{v}.\zeta dx \tag{2.9}$$

$$\geq \lambda_2 \|b\|_\infty g \left( C \left( \frac{1}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right)$$

$$+\mu_2 \|\beta\|_\infty \gamma \left( \left\{ \left( \frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1-\nu_q^{q-1}} \right) g \left( C \left( \frac{1}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right\}^{\frac{1}{q-1}} \|\omega_q\|_\infty \right)$$

$$\geq \int_\Omega [b(x) g(\overline{u}) + \mu_2 \beta(x) \gamma(\overline{v})] \zeta dx.$$

According to $(2.7)$ and $(2.8)$, we can conclude that $(\overline{u}, \overline{v})$ is a supersolution of $(2.1)$. Further $\overline{u} \geq \underline{u}$ and $\overline{v} \geq \underline{v}$ for $C$ large, Thus, there exists a solution $(u,v) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ of $(2.1)$ with $\underline{u} \leq u \leq \overline{u}$, and $\underline{v} \leq v \leq \overline{v}$. This completes the proof of Theorem 2.1. ■

Now we show that the more general system $(2.1)$ has also at least distinct three positive solutions.

**Theorem 2.2** [14] *Let $(H1)-(H4)$ hold. Further let $f, g, h,$ and $\gamma$ be sufficiently smooth functions in the neighborhood of zero with $f(0) = h(0) = g(0) = \gamma(0) = 0 = f^{(k)}(0) = h^{(k)}(0) = g^{(l)}(0) = \gamma^{(l)}(0)$ for $k = 1, 2, .... [p-1]$, $l = 1, 2, .... [q-1]$, where $[s]$ denotes the integer part of $s$. Then $(2.1)$ has at least two positive solutions provided $\lambda_i + \mu_i$ are large; $i = 1, 2$.*

**Proof** we will construct a subsolution $(\psi_1,\psi_2)$, a strict supersolution $(\zeta_1,\zeta_2)$, a strict subsolution $(\omega_1,\omega_2)$, and a supersolution $(z_1,z_2)$ for (2.1) such that $(\psi_1,\psi_2)\leq(\zeta_1,\zeta_2)\leq(z_1,z_2)$, $(\psi_1,\psi_2)\leq(\omega_1,\omega_2)\leq(z_1,z_2)$, and $(\omega_1,\omega_2)\nleq(\zeta_1,\zeta_2)$. Then (2.1) has at least three distinct solutions $(u_i,v_i)$, $i=1,2,3$, such that $(u_1,v_1)\in[(\psi_1,\psi_2),(\zeta_1,\zeta_2)]$, $(u_2,v_2)\in[(\omega_1,\omega_2),(z_1,z_2)]$, and $(u_3,v_3)\in[(\psi_1,\psi_2),(z_1,z_2)]\setminus([(\psi_1,\psi_2),(\zeta_1,\zeta_2)]\cup(\omega_1,\omega_2),(z_1,z_2))$.

We first note that $(\psi_1,\psi_2)=(0,0)$ is a solution (hence a subsolution). we can always construct a large supersolution $(z_1,z_2)=(\overline{u},\overline{v})$. We next consider

$$\begin{cases} -\triangle_p\omega_1-|\omega_1|^{p-2}\omega_1=\lambda_1 a(x)\widetilde{f}(\omega_2)+\mu_1\alpha(x)\widetilde{h}(\omega_1) \text{ in } \Omega, \\ -\triangle_q\omega_2-|\omega_2|^{q-2}\omega_2=\lambda_2 b(x)\widetilde{g}(\omega_1)+\mu_2\beta(x)\widetilde{\gamma}(\omega_2) \text{ in } \Omega, \\ \omega_1=\omega_2=0 \text{ on } \partial\Omega, \end{cases} \tag{2.10}$$

where $\widetilde{f}(s)=f(s)-1$, $\widetilde{h}(s)=h(s)-1$, $\widetilde{g}(s)=g(s)-1$, $\widetilde{\gamma}(s)=\gamma(s)-1$. Then by Theorem 2.1, (2.10) has a positive solution $(\omega_1,\omega_2)$ when $\lambda_i+\mu_i$ are large; $i=1,2$. Clearly this $(\omega_1,\omega_2)$ is a strict subsolution of (2.1). Finally we construct the strict supersolution $(\zeta_1,\zeta_2)$.

Let $\phi_p$, $\phi_q$ the corresponding eigenfunction with operator $-\triangle_p$ and $-\triangle_q$. We first note that there exist positive constants $C_1$ and $C_2$ such that

$$\phi_p\leq C_1\phi_q \quad \text{and} \quad \phi_q\leq C_2\phi_p \tag{2.11}$$

Let $(\zeta_1,\zeta_2)=(\rho\phi_p,\rho\phi_q)$ ,where $\rho>0$. Let

$$G_p(x) := (\sigma_p-1)x^{p-1}-\lambda_1 f(C_2x)-\mu_1 h(x)$$

and

$$G_q(x) := (\sigma_q-1)x^{q-1}-\lambda_2 g(C_1x)-\mu_2\gamma(x).$$

Observe that $G_p(0)=G_q(0)=0$, $G_p^{(k)}(0)=G_q^{(l)}(0)=0$ for $k=1,2,....[p-1]$ and $l=1,2,....[q-1]$. $G_p^{(p-1)}(0)>0$ and $G_q^{(q-1)}(0)>0$ if $p,q$ are integers, while $\lim_{r\to+\infty}G_p^{([p])}(r)=+\infty=\lim_{r\to+\infty}G_p^{([q])}(r)$ if $p,q$ are not integers. Thus there exists $\theta$ such that $G_p(x)>0$ and $G_q(x)>0$ for $x\in(0,\theta]$. Hence for $0<\rho\leq\theta$ we have

$$(\sigma_p-1)\,\zeta_1^{p-1}=(\sigma_p-1)\left(\rho\phi_p\right)^{p-1}>\lambda_1 f\left(C_2\rho\phi_p\right)-\mu_1 h\left(\rho\phi_p\right),$$

By $(2.11)$ and $f$ is monotone functions, we have

$$\begin{aligned}(\sigma_p-1)\,\zeta_1^{p-1} &= (\sigma_p-1)\left(\rho\phi_p\right)^{p-1}>\lambda_1 f\left(C_2\rho\phi_p\right)-\mu_1 h\left(\rho\phi_p\right)\\ &\geq \lambda_1 f\left(\rho\phi_q\right)-\mu_1 h\left(\rho\phi_p\right)\\ &= \lambda_1 f\left(\zeta_2\right)-\mu_1 h\left(\zeta_1\right),\quad x\in\Omega,\end{aligned}\tag{2.12}$$

and similarly we get

$$\begin{aligned}(\sigma_q-1)\,\zeta_2^{q-1} &= (\sigma_q-1)\left(\rho\phi_q\right)^{q-1}>\lambda_2 g\left(C_1\rho\phi_q\right)-\mu_2 \gamma\left(\rho\phi_q\right)\\ &\geq \lambda_2 g\left(\rho\phi_p\right)-\mu_2 \gamma\left(\rho\phi_q\right)\\ &= \lambda_2 g\left(\zeta_1\right)-\mu_2 \gamma\left(\zeta_2\right),\quad x\in\Omega,\end{aligned}\tag{2.13}$$

Using the inequalities $(2.12)$ and $(2.13)$, we have

$$\begin{aligned}&\int_\Omega \left|\nabla\zeta_1\right|^{p-2}\nabla\zeta_1.\nabla\xi dx-\int_\Omega \left|\zeta_1\right|^{p-2}\zeta_1.\xi dx\\ &= \rho^{p-1}\left\{\int_\Omega \left|\nabla\phi_p\right|^{p-2}\nabla\phi_p.\nabla\xi dx-\int_\Omega \left|\phi_p\right|^{p-2}\phi_p.\xi dx\right\}\\ &= \int_\Omega\left\{\sigma_p\left(\rho\phi_p\right)^{p-1}-\left(\rho\phi_p\right)^{p-1}\right\}\xi dx,\quad \text{because } \phi_p>0,\\ &= \int_\Omega\left\{(\sigma_p-1)\left(\rho\phi_p\right)^{p-1}\right\}\xi dx\\ &= \int_\Omega (\sigma_p-1)\,\zeta_1^{p-1}\xi dx\\ &> \lambda_1\int_\Omega f\left(\zeta_2\right)\xi dx-\mu_1\int_\Omega h\left(\zeta_1\right).\xi dx,\end{aligned}$$

Similarly we have

$$\begin{aligned}&\int_\Omega \left|\nabla\zeta_2\right|^{q-2}\nabla\zeta_2.\nabla\xi dx-\int_\Omega \left|\zeta_2\right|^{q-2}\zeta_2.\xi dx\\ &> \lambda_2\int_\Omega g\left(\zeta_1\right)\xi dx-\mu_2\int_\Omega \gamma\left(\zeta_2\right)\xi dx.\end{aligned}$$

Thus $(\zeta_1,\zeta_2)$ is a strict supersolution. Here we can choose $\rho$ small so that $(\omega_1,\omega_2) \nleq (\zeta_1,\zeta_2)$. Hence there exist solutions $(u_1,v_1) \in [(\psi_1,\psi_2),(\zeta_1,\zeta_2)]$, $(u_2,v_2) \in [(\omega_1,\omega_2),(z_1,z_2)]$, and $(u_3,v_3) \in [(\psi_1,\psi_2),(z_1,z_2)] \setminus ([(\psi_1,\psi_2),(\zeta_1,\zeta_2)] \cup (\omega_1,\omega_2),(z_1,z_2))$.

Since $(\psi_1,\psi_2) \equiv (0,0)$ is a solution it may turn out that $(u_1,v_1) \equiv (\psi_1,\psi_2) \equiv (0,0)$. In any case we have two positive solutions $(u_2,v_2)$ and $(u_3,v_3)$. Hence Theorem 2.2 holds. ■

## 2.4 Example 1

**Example 2.1** [14] *Let*

$$\begin{aligned} f(x) &= \sum_{i=1}^{m} a_i x^{p_i} - c_1,\ g(x) = \sum_{j=1}^{n} b_j x^{q_j} - c_2 \\ h(x) &= \sum_{k=1}^{s} \alpha_k x^{r_k} - c_3,\ \gamma(x) = \sum_{l=1}^{\tau} \beta_l x^{d_l} - c_4, \end{aligned}$$

*where* $a_i, b_j, \alpha_k, \beta_l, p_i, q_j, r_k, d_j, c_1, c_2, c_3, c_4 \geq 0,\ p_i q_j < (p-1)(q-1),\ r_k < (p-1),\ d_j < (q-1)$.

*Then it is easy to see that* $f, g, h$ *and* $\gamma$ *satisfy the hypotheses of Theorem* 2.1.

## 2.5 Example 2

**Example 2.2** [14] *Let*

$$\begin{aligned} f(x) &= \begin{cases} x^{p_1}, & x \leq 1, \\ \frac{p_1}{p_2} x^{p_2} + \left(1 - \frac{p_1}{p_2}\right), & x > 1, \end{cases},\ h(x) = \begin{cases} x^{p_3}, & x \leq 1, \\ \frac{p_3}{p_4} x^{p_4} + \left(1 - \frac{p_3}{p_4}\right), & x > 1, \end{cases}, \\ g(x) &= \begin{cases} x^{q_1}, & x \leq 1, \\ \frac{q_1}{q_2} x^{q_2} + \left(1 - \frac{q_1}{q_2}\right), & x > 1, \end{cases},\ \gamma(x) = \begin{cases} x^{q_3}, & x \leq 1, \\ \frac{q_3}{q_4} x^{q_4} + \left(1 - \frac{q_3}{q_4}\right), & x > 1. \end{cases} \end{aligned}$$

*Here we assume* $p_1, p_3 > p-1$ *if* $p$ *is an integer,* $p_1, p_3 > [p]$ *if* $p$ *is not an integer,* $q_1, q_3 > q-1$ *if* $q$ *is an integer,* $q_1, q_3 > [q]$ *if* $q$ *is not an integer,* $p_2 q_2 < (p-1)(q-1), p_4 < p-1$ *and* $q_4 < q-1$. *Then it is easy to see that* $f, g, h$ *and* $\gamma$ *satisfy the hypotheses of Theorem* 2.2.

# Chapter 3

# Existence and nonexistence of positive weak solution to the quasilinear elliptic system

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## 3.1 Introduction

In this chapter, we are concerned with the existence and nonexistence of positive weak solution to the quasilinear elliptic system

$$\begin{cases} -\Delta_{p_i} u_i - |u_i|^{p_i-2} u_i = \lambda_i f_i (u_1, ..., u_m) \text{ in } \Omega, 1 \le i \le m \\ \\ u_i = 0 \text{ on } \partial\Omega, \forall i, 1 \le i \le m \end{cases} \tag{3.1}$$

where $\Delta_p z = div \left( |\nabla z|^{p-2} \nabla z \right), p \ge 1, \lambda_i, 1 \le i \le m$ are a positive parameter, and $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We prove the existence of a positive weak solution for $\lambda_i > \lambda_i^*, 1 \le i \le m$ when

$$\lim_{t \to +\infty} \frac{f_i \left(\hat{t}\right)}{t^{p_i - 1}} = 0, \hat{t} = \underbrace{(t, ..., t)}_{m \text{ time}}, \forall i, 1 \le i \le m$$

## 3.2 Definitions and notations

Let

$$W^{1,p} (\Omega) = \{u \in L^p (\Omega) : |\nabla u| \in L^p (\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left( \int_\Omega (|u|^p + |\nabla u|) \, dx \right)^{\frac{1}{p}}$$

then $W^{1,p} (\Omega)$ is a Banach space. We denote by $W_0^{1,p} (\Omega)$ the closure of $C_0^\infty (\Omega)$ in $W^{1,p} (\Omega)$.

Throughout this paper, we let $X$ be the Cartesian product of $m$ spaces $W_0^{1,p_i} (\Omega)$ for $1 \le i \le m$, i.e, $X = W_0^{1,p_1} (\Omega) \times ... \times W_0^{1,p_m} (\Omega)$. We give the definition of weak solution and sub-super solution of $(3.1)$ as follows.

**Definition 3.1** *We called positive weak solution $u = (u_1, ..., u_m) \in X$ of $(3.1)$ such that satisfies*

$$\int_\Omega |\nabla u_i|^{p_i-2} u_i \nabla \phi_i dx - \int_\Omega |u_i|^{p_i-2} u_i \phi_i dx = \lambda_i \int_\Omega f_i (u_1, ..., u_m) \, \phi_i dx,$$

*for all* $\phi = (\phi_1, ..., \phi_m)$ *with* $\phi_i \geq 0, 1 \leq i \leq m.$

**Definition 3.2** *We called positive weak subsolution* $(\psi_1, ..., \psi_m) \in X$ *and supersolution* $(z_1, ..., z_m) \in X$ *of* $(3.1)$ *such that* $\psi_i \leq z_i$, $\forall 1 \leq i \leq m$, *satisfies*

$$\int_\Omega |\nabla \psi_i|^{p_i-2} \psi_i \nabla \phi_i dx - \int_\Omega |\psi_i|^{p_i-2} \psi_i \phi_i dx \leq \lambda_i \int_\Omega f_i (\psi_1, ..., \psi_m) \phi_i dx,$$

*and*

$$\int_\Omega |\nabla z_i|^{p_i-2} z_i \nabla \phi_i dx - \int_\Omega |z_i|^{p_i-2} z_i \phi_i dx \geq \lambda_i \int_\Omega f_i (z_1, ..., z_m) \phi_i dx,$$

*for all* $\phi = (\phi_1, ..., \phi_m)$ *with* $\phi_i \geq 0, 1 \leq i \leq m.$

**Lemma 3.1** (*See Haghaieghi and Afrouzi* [20]). *Suppose there exist subsolutions and supersolutions* $(\psi_1, ..., \psi_m)$ *and* $(z_1, ..., z_m)$ *respectively of system* $(3.1)$ *such that* $(\psi_1, ..., \psi_m) \leq (z_1, ..., z_m)$. *Then system* $(3.1)$ *has a solution* $(u_1, ..., u_m)$ *such that* $(u_1, ..., u_m) \in [(\psi_1, ..., \psi_m), (z_1, ..., z_m)]$.

**Lemma 3.2** *(generalized Young inequality). Let* $a_k > 0$ *and* $1 < p_k < \infty$ *for* $1 \leq k \leq m$, *numbers conjugates. Then*

$$\forall a_k > 0 : \Pi_{k=1}^m a_k \leq \sum_{k=1}^m \frac{a_k^{p_k}}{p_k}, \text{ where } \sum_{k=1}^m \frac{1}{p_k} = 1.$$

**Proof** We can see that

$$\Pi_{k=1}^m a_k = \exp^{\ln\left(\Pi_{k=1}^m a_k\right)} = \exp\left(\sum_{k=1}^m \frac{\ln\left(a_k^{p_k}\right)}{p_k}\right),$$

according to the convexity of the exponential function, Then

$$\begin{aligned}\Pi_{k=1}^{m} a_k &= \exp\left(\sum_{k=1}^{m} \frac{\ln\left(a_k^{p_k}\right)}{p_k}\right) \\ &\leq \sum_{k=1}^{m} \frac{1}{p_k} \exp\left(\ln\left(a_k^{p_k}\right)\right) \\ &\leq \sum_{k=1}^{m} \frac{a_k^{p_k}}{p_k}, \text{ where } \sum_{k=1}^{m} \frac{1}{p_k} = 1.\end{aligned}$$

■

We suppose that $f_i : ([0,\infty[)^m \to \mathbb{R}$ are in $L^{p_i^*}(X), p_i^* = \frac{Np_i}{N-p_i}, 1 \leq i \leq m$, verify the following assumptions;

$$\begin{cases} 1)\ f_i \text{ are quasi-monotone nondecreasing with respect to } t_k \\ \text{i.e; } f_i\left(t_1, ..., t_k^1, ..., t_m\right) \leq f_i\left(t_1, ..., t_k^2, ..., t_m\right), \forall t_k^1 \leq t_k^2, \forall k, 1 \leq k \leq m \\ 2)\ \lim\limits_{t_k \to +\infty} f_i\left(t_1, ..., t_m\right) = +\infty, \forall k, 1 \leq k \leq m, \\ 3)\ \exists \theta > 0 : f_i\left(t_1, ..., t_m\right) \geq -\theta\ , \forall t_1, ..., t_m \geq 0, \end{cases} \tag{H1}$$

$$\forall i, 1 \leq i \leq m : \lim_{t \to +\infty} \frac{f_i\left(\hat{t}\right)}{t^{p_i - 1}} = 0, \hat{t} = \underbrace{(t, ..., t)}_{m \text{ time}} \in \mathbb{R}^m, \tag{H2}$$

$$\exists \xi_{i,j} > 0, , 1 \leq i, j \leq m : f_i\left(t_1, ..., t_m\right) \leq \sum_{j=1}^{m} \xi_{i,j} t_j^{\left(\frac{p_i - 1}{p_i}\right) p_j} \tag{H3}$$

An example:

$$f_i(t) = f_i\left(t_1, ..., t_m\right) = a_i \prod_{j=1}^{m} t_j^{\sigma_{i,j}} - C_i, \text{ where } \sigma_{i,j}, a_i, C_i > 0.$$

It is easy to see that $f_i$ satisfy $(H1)$ and $(H2)$ if $\sum_{j=1}^{m} \sigma_{i,j} < p_i - 1, 1 \leq i, j \leq m$ and satisfy $(H3)$ if $\sigma_{i,j} = \frac{p_i - 1}{p_i}, 1 \leq i, j \leq m$ .

Let $\lambda_{p_i}$ be the first eigenvalue of $-\Delta_{p_i}$ with Dirichlet boundary conditions and $\varphi_i$ the corresponding positive eigenfunction with $\|\varphi_i\|_\infty = 1$, and $M_i, \sigma_i, \delta > 0, 1 \leq i \leq m$ such that

$$\forall i, 1 \leq i \leq m : \begin{cases} |\nabla \varphi_i|^{p_i} - \lambda_{p_i} \varphi_i^{p_i} \geq M_{p_i}, \text{ on } \overline{\Omega}_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\} \\ \\ \varphi_i \geq \sigma_i \text{ on } \Omega \backslash \overline{\Omega}_\delta, \end{cases}$$

The assumption $(H1)$ assume that

$$\exists \eta_i = (\eta_i^1, ..., \eta_i^m), \eta_i^k > 0 : \ f_i(\eta_i) = \frac{\theta \lambda_{p_i}}{M_{p_i}}, \forall i, k, 1 \leq i, k \leq m$$

## 3.3 Main results

**Theorem 3.1** [10] *Let* $(H1)$ *and* $(H2)$ *hold. Then for* $\lambda_i > \lambda_i^*, \forall i, 1 \leq i \leq m$*, the system* $(3.1)$ *has a large positive solution* $u = (u_1, ..., u_m) \in X$*, where*

$$\lambda_i^* = \frac{M_{p_i}}{\theta \sigma_i^{p_i}} \left( \frac{p_i}{p_i - 1} \right)^{p_i - 1} \left( \max_{k=1,\dots,m} (\eta_k^i) \right)^{p_i - 1}$$

**Proof** We shall verify that $\psi_i, 1 \leq i \leq m$, where

$$\psi_i = \left( \frac{\theta \lambda_i}{M_{p_i}} \right)^{\frac{1}{p_i - 1}} \left( \frac{p_i - 1}{p_i} \right) \varphi_i^{\frac{p_i}{p_i - 1}},$$

is a subsolution of $(3.1)$ for $\lambda_i, 1 \leq i \leq m$ large. Let $\phi_i \in W_0^{1,p_i}(\Omega)$ with $\phi_i \geq 0, 1 \leq i \leq m$.A calculation shows that

$$\begin{aligned} \int_\Omega |\nabla \psi_i|^{p_i - 2} \psi_i \nabla \phi_i dx - \int_\Omega |\psi_i|^{p_i - 2} \psi_i \phi_i dx &\leq \frac{\theta \lambda_i}{M_{p_i}} \int_\Omega \varphi_i |\nabla \varphi_i|^{p_i - 2} \nabla \varphi_i \nabla \phi_i dx \\ &= \frac{\theta \lambda_i}{M_{p_i}} \left\{ \int_\Omega |\nabla \varphi_i|^{p_i - 2} \nabla \varphi_i \nabla (\varphi_i \phi_i) dx - \int_\Omega |\nabla \varphi_i|^{p_i} \phi_i dx \right\} \\ &= \frac{\theta \lambda_i}{M_{p_i}} \int_\Omega (\lambda_{p_i} \varphi_i^{p_i} - |\nabla \varphi_i|^{p_i}) \phi_i dx. \end{aligned}$$

Now, on $\overline{\Omega}_\delta$ we have $|\nabla \varphi_i|^{p_i} - \lambda_{p_i} \varphi_i^{p_i} \geq M_{p_i}$, which implies that

$$\frac{\theta}{M_{p_i}} (\lambda_{p_i} \varphi_i^{p_i} - |\nabla \varphi_i|^{p_i}) - f_i(\psi_1, ..., \psi_m) \leq 0.$$

Hence

$$\int_{\overline{\Omega}_\delta} |\nabla\psi_i|^{p_i-2}\psi_i\nabla\phi_i dx - \int_{\overline{\Omega}_\delta} |\psi_i|^{p_i-2}\psi_i\phi_i dx \leq \lambda_i \int_{\overline{\Omega}_\delta} f_i(\psi_1,...,\psi_m)\phi_i dx. \tag{3.2}$$

Next, on $\Omega\backslash\overline{\Omega}_\delta$, we have $\varphi_k \geq \sigma_k$ for some $\sigma_k > 0$, and therefore for $\lambda_k \geq \lambda_k^*$,

$$\psi_k \geq \left(\frac{\theta\lambda_k^*}{M_{p_k}}\right)^{\frac{1}{p_k-1}} \left(\frac{p_k-1}{pk}\right)\sigma_k^{\frac{p_k}{p_k-1}} = \left(\max_{1\leq k\leq m}\eta_i^k\right) \geq \eta_i^k, \quad \forall i,k, 1\leq i,k\leq m.$$

then

$$f_i(\psi_1,...,\psi_m) \geq f_i\left(\eta_i^1,...,\eta_i^m\right) = f_i(\eta_i) = \frac{\theta\lambda_{p_i}}{M_{p_i}} \geq \frac{\theta}{M_{p_i}}\left(\lambda_{p_i}\varphi_i^{p_i} - |\nabla\varphi_i|^{p_i}\right), \ \forall i, \ 1\leq i\leq m.$$

Hence

$$\int_{\Omega\backslash\overline{\Omega}_\delta} |\nabla\psi_i|^{p_i-2}\psi_i\nabla\phi_i dx - \int_{\Omega\backslash\overline{\Omega}_\delta} |\psi_i|^{p_i-2}\psi_i\phi_i dx \leq \lambda_i \int_{\Omega\backslash\overline{\Omega}_\delta} f_i(\psi_1,...,\psi_m)\phi_i dx. \tag{3.3}$$

Finaly, by $(3.2)$ and $(3.3)$ we have

$$\begin{aligned}\int_\Omega |\nabla\psi_i|^{p_i-2}\psi_i\nabla\phi_i dx - \int_\Omega |\psi_i|^{p_i-2}\psi_i\phi_i dx &= \int_{\overline{\Omega}_\delta} |\nabla\psi_i|^{p_i-2}\psi_i\nabla\phi_i dx - \int_{\overline{\Omega}_\delta} |\psi_i|^{p_i-2}\psi_i\phi_i dx \\ &\quad + \int_{\Omega\backslash\overline{\Omega}_\delta} |\nabla\psi_i|^{p_i-2}\psi_i\nabla\phi_i dx - \int_{\Omega\backslash\overline{\Omega}_\delta} |\psi_i|^{p_i-2}\psi_i\phi_i dx \\ &\leq \lambda_i \int_\Omega f_i(\psi_1,...,\psi_m)\phi_i dx.\end{aligned}$$

i.e , $\psi = (\psi_1,...,\psi_m) \in X$ is a subsolution of $(3.1)$ .

Next, let $\omega_i$ be the solution of

$$\begin{cases} -\triangle_{p_i}\omega_i = 1 \text{ in } \Omega, \\ \omega_i = 0 \text{ on } \partial\Omega. \end{cases}$$

Let

$$z_i = \left(\frac{\lambda_i}{1-\|\omega_i\|_\infty^{p_i-1}}\right)^{\frac{1}{p_i-1}} \left(f_i\left(C\lambda_1^{\frac{1}{p_1-1}},...,C\lambda_1^{\frac{1}{p_1-1}}\right)\right)^{\frac{1}{p_i-1}}\omega_i, \ 1\leq i\leq m,$$

where $C \geq 0$ is a large number to be chosen later.We shall verify that $z = (z_1,...,z_m) \in X$ is a supersolution of $(3.1)$ for $\lambda_i$ large. To this end, let $\phi_i \geq 0$, $1 \leq i \leq m$.

By $(H1)$ and $(H2)$ we can choose $C$ large enough so that

$$\begin{aligned}\left(C\lambda_1^{\frac{1}{p_1-1}}\right)^{p_i-1} &\geq \mu_i^{p_i-1}\frac{\lambda_i}{1-\mu_i^{p_i-1}}f_i\left(C\lambda_1^{\frac{1}{p_1-1}},...,C\lambda_1^{\frac{1}{p_1-1}}\right)\\ &\geq \frac{\lambda_i}{1-\mu_i^{p_i-1}}f_i\left(C\lambda_1^{\frac{1}{p_1-1}},...,C\lambda_1^{\frac{1}{p_1-1}}\right)\omega_i^{p_i-1}\\ &\geq z_i^{p_i-1},\end{aligned}$$

where $\mu_i = \|\omega_i\|_\infty$ . Then

$$C\lambda_1^{\frac{1}{p_1-1}} \geq z_i,\ \forall i,\ 1 \leq i \leq m,$$

which imply that

$$f_i\left(C\lambda_1^{\frac{1}{p_1-1}},...,C\lambda_1^{\frac{1}{p_1-1}}\right) \geq f_i\left(z_1,...,z_m\right),\ \forall i,\ 1 \leq i \leq m,$$

Then we have

$$\begin{aligned}\int_\Omega |\nabla z_i|^{p_i-2} z_i \nabla\phi_i dx - \int_\Omega |z_i|^{p_i-2} z_i\phi_i dx &= \frac{\lambda_i}{1-\|\omega_i\|_\infty^{p_i-1}}f_i\left(C\lambda_1^{\frac{1}{p_1-1}},...,C\lambda_1^{\frac{1}{p_1-1}}\right)\int_\Omega |\nabla\omega_i|^{p_i-2}\omega_i\nabla\phi_i dx\\ &\quad -\frac{\lambda_i}{1-\|\omega_i\|_\infty^{p_i-1}}f_i\left(C\lambda_1^{\frac{1}{p_1-1}},...,C\lambda_1^{\frac{1}{p_1-1}}\right)\int_\Omega |\omega_i|^{p_i-2}\omega_i\phi_i dx\\ &\geq \frac{\lambda_i}{1-\|\omega_i\|_\infty^{p_i-1}}f_i\left(C\lambda_1^{\frac{1}{p_1-1}},...,C\lambda_1^{\frac{1}{p_1-1}}\right)\int_\Omega \left(-\triangle_{p_i}\omega_i - \|\omega_i\|_\infty^{p_i-1}\right)\phi_i d\\ &\geq \frac{\lambda_i}{1-\|\omega_i\|_\infty^{p_i-1}}f_i\left(C\lambda_1^{\frac{1}{p_1-1}},...,C\lambda_1^{\frac{1}{p_1-1}}\right)\int_\Omega \left(1 - \|\omega_i\|_\infty^{p_i-1}\right)\phi_i dx\\ &= \lambda_i\int_\Omega f_i\left(C\lambda_1^{\frac{1}{p_1-1}},...,C\lambda_1^{\frac{1}{p_1-1}}\right)\phi_i dx.\end{aligned}$$

i.e., $(z_1,...,z_m) \in X$ is a supersolution of $(3.1)$ with $z_i \geq \psi_i$, $1 \leq i \leq m$, for $C$ large. Thus, there exists a solution $u = (u_1,...,u_m) \in X$ of $(3.1)$ with $\psi_i \leq u_i \leq z_i$, $1 \leq i \leq m$.This completes the proof. ■

**Theorem 3.2** [10] *If $f_i$, $1 \leq i \leq m$, verify $(H3)$ the system $(3.1)$ has not nontrivial positive solutions for $0 < \lambda_i^0 < \lambda_{p_i}$, where*

$$\lambda_i^0 = \left(1 + \frac{\lambda_i}{p_i}\sum_{j=1}^m \xi_{i,j}\right) + \left(\sum_{k=1}^m \left(\frac{p_k-1}{p_k}\right)\lambda_k\xi_{k,i}\right)$$

**Proof** Multiplying Eq. $(3.1)$ by $u_i$ and integrating over $\Omega$, we obtain

$$\begin{aligned}
\int_\Omega |\nabla u_i|^{p_i}\,dx - \int_\Omega |u_i|^{p_i}\,dx &= \lambda_i \int_\Omega f_i(u_1,...,u_m)\,u_i dx \\
&\leq \lambda_i \int_\Omega \sum_{j=1}^m \xi_{i,j} u_j^{\left(\frac{p_i-1}{p_i}\right)p_j} u_i dx \\
&\leq \frac{\lambda_i}{p_i}\left[\int_\Omega \sum_{j=1}^m \xi_{i,j}\left(u_i^{p_i} + (p_i-1)\,u_j^{p_j}\right)\right] dx \\
&= \frac{\lambda_i}{p_i}\left[\left(\sum_{j=1}^m \xi_{i,j}\right)\|u_i\|_{p_i}^{p_i} + (p_i-1)\sum_{j=1}^m \xi_{i,j}\|u_j\|_{p_i}^{p_i}\right]
\end{aligned}$$

It follows that

$$\int_\Omega |\nabla u_i|^{p_i}\,dx \leq \left(1+\frac{\lambda_i}{p_i}\left(\sum_{j=1}^m \xi_{i,j}\right)\right)\|u_i\|_{p_i}^{p_i} + \sum_{k=1}^m \left(\frac{p_k-1}{p_k}\right)\lambda_k \xi_{k,i}\|u_k\|_{p_i}^{p_i},$$

in an other hand

$$\lambda_{p_i} = \inf \frac{\|\nabla u_i\|_{p_i}^{p_i}}{\|u_i\|_{p_i}^{p_i}},\ u_i \in W_0^{1,p_i}\setminus\{0\}$$

Then, we have

$$\sum_{i=1}^m \left(\lambda_{p_i} - \lambda_i^0\right)\|u_i\|_{p_i}^{p_i} \leq 0,$$

but this is contradiction if $0 < \lambda_i^0 < \lambda_{p_i}$. This completes the proof. ■

**Corollary 3.1** [10] *Consider the following system in $X$*

$$\begin{cases} -\Delta_{p_i} u_i - |u_i|^{p_i-2} u_i = \lambda_i \prod_{k=1}^m u_k^{\beta_{i,k}} \text{ in } \Omega, \forall i, 1 \leq i \leq m, \\ \\ u_i = 0 \text{ on } \partial\Omega, \forall i, 1 \leq i \leq m \end{cases} \tag{3.4}$$

*1) The system $(3.4)$ has a positive weak solution if*

$$\sum_{k=1}^m \beta_{i,k} < p_i - 1, \forall i, 1 \leq i \leq m \tag{3.5}$$

*2) The system $(3.4)$ has not positive weak solution if $\bar{\lambda}_i < \lambda_{p_i}$ and*

$$\sum_{k=1}^m \frac{\beta_{i,k} + \delta_{i,k}}{p_k} = 1 \tag{3.6}$$

*where*

$$\bar{\lambda}_i = 1 + \sum_{k=1}^{m} \frac{\beta_{i,k} + \delta_{i,k}}{p_k} \lambda_k, \text{ with } \delta_{i,k} = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{cases}$$

**Proof** 1. Using Theorem 3.1, the assumption (3.5) imply the desired result.

2. Under (3.6) and the generalized Young inequality ( Lemma 2) ,we can deduce that

$$u_i \Pi_{k=1}^{m} u_k^{\beta_{i,k}} \leq \sum_{k=1}^{m} \frac{\beta_{i,k} + \delta_{i,k}}{p_k} u_k^{p_k}, \quad \forall i, 1 \leq i \leq m. \tag{3.7}$$

Multiplying the equation (3.4) by $u_i$ and integrating over $\Omega$ we obtain by using (3.7)

$$\begin{aligned} \int_{\Omega} |\nabla u_i|^{p_i} \, dx - \int_{\Omega} |u_i|^{p_i} \, dx &\leq \lambda_i \int_{\Omega} \sum_{k=1}^{m} \frac{\beta_{i,k} + \delta_{i,k}}{p_k} u_k^{p_k} dx \\ &= \lambda_i \sum_{k=1}^{m} \frac{\beta_{i,k} + \delta_{i,k}}{p_k} \|u_k\|_{p_k}^{p_k}, \end{aligned}$$

Then

$$\int_{\Omega} |\nabla u_i|^{p_i} \, dx \leq \left[ 1 + \lambda_i \sum_{k=1}^{m} \frac{\beta_{i,k} + \delta_{i,k}}{p_k} \right] \|u_k\|_{p_k}^{p_k},$$

in an other hand

$$\lambda_{p_i} \|u_i\|_{p_i}^{p_i} \leq \|\nabla u_i\|_{p_i}^{p_i} \leq \left[ 1 + \lambda_i \sum_{k=1}^{m} \frac{\beta_{i,k} + \delta_{i,k}}{p_k} \right] \|u_k\|_{p_k}^{p_k},$$

Then

$$\sum_{i=1}^{m} \left( \lambda_{p_i} - \overline{\lambda_i} \right) \|u_i\|_{p_i}^{p_i} = \sum_{i=1}^{m} \left[ \lambda_{p_i} - \left( 1 + \sum_{k=1}^{m} \frac{\beta_{i,k} + \delta_{i,k}}{p_k} \lambda_k \right) \right] \|u_i\|_{p_i}^{p_i} \leq 0.$$

which is a contradiction if $\overline{\lambda_i} \leq \lambda_{p_i}$. This completes the proof. ■

# Conclusion

The sub and super solution method is a tool to show the existence of at least a weak solution of semilinear problems thanks to the eigenfunction associated with the $p$-Laplacian operator, nevertheless the uniqueness remains open by this method.

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