People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
Larbi University Tebessi – Tebessa
Faculty of the exact sciences and sciences of nature and life
Department: Mathematics and Informatic

Final dissertation
For the graduation of Master
Domain: Mathematics and Informatic
Field: Mathematics
Option: Partial Derivative Equations and Applications
Theme

Recent advences in averaged controllability of hyperbolic PDEs

Presented by :

Haithem Amrani

In front of the jury:

Mr, Salem Abdelmalek	MCA Larbi University Tebessi President
Mr, Nouri Boumaza	MCA Larbi University Tebessi Examiner
Mr , Abdelhak Hafdallah	MCA Larbi University Tebessi Supervisor

Date of dissertation : 19/06/2021

بسم الله الرحمن الرحيم

Contents

Abstract

The aim of this memory is to study the averaged controllability for parameter-dependent systems. We discuss the notion of averaged control which has been introduced recently by E.Zauzua [12]. We apply the method HUM where it is based on uniqueness criteria (the direct and inverse inequalities) to some hyperbolic equations with an unknown parameter.

First, we consider the problem of controllability for linear finite and infinite dimensional systems and we give the averaged rank condition for averaged controllability.

Secondly, we study the averaged controllability of wave equation depending on a parameter when the control is applied on the boundary, by using the Hilbert Uniqueness Method (the method HUM) which has been introduced by J.L. Lions [8], where this method allows us to design the avraged control of our parameter-dependent wave equation.

Finally, we study with the same argument of the HUM method the problem of vibrating plate equation depending on a parameter when the control is applied on the boundary also.

Keywords: Averaged control, hyperbolic equation, the Hilbert Uniqueness Method-(HUM), parameter-dependent wave equations, parameter-dependent vibrating plate equations, averaged energy, averaged direct inequality, averaged inverse inequality.

Résumé

L'objectif de ce mémoire est d'étudier la contrôlabilité moyenne pour des systèmes dépendants de paramètres. Nous discutons la notion de contrôle moyenné récemment introduite par E.Zauzua [12] . Nous appliquons la méthode HUM où elle est basée sur des critères d'unicité (les inégalités directes et inverses) à quelques équations hyperboliques avec un paramètre inconnu.

Premièrement, nous considérons le problème de la contrôlabilité pour les systèmes linéaires de dimension finie et infinie et nous donnons la condition de rang moyen pour la contrôlabilité moyenne.

Deuxièmement, nous étudions la contrôlabilité moyenne de l'équation d'onde en fonction d'un paramètre lorsque le contrôle est appliqué sur la frontière, en utilisant la méthode d'unicité de Hilbert (la méthode HUM) qui a été introduite par J.L. Lions [8] , où cette méthode nous permet de concevoir le contrôle moyen de notre équation d'onde dépendante des paramètres.

Enfin, nous étudions avec le même argument de la méthode HUM le problème de l'équation de la plaque vibrante dépendant d'un paramètre lorsque le contrôle est appliqué sur la frontière aussi.

Mots-clés : Contrôle moyenné, équation hyperbolique, méthode d'unicité de Hilbert (HUM), équations d'onde dépendantes des paramètres, équations de plaque vibrante dépendantes des paramètres , énergie moyenne, inégalité directe moyenne, inégalité inverse moyenne.

ملخص

الهدف من هذه الذاكرة هو دراسة متوسط القدرة على التحكم للأنظمة المعتمدة على المعلمات. نناقش مفهوم التحكم المتوسط الذي تم تقديمه مؤخرًا بواسطة E.Zauzua[12]. نطبق طريقة HUM حيث تعتمد على معايير التفرد (المباشر والعكسي عدم المساواة) لبعض المعادلات القطعية ذات المعلمة غير المعروفة.

أولاً ، نعتبر مشكلة قابلية التحكم للأنظمة الخطية المحدودة وغير المحدودة الأبعاد ونعطي شرط الترتيب المتوسط لقابلية التحكم المتوسطة.

ثانيًا ، ندرس متوسط قابلية التحكم في معادلة الموجة اعتمادًا على معلمة متى يتم تطبيق عنصر التحكم على الحدود باستخدام طريقة هيلبرت الفريدة (طريقة HUM) الذي تم تقديمه بواسطة JL Lions [8] ، حيث تسمح لنا هذه الطريقة بتصميم ملف يتم التحكم في معادلة الموجة المعتمدة على المعلمة.

أخيرًا ، ندرس بنفس حجة طريقة HUM مشكلة اهتزاز اللوحة تعتمد المعادلة على معلمة عند تطبيق عنصر التحكم على الحدود أيضًا.

الكلمات المفتاحية: التحكم المتوسط ، المعادلة القطعية ، طريقة هيلبرت الفريدة- (HUM) ، معادلات الموجة المعتمدة على المعلمة ، معادلات لوحة الاهتزاز المعتمدة على المعلمة ، متوسط الطاقة ، متوسط عدم المساواة المباشرة ، متوسط عدم المساواة العكسية.

Dedication

This memory is dedicated
To... My great parents, who never stop giving of themselves in countless ways.
To... My beloved brothers and sisters.
To... all the people in my life who touch my heart.

Acknowledgments

In preamble to this research memory we thank **ALLAH** who helps us and gives us patience and courage during these long years of study.

To my dear parents in recognition of their various sacrifices, their precious advice, moral support and encouragement throughout my studies. I can not thank them enough for everything they did to me.

We wish to express our sincere thanks to the people who helped us and contributed to the development of this dissertation and to the success of this wonderful academic year.

These thanks go first to the faculty and administration of the Faculty of Math and computer science L'arbi Tébessi, for the richness and quality of their teaching and who make great efforts to provide their students with updated training.

We would like to sincerely thank **Dr.Abdelhak Hafdallah** who, as supervisor of the memory, Whom we respect deeply, has always been very attentive and available throughout the realization of this memory, inspiration, help and time that we have been dedicated. and without him this memroy would never have been born.

All the grateful to the members of the jury **Dr.Salem Abdelmalek** and **Dr.Nouri Boumaza** who accepted to examine our research work.

Finally, we express our sincere thanks to all our relatives and friends who have always encouraged us during these memos.

Thank you all.

Amrani Haithem.

Notations

$\mathbb{R}$	Set of real numbers.
$\Vert.\Vert_H$	A norm in space H.
$\langle .,.\rangle_H$	A scalar product in Hilbert space H.
$\langle .,.\rangle_{H^*,H}$	Duality product between H and H^*.
C^2	The class of functions with continuous first and second derivative.
$\frac{\partial y}{\partial \nu} = \nabla y.\nu$	The conormal derivative.
$\Delta = \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}$	The laplacien operator.
$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)^T$	The gradient operator.
div	Divergence.
$\mathcal{A}^*$	The adjoint operator of $\mathcal{A}$.
$d\Gamma$	Lebesgue measure on boundary Γ.
χ_ω	Characteristic function of the set ω.
$\mathcal{L}(E, F)$	The space of linear bounded operators from E to F.
$\mathcal{D}(Q)$	The space of functions in C^∞ with a compact support tin Q.
ODE	Ordinary differential equation.
PDE	Partial differential equation.
BI problem	boundary initial problem.
iff	If and only if.
a.e.	Almost every where.
HUM	the Hilbert Uniqueness Method

Figures list

Introduction

The evolution over time of many physical, biological, economic or mechanics is modeled by partial differential equations (PDE) and ordinary differential equations (ODE) , the interest of modeling of a phenomenon of mathematical objects is the possibility of understanding the phenomenon and from the influence of different parameters, also to forecasting through simulation. Often we seeks to investigate the possibility of acting on a system so that it functions for a desired purpose, or "At best", "at least cost" etc. It is the object of the theory of control which is a theory mathematics allowing to determine the laws of guidance, of action, on a given system. The controllability of partial differential equations is a subject in full development. His history began with the case of the finite dimension, its extension to the infinite dimension has experienced several times. This subject has undergone very significant development since the work of Jacques-Louis Lions [8] in the late 1970. Specifically, J.L. Lions introduced the so-called Hilbert Uniqueness Method (the HUM method [8]) and this was the point start of a fruitful period on the subject. The 90's are marked by highlights among them, C. Bardos, G. Lebeau and J. Rauch give a condition (almost) necessary and sufficient exact controllability of the wave equation controlled on part of the edge or domain using microlocal analysis results.
The issues identifying with nature, climate, and environment are currently the focal point of numerous researchers, residents, ideological groups, organizations, and states due to its worldwide impact and interest. The environment assumes a critical part at all levels, particularly its significant change "an Earth-wide temperature boost". It is notable that air and water are genuine wellsprings of life for verdure, fauna, and people. In this way, as their inclinations are defiled by natural assaults, they become threats for living creatures. This may incorporate vegetative issues for greenery and inebriation or even instances of sicknesses for people. Researchers are attempting to decide the best palliatives for the security and sterilization of these regular assets. They can not, hence, prevail without interdisciplinary participation. From data observed by naturalists to models of equations designed by mathematicians to the expertise of computer engineers, digital simulation plays a very important role in the mediation between scientific disciplines.
In reality, when modeling those phenomenon and other phenomenon (for instance in physical science, statics, dynamic populace and numerous different strengths) we experience some missing information, due to unavailability of some information or due to different reasons, for instance, in practically every one of the issues of meteorology or oceanography, we never know the underlying information, we have an incredible assortment of conceivable outcomes while picking

the underlying second. Same thing for the issues of contamination in a lake, a stream or in an estuary.

In addition, boundary conditions may also be unknown or only partially known on a part of the boundary that may, for example, be inaccessible to measurements whether biomedical situations or situations corresponding to accidents. The same goes for source terms that can be difficult to access, the same for the structure of the domain, which can also be imperfectly known, for example in oil well management where part of the boundary of the domain is unknown.

Hence, their modeling leads to PDEs with some incomplete data (or missing data). In system analysis, incomplete data means that the initial conditions, boundary conditions, second member of the equation or some of the parameters in the main operator in the system are unknown. One of the objectives in the study of those problems is to control her regardless of the missing terms in their associated mathematical model. Throughout this thesis, we use the terms 'missing data', 'incomplete data' or 'uncertainty' equivalently.

In this memory, we are interested in the *averaged control* [12]of distributed systems with incomplete data.

However, average control is a new concept in control theory introduced by E. Zuazua (2014)[12] to control systems containing an unknown parameter. A natural idea is to solve those problems is to look for a robust control i.e. looking for control independently of the unknown parameter. Simply, the idea consists on controlling the average of state with respect to the unknown parameter to be equal or closed to a fixed target, then in (Lazar & Zuazua, 2014) authors studied the problem of averaged controllability and observability both for a wave equation, and in (Lohéac & Zuazua, 2017) authors treated the problem of averaged controllability for a general control systems.

Note that in the previous studies of control problems with missing data, authors take into account an unknown parameter in the main equation. and for find the robust control they applied the average control.

In our case, we treat the case where the considered model contains an unknown parameter in the main equation. We introduce the notions of *averaged control*, *averaged rank condition and HUM method* to study such kind of control problems with missing data.

Memory overview

This memory is divided into three chapters:

In the first chapter, we give a brief overview of the classical control theory for distributed systems. Then, we outline the notions of average control and the equivalent notion of controllability and observabillty with a characterization of each one in the case of an abstract equation. And we give applications of the concept of average control. We finish the chapter by the basic principles of the

Hilbert Uniqueness Method (HUM).

In the second chapter, we present the wave equation with a parameter depending, the averaged energy associated to the wave equation and the direct & inverse inequalities which plays an important role in the application of HUM method.

In the last chapter, we treat the averaged null controllability for a parameter-dependent vibrating plate equation by the same argument of HUM method.

Chapter 1

Averaged controllability

We are interested in this chapter to talk about the problem of controlling systems submitted to parametrized perturbations, either finite or infinite dimensional systems. We have also given examples of average control system heat equation and the inverted pendulum and we analyze the problem of averaged observability, this topic is motivated by the control of parameter-dependent systems. We look for controls ensuring the controllability of the averages of the states with respect to the parameter. This turns out to be equivalent to the problem of averaged observation in which one aims at recovering the energy of the initial data of the adjoint system by measurements done on its averages, under the assumption that the initial data of all the components of the adjoint system coincide.

1.1 Averaged controllability of finite dimensional systems

1.1.1 Problem position

We consider the finite dimensional control system in the linear case by the following form

$$\begin{cases} y'(t) = A(\nu)y(t) + B(\nu)u(t), 0 < t < T, \\ \qquad\qquad y(0) = y^0. \end{cases} \tag{1.1}$$

where $A(\nu)$ is a matrix in $\mathcal{M}_{N\times N}(\mathbb{R})$, $u := u(t)$ is a M-component control vector in $\mathbb{R}^M$ such that $M \leq N$, entering and acting on the system through the control matrix $B(\nu)$, a $N \times M$ parameter-dependent matrix. The parameter $\nu \in R$, for simplicity, we'll suppose ν in the interval $(0, 1)$. In principle, the initial datum $y^0 \in \mathbb{R}^N$ is supposed independent of the parameter ν. The state it self $y(t, \nu)$ depends on ν. In (1.1), the vector valued function $y(t, \nu) := (y^1(t, \nu), ..., y^N(t, \nu)) \in \mathbb{R}^N$ represent the state of the system.

Definition 1.1 *[12] We say that the system (1.1) is controllable in average if for final target $y_d \in \mathbb{R}^N$ and arbitrary initial data y^0 there exist control time $T > 0$ and a control function u independent of ν such that the solution of (1.1) satisfies*

$$\int_0^1 y(T,\nu)d\nu = y_d. \tag{1.2}$$

1.1.2 Averaged controllability

Theorem 1.1 (The averaged rank condition) ***[12]*** *The system (1.1) is averaged controllable if and only if the following rank condition satisfied:*

$$\text{Rank}\left[\int_0^1 [A(\nu)]^j B(\nu)d\nu\right] = N \text{ , for all } j = \overline{0 \cdots N-1}. \tag{1.3}$$

Remark 1.1 *The averaged rank condition can be simplified when all the matrices $A(\nu)$,$B(\nu)$ are multiples of the same constant matrices A,B: $A(\nu) = \alpha(\nu)A$, $B(\nu) = \beta(\nu)B$. So in this case,*

$$\int_0^1 [A(\nu)]^j B(\nu)d\nu = A^j B \int_0^1 [\alpha(\nu)]^j \beta(\nu)d\nu, \forall j \geq 0,$$

and

$$\left[\int_0^1 [A(\nu)]^j B(\nu)d\nu\right] = \left[A^j B \int_0^1 [\alpha(\nu)]^j \beta(\nu)d\nu\right] \text{ for all } j = \overline{0 \cdots N-1}.$$

Thus, under the further assumption that : $\int_0^1 [\alpha(\nu)]^j \beta(\nu)d\nu \neq 0$, $j = 1, .., N-1$, the averaged rank condition is equivalent to the classical one, see the appendix (3.5)

$$\text{Rank}\left[B, AB, A^2B, ..., A^{N-1}B\right] = N.[4] \tag{1.4}$$

Involving only powers of A up to order $N-1$. If some of the integrals $\int_0^1 [\alpha(\nu)]^j \beta(\nu)d\nu$ vanish, then the condition differs from the classical one. See [5] for the classical kalman rank condition.

1.1.3 The averaged controllability of inverted pendulum equation depending on a parameter

★ What is inverted pendulum control system?

An inverted pendulum is a pendulum that has its center of mass above its pivot point. It can be suspended stably in this inverted position by using a control system to monitor the angle of the pole and move the pivot point horizontally back under the center of mass when it starts to fall over, keeping it balanced.

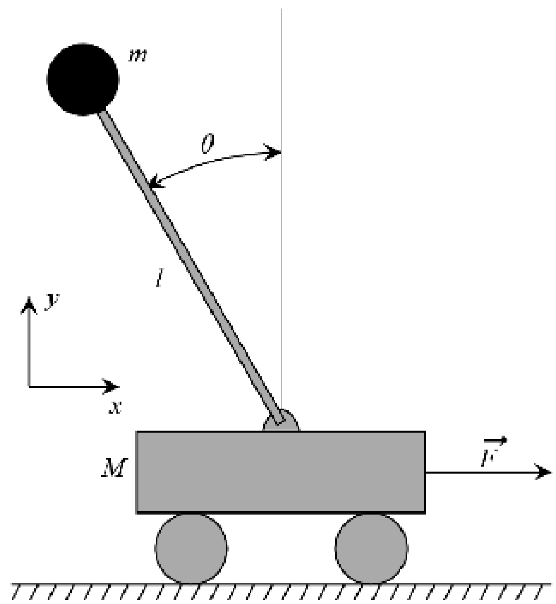
The inverted pendulum is a classic problem in dynamics and control theory and is used as a benchmark for testing control strategies.

★ Equations of the inverted pendulum motion [10]

In a configuration where the pivot point of the pendulum is fixed in space, The equation of motion below assumes no friction or any other resistance to movement, a rigid massless rod, and the restriction to 2-dimensional movement.

$$\theta'' = \frac{g}{l}\sin\theta$$

Where θ'' is the angular acceleration of the pendulum, g is the standard gravity on the surface of the Earth, l is the length of the pendulum, and θ is the angular displacement measured from the equilibrium position, see the Figure (1).



Figure(1):The Inverted Pendulum.

We will consider here a specific linear parameter-dependent problem about this **inverted pendulum [4],** so let's talk about this linearized cart-inverted pendulum system

$$\begin{pmatrix} x'_\nu \\ v'_\nu \\ \theta'_\nu \\ \omega'_\nu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{-\nu}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\nu+M}{M} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_\nu \\ v_\nu \\ \theta_\nu \\ \omega_\nu \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} u,$$

It describes the dynamically behavior of a system composed of a cart of mass M and a rigid pendulum of length l . Both M and l will be a fixed values. The pendulum is anchored to the cart and at the free extremity it is placed a **(small)** variable mass described by the parameter ν. This is the main idea of our problem here where the mass m of the ball is neglected, see the Figure (2).

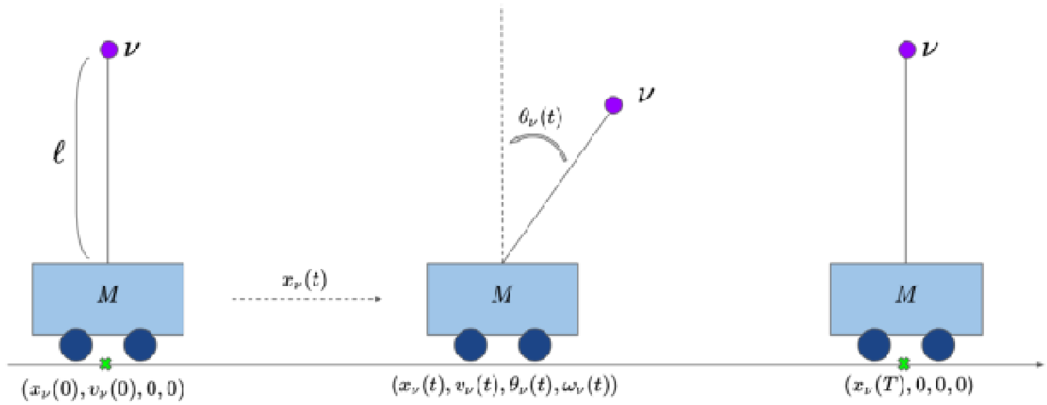


Figure (2) : Cart-Inverted Pendulum System .

Where the cart moves on a horizontal plane, the states $x_\nu(t)$ describe its position and $v_\nu(t)$ describe its velocity, During the motion of the cart the pendulum deviates from the initial vertical position by an angle $\theta_\nu(t)$ with an angular velocity $\omega_\nu(t)$.

The moving starting from an initial state $(x_\nu(0), v_\nu(0), 0, 0)$, and our goal will be to find a parameter-independent control function u and our practical method to check the controllability of our system is if the average rank condition satisfied it's means the rank is equal to the number of state variables for any initial state values. The acceptable control effort u can directing the state to any final state $(x_\nu(T), 0, 0, 0)$ values within some finite time.

We will calculate the averaged rank (1.3) for this system to see if it is controllable or not, so we need to calculate

$$\text{Rank}\left[\int_0^1 [A(\nu)]^j B(\nu)d\nu\right], \text{ for all } j = \overline{0\cdots 3}\,.$$

Where

$$A(\nu) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{-\nu}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\nu+M}{M} & 0 & 0 \end{pmatrix} \text{ and, } B(\nu) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$

so let's calculate the column vector C_j

$$C_j = \int_0^1 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{-\nu}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\nu+M}{M} & 0 & 0 \end{pmatrix}^j \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} d\nu \text{ ,for all } j = \overline{0\cdots 3},$$

then for $j = \overline{0 \cdots 3}$ we have

$$C_0 = \int_0^1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} d\nu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$

$$C_1 = \int_0^1 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -\frac{\nu}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\nu+M}{M} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} d\nu = \begin{pmatrix} 0 \\ -\frac{1}{2M} \\ -1 \\ \frac{M+1}{M} \end{pmatrix},$$

$$C_2 = \int_0^1 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{-\nu}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\nu+M}{M} & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} d\nu = \begin{pmatrix} \frac{-2M+1}{M} \\ \frac{1}{3M^2} \\ \frac{2M-1}{2M} \\ \frac{-2-3M}{6M^2} \end{pmatrix},$$

$$C_3 = \int_0^1 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{-\nu}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\nu+M}{M} & 0 & 0 \end{pmatrix}^3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} d\nu = \begin{pmatrix} \frac{-4M^2+6M-3}{12M^3} \\ \frac{1}{5M^4} \\ \frac{4M-3}{12M^3} \\ \frac{4-5M}{20M^4} \end{pmatrix},$$

so we just need to calculate the rank of the matrix $[C_j]$, for all $j = \overline{0 \cdots 3}$, we get

$$\mathrm{Rank} \begin{pmatrix} 0 & 0 & \frac{-2M+1}{M} & \frac{-4M^2+6M-3}{12M^3} \\ 1 & -\frac{1}{2M} & \frac{1}{3M^2} & \frac{1}{5M^4} \\ 0 & -1 & \frac{2M-1}{2M} & \frac{4M-3}{12M^3} \\ -1 & \frac{M+1}{M} & \frac{-2-3M}{6M^2} & \frac{4-5M}{20M^4} \end{pmatrix} = 4\,.$$

With the condition

$$2M - 1 \neq 0, M \neq 0, 80M^4 - 320M^3 + 340M^2 - 212M + 51 \neq 0\,.$$

So we have four state variables then the average rank condition is satisfied, then our system is average controllable.

1.1.4 Averaged observability inequality

Another characterization of the controllability property is provided by the dual problem of observability of **the $\nu-$ dependent adjoint system:**

$$\begin{cases} -\varphi'(t) = A^*(\nu)\varphi(t), 0 < t < T, \\ \varphi(T) = \varphi^0. \end{cases} \tag{1.5}$$

for all values of the parameter ν we take for simplicity the same datum for φ at $t = T$, and the solution $\varphi = \varphi(t,\nu)$of the adjoint system depends on the parameter ν.

Remark 1.2 *[12] Let φ the corresponding solution of (1.5) and by multiplying (1.1) by $\varphi = \varphi(t,\nu)$ and (1.5) by $y = y(t,\nu)$ we deduce that:*

$$\langle y', \varphi\rangle_{\mathbb{R}^N} = \langle A(\nu)y, \varphi\rangle_{\mathbb{R}^N} + \langle B(\nu)u, \varphi\rangle_{\mathbb{R}^N} \text{ and } \langle -y, \varphi'\rangle_{\mathbb{R}^N} = \langle A^*(\nu)\varphi, y\rangle_{\mathbb{R}^N} \ ,$$

then we have $\frac{d}{dt}\langle y, \varphi\rangle_{\mathbb{R}^N} = \langle B(\nu)u, \varphi\rangle_{\mathbb{R}^N} = \langle u, B^*(\nu)\varphi\rangle_{\mathbb{R}^N}$ we suppose that

$$u(t) = \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu,$$

where φ is the solution of the adjoint system and we integrating with respect to $t \in (0,T)$ and $\nu \in (0,1)$ we get the following

$$\begin{aligned} \int_0^T \left\langle u(t), \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu \right\rangle_{\mathbb{R}^N} dt &= \int_0^T \int_0^1 \langle B(\nu)u(t), \varphi(t,\nu)\rangle_{\mathbb{R}^N} d\nu dt \\ &= \left\langle \int_0^1 y(T,\nu)d\nu, \varphi^0 \right\rangle_{\mathbb{R}^N} - \left\langle y^0, \int_0^1 \varphi(0,\nu)d\nu \right\rangle_{\mathbb{R}^N} \end{aligned}$$

but we have here the equation satisfied by the state $y(t,\nu)$ and the adjoint one $\varphi(t,\nu)$ so we get the following

$$\begin{aligned} \int_0^T \int_0^1 \langle B(\nu)u(t), \varphi(t,\nu)\rangle_{\mathbb{R}^N} d\nu dt &= \int_0^T \int_0^1 \langle y' - A(\nu)y, \varphi(t,\nu)\rangle_{\mathbb{R}^N} d\nu dt \\ &= \int_0^1 \langle y(T,\nu), \varphi^0\rangle_{\mathbb{R}^N} d\nu - \int_0^1 \langle y^0, \varphi(0,\nu)\rangle_{\mathbb{R}^N} d\nu \\ &\quad + \int_0^T \int_0^1 \langle y, (-\varphi' + A(\nu)^*)\varphi\rangle_{\mathbb{R}^N} d\nu dt \\ &= \int_0^1 \langle y(T,\nu), \varphi^0\rangle_{\mathbb{R}^N} d\nu - \int_0^1 \langle y^0, \varphi(0,\nu)\rangle_{\mathbb{R}^N} d\nu \end{aligned}$$

In other words, we have the duality identity

$$\int_0^1 \langle y(T,\nu), \varphi^0\rangle_{\mathbb{R}^N} d\nu = \int_0^T \left\langle u(t), \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu \right\rangle_{\mathbb{R}^N} dt + \int_0^1 \langle y^0, \varphi(0,\nu)\rangle_{\mathbb{R}^N} d\nu.$$

And with the controllability definition (1.2) can be recast as follows

$$\langle y^1, \varphi^0\rangle = \int_0^T \left\langle u(t), \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu \right\rangle dt + \int_0^1 \langle y^0, \varphi(0,\nu)\rangle d\nu, \forall \varphi^0 \in \mathbb{R}^N.$$

So, we get the following

$$\left\langle y^1, \varphi^0 \right\rangle = \int_0^T \left\langle \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu, \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu \right\rangle dt + \int_0^1 \left\langle y^0, \varphi(0,\nu) \right\rangle d\nu,$$

then

$$\left\langle y^1, \varphi^0 \right\rangle = \int_0^T \left| \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu \right|^2 dt + \int_0^1 \left\langle y^0, \varphi(0,\nu) \right\rangle d\nu,$$

and we define the quadratic functional over the class of solutions of the adjoint system (1.5) according to the Euler-Lagrange equation associated to the minimization of a suitable quadratic functional so we write here

$$J(\varphi^0) = \frac{1}{2} \int_0^T \left| \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu \right|^2 dt + \int_0^1 \left\langle y^0, \varphi(0,\nu) \right\rangle d\nu - \left\langle y^1, \varphi^0 \right\rangle. \tag{1.6}$$

Remark 1.3 (Averaged observability inequality) *[12] The system (1.1) is said to be observable in time $T > 0$ if there exists $C > 0$ such that*

$$\left|\varphi^0\right|^2 \leq C \int_0^T \left| \int_0^1 B^*(\nu)\varphi(t,\nu)d\nu \right|^2 dt, \tag{1.7}$$

for all $\varphi \in \mathbb{R}^N$, φ being the corresponding solution of (1.5).

Remark 1.4 *Then the problem is reduced to prove the existence of the minimizer of J and for this it is sufficient to prove the coercivity of the functional J or, in other words, the existence of a positive constant $C > 0$ such that the observability inequality (1.7) holds . See theorem 2.1.1 in [13]*

1.2 Averaged controllability of infinite dimensional systems

1.2.1 Problem position

Consider the following Cauchy problem which depends on parameter ν

$$\begin{cases} y'(t) = A(\nu)y(t) + B(\nu)u(t); \\ y(0) = y^0 \in D(A(\nu)) \subset X; \end{cases} \tag{1.8}$$

Where $\nu \in (0,1)$, $A(\nu)$ is an operator on the separable Hilbert space X the state space, $B(\nu)$ is a control operator on $\mathcal{L}(U,X)$, $y(t,\nu) \in X$,the parameter dependent state variable , $u(t) \in U$ is the control variable (independent of the parameter ν), U being the separable Hilbert control space. For all $\nu \in (0,1)$, the operator $A(\nu)$ is generate a C_0-semigroup $\{S(t,\nu)\}_{t\geq 0}$.
According to the method of varying the constants we can write $y(t,\nu)$ the solution of (1.8) in the following form :

$$y(t,\nu) = S(t,\nu)y^0 + \int_0^t S(t-s,\nu)B(\nu)u(s)ds, \nu \in (0,1), t \geq 0.$$

1.2.2 Averaged controllability

Definition 1.2 (Exact averaged controllability) *[13] The system (1.8) is exactly averaged controllable on X if for initial conditions $y^0 \in X$ and every final target $y_d \in X$,there exists a control $u \in L^2([0,T],U)$ (independent of the parameter ν) such that*

$$\int_0^1 y(T,\nu)d\nu = y_d. \tag{1.9}$$

Definition 1.3 (Approximate averaged controllability) *[13] The system (1.8) is weakly averaged controllable on X if for initial conditions $y^0 \in X$ and every final target $y_d \in X$, for every $\varepsilon > 0$ there exists a control $u \in L^2([0,T],U)$ such that*

$$\left\| \int_0^1 y(T,\nu)d\nu - y_d \right\|_X \leq \varepsilon. \tag{1.10}$$

Definition 1.4 (Null averaged controllability) *[13] The system (1.8) is null averaged controllable on X if for initial conditions $y^0 \in X$ and every final target $y_d \in X$,there exists a control $u \in L^2([0,T],U)$ (independent of the parameter ν) such that*

$$\int_0^1 y(T,\nu)d\nu = 0. \tag{1.11}$$

Remark 1.5 *The basic idea about the average control is to find an average trajectory independing to the unknown parameter ν among several trajectories resulting from the unknown parameter ν resulting from the mathematical modeling of a physical or biomedical phenomena to control our system , for example in the biomedical phenomena is the X-rays. see the Figure (3).*

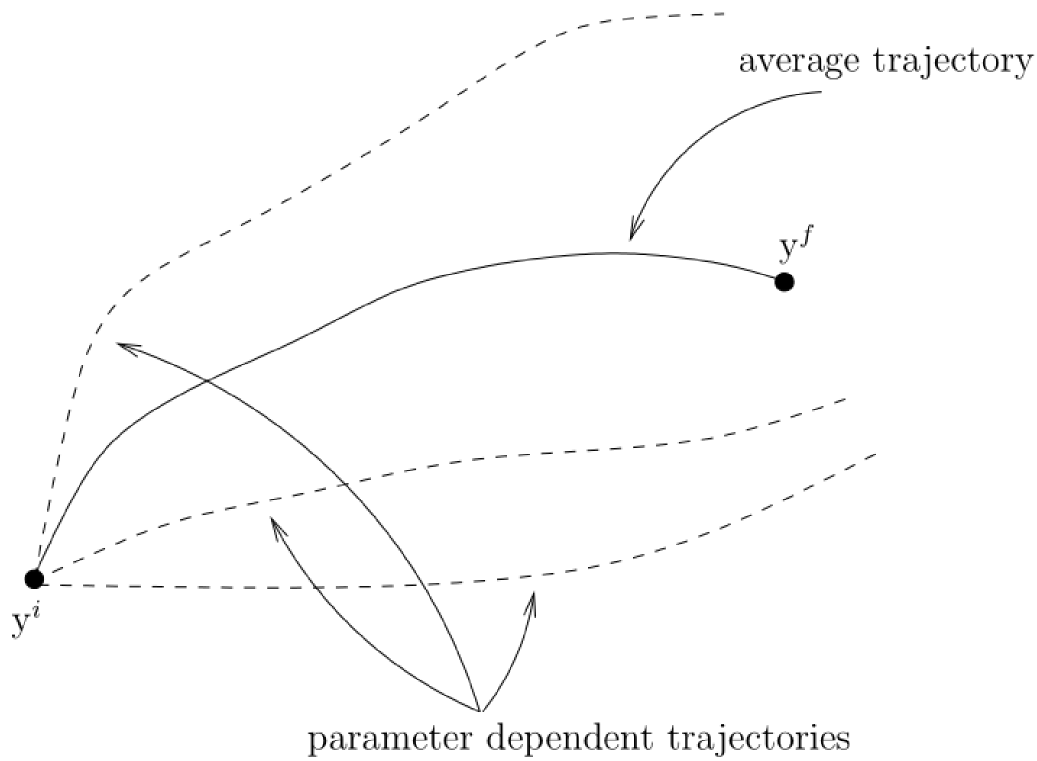


Figure (2) : Averaged Controllability[7] .

1.2.3 Characterizations of averaged controllability

Let us introduce two separable Hilbert spaces, namely the state space X and the control space U, each of them being identified with its dual . For every $\nu \in (0,1)$, consider the operator $A(\nu)$ on X with domain $D(A(\nu))$ and assume that:
(i) $A(\nu)$ has a non-empty resolvent $\rho(A(\nu))$;
(ii) $A(\nu)$ generates a strongly continuous *semigroup* $S(t,\nu)$ on X;
Let us now introduce the control operators. For every $\nu \in (0,1)$, we set $B(\nu) \in L(U,X)$ For every $\nu \in (0,1)$, we introduce the input to state map $\Phi \in \mathcal{L}(L^2(\mathbb{R},U),X)$ defined by

$$\Phi(t,\nu)u(t) = \int_0^t S(t-s,\nu)B(\nu)u(s)ds,$$

Then the solution of (1.8) is

$$y(t,\nu) = S(t,\nu)y^0 + \Phi(t,\nu)u(t), (t>0, u \in L^2((0,T),U)),$$

Taking the average of (1.8) with respect to ν, we obtain

$$\int_0^1 y(t,\nu)d\nu = \int_0^1 S(t,\nu)y^0 d\nu + \mathbf{F}_t u, (t>0, u\in L^2((0,T),U)),$$

Where we have defined the averaged input

$$\mathbf{F}_t u = \int_0^1 \Phi(t,\nu)u(t)d\nu, (t>0, u\in L^2((0,T),U)),$$

See [6] for more details.

Proposition 1.1 *We Consider the system (1.8) then*
(i) The system is exactly averaged controllable iff the operator $\mathbf{F}_t$ is surjective , i.e.: $\mathrm{Im}(\mathbf{F}_t) = X$.
(ii) The system is weakly averaged controllable iff the Image of $\mathbf{F}_t$ is dense , i.e.: $\overline{\mathrm{Im}(\mathbf{F}_t)} = X$.

Proof. (i) the system (1.8) is exactly averaged controllable
$\Leftrightarrow \forall y^0, y^1 \in X, \exists\, u \in L^2((0,T);U) : y^1 = \int_0^1 y(t,\nu)d\nu = \int_0^1 S(t,\nu)y^0 d\nu + \int_0^1\int_0^t S(t-s,\nu)B(\nu)u(s)dsd\nu$
we take for simplified $y^0 = 0$, $so : y^1 = \int_0^1 y(t,\nu)d\nu = \int_0^1\int_0^t S(t-s,\nu)B(\nu)u(s)dsd\nu = \mathbf{F}_t u$
$\Leftrightarrow$ $\mathbf{F}_t$ surjective
$\Leftrightarrow \mathrm{Im}(\mathbf{F}_t) = X.$
(ii) the system (1.8) is weakly averaged controllable
$\Leftrightarrow \forall y^0, y^1 \in X, \exists\, u \in L^2((0,T);U) : \left\| \int_0^1 y(t,\nu)d\nu - y^1 \right\| < \varepsilon \,, \forall \varepsilon > 0,$
$\Leftrightarrow \forall y^0, y^1 \in X, \exists\, u \in L^2((0,T);U) : \left\| \mathbf{F}_t u - (\int_0^1 S(t,\nu)y^0 d\nu + y^1) \right\| < \varepsilon \,, \forall \varepsilon > 0,$
$\Leftrightarrow \overline{\mathrm{Im}(\mathbf{F}_t)} = X.$ ■

1.2.4 Application to PDEs depending on a parameter

Example of heat equation with parameter depending

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$. Consider the controlled heat equation

$$\begin{cases} y_t - \mathrm{div}(a(x,\nu)\nabla y) = u(x,t)\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0 & \text{in } \Omega. \end{cases} \tag{1.12}$$

Where $Q = \Omega \times (0,T)$ stands for the space-time cylinder where the equation holds,and $\Sigma = \partial\Omega \times (0,T)$ for the lateral boundary the diffusivity coefficients $a(x,\nu)$, taken to be scalar for to be simplified, and to depend on the parameter $\nu \in (0,1)$, and ω be an open non-empty subset of Ω. We assume that $y^0 \in L^2(\Omega)$ and $u \in L^2(\Omega \times (0,T))$ and $y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, for all $\nu \in (0,1)$.

★ If we study the problem of averaged null controllability of (1.12) that's leads to find a control u such that the solution of (1.12) satisfies

$$\int_0^1 y(T,\nu)d\nu = 0.$$

Then the problem can be shown to be equivalent to an averaged observability inequality for the adjoint system

$$\begin{cases} \varphi_t - \operatorname{div}(a(x,\nu)\nabla\varphi) = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi^0 & \text{in } \Omega. \end{cases} \tag{1.13}$$

So, the control function can be shown to be of the form

$$u(x,t) = \int_0^1 \varphi(x,t,\nu)d\nu.$$

Where φ is a distinguished solution of the adjoint system determined by the datum φ^0 minimizer of the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \left|\int_0^1 \varphi(x,t,\nu)d\nu\right|^2 dxdt + \int_\Omega y^0 \int_0^1 \varphi(x,0,\nu)d\nu dxdt,$$

We observe that, to prove the coercivity of the functional J, the following averaged observability inequality is needed

$$\exists C > 0 : \left\|\int_0^1 \varphi(x,0,\nu)d\nu\right\|^2 \leq C\int_0^T \int_\omega \left|\int_0^1 \varphi(x,t,\nu)d\nu\right|^2 dxdt$$

The case where the control set ω is the whole domain Ω does not seem to be an easy one that's why we take the simple case where $\omega \subset \Omega$,

★ If we study the problem of average approximate controllability that's leads to: For given an initial condition $y^0 \in L^2(\Omega)$,a final target $y^1 \in L^2(\Omega)$ and $\varepsilon > 0$ then should we find a control function $u \in L^2(\omega \times (0,T))$ such that the solution of (1.12) satisfies:

$$\left\|\int_0^1 y(T,\nu)d\nu - y^1\right\|_{L^2(\Omega)} \leq \varepsilon.$$

The wave equation example

Let us consider the following controlled wave equation [6]

$$\begin{cases} y_{tt} - \operatorname{div}(a(x,\nu)\nabla y) = u(x,t)\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0 \ , \ y_t(x,0) = y^1 & \text{in } \Omega. \end{cases} \tag{1.14}$$

Where Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$ and $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$ for the uniformly bounded from below and above by positive constants independent of ν the diffusivity coefficients $a(x,\nu)$ taken to depend on the parameter $\nu \in (0,1)$ and ω be an open non-empty subset of Ω and $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$.
We assume that $u \in L^2(\omega)$ and $y \in X = H_0^1(\Omega) \times L^2(\Omega)$,the scalar product on state space X

$$\left\langle \begin{bmatrix} z^0 \\ z^1 \end{bmatrix}, \begin{bmatrix} y^0 \\ y^1 \end{bmatrix} \right\rangle_{H_0^1(\Omega)\times L^2(\Omega)} = \left\langle \nabla z^0, \nabla y^0 \right\rangle_{(L^2(\Omega))^d} + \left\langle z^1, y^1 \right\rangle_{L^2(\Omega)}, \text{ for all } (z^0, z^1) \text{ and } (y^0, y^1).$$

Let us define the operator $A(\nu)$ on $L^2(\Omega)$ by

$$D(A(\nu)) = H^2(\Omega) \times H_0^1(\Omega) \text{ and } A(\nu)f = -\operatorname{div}(a(x,\nu)\nabla f) \text{ for all } f \in D(A(\nu)),$$

And let us define the space $X(\nu) = X$ endowed with the scalar product

$$\begin{aligned} \left\langle \begin{bmatrix} z^0 \\ z^1 \end{bmatrix}, \begin{bmatrix} y^0 \\ y^1 \end{bmatrix} \right\rangle_{X(\nu)} &= \left\langle \sqrt{A(\nu)} z^0, \sqrt{A(\nu)} y^0 \right\rangle_{L^2(\Omega)} + \left\langle z^1, y^1 \right\rangle_{L^2(\Omega)} \\ &= \left\langle \sqrt{a(x,\nu)} z^0, \sqrt{a(x,\nu)} y^0 \right\rangle_{(L^2(\Omega))^d} + \left\langle z^1, y^1 \right\rangle_{(L^2(\Omega))^d}. \end{aligned}$$

Since $a(x,\nu)$ is uniformly bounded from above and below then the $X(\nu)$ and X-norms are equivalent.
For every $\nu \in (0,1)$, let us now define the operator $A(\nu)$ on X by

$$\begin{aligned} D(A(\nu)) &= H_0^1(\Omega) \times L^2(\Omega) \text{ and} \\ A(\nu) \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} &= \begin{bmatrix} 0 & Id \\ -A(\nu) & 0 \end{bmatrix} \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} = \begin{bmatrix} z^1 \\ \operatorname{div}(a(x,\nu)\nabla z^0) \end{bmatrix}, \left(\begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \in D(A(\nu)) \right) \end{aligned}$$

With these definitions, $A(\nu)$ is skew adjoint on $X(\nu)$ and generates a semigroup on $X(\nu)$,let us now define the bounded control operator $B \in L(L^2(\omega), H_0^1(\Omega) \times L^2(\Omega))$ independent of ν, by:

$$Bu = \begin{bmatrix} 0 \\ u\chi_\omega \end{bmatrix} \qquad \big(u \in L^2(\omega)\big)\,.$$

So the system (1.14) can be expressed in the condensed form:

$$\begin{aligned} Y_t &= A(\nu)Y + Bu \\ Y(0) &= \begin{bmatrix} y^0 \\ y^1 \end{bmatrix} \in H_0^1(\Omega) \times L^2(\Omega)\text{ , where } Y(t) = \begin{bmatrix} y(t) \\ y_t(t) \end{bmatrix}. \end{aligned}$$

1.3 Basic principles of the Hilbert Uniqueness Method

1.3.1 A boundary control Model

We consider the following wave equation with a boundary control action:

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } Q, \\ y = u & \text{on } \Sigma_0, \\ y = 0 & \text{on } \tilde{\Sigma}_0, \\ y(x,0) = y^0\ ,\ y_t(x,0) = y^1 & \text{in } \Omega. \end{cases} \tag{1.15}$$

Where Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$ and $\Gamma = \partial\Omega$ and $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$ and$\Sigma_0 = \Gamma_0 \times (0,T)$ where Γ_0 open part of Γ and $\tilde{\Sigma}_0 = \tilde{\Gamma}_0 \times (0,T)$ where $\tilde{\Gamma}_0 = \Gamma - \Gamma_0$.
The problem of exact controllability of our system is to find u such that

$$y(T) = y_t(T) = 0.$$

If such a u exists, we say there is null controllability of the system.

1.3.2 The Hilbert Uniqueness Method

We describe the HUM method by the following steps [9]:

★ **Homogeneous system**

We consider the following equation with $\Sigma = \Gamma \times (0,T)$ and $\phi^1, \phi^0 \in D(\Omega)$

$$\begin{cases} \phi_{tt} - \Delta\phi = 0 & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \phi(x,0) = \phi^0\ ,\ \phi_t(x,0) = \phi^1 & \text{in } \Omega. \end{cases} \tag{1.16}$$

★ **The backward equation**

We consider the following adjoint system

$$\begin{cases} \psi_{tt} - \Delta\psi = 0 & \text{in } Q, \\ \psi = \frac{\partial\phi}{\partial\eta} & \text{on } \Sigma_0, \\ \psi = 0 & \text{on } \tilde{\Sigma}_0, \\ \psi(T) = 0\ ,\ \psi_t(T) = 0 & \text{in } \Omega. \end{cases} \tag{1.17}$$

Where η the normal vector , $\frac{\partial}{\partial\eta}$ the derivative in that direction ,(i.e) : $\frac{\partial\phi}{\partial\eta} = \nabla\phi.\eta = \sum_{i=1}^{n} \frac{\partial\phi}{\partial x_i}\eta_i$,or, with the summation convention of repeated indices : $\frac{\partial\phi}{\partial\eta} = \frac{\partial\phi}{\partial x_i}\eta_i$.

★ **The operator Λ**

We define a linear operator Λ which associates with $\{\phi^1, \phi^0\}$by

$$\Lambda\{\phi^1, \phi^0\} = \{\psi_t(0), -\psi(0)\}.$$

And by multiplying the equation of (1.17) by $\phi = \phi(x,t)$ and by integrating on Q , we get

$$\underbrace{\int_Q \psi_{tt}\phi dxdt}_{(I)} - \underbrace{\int_Q \phi\Delta\psi dxdt}_{(II)} = 0.$$

For the first integral (I),we have

$$(\psi_t\phi)_t = \psi_{tt}\phi + \psi_t\phi_t,$$

Then we integrating on $(0,T)$, we obtain

$$\langle\psi_t(T), \phi(T)\rangle_{L^2(\Omega)} - \langle\psi_t(0), \phi(0)\rangle_{L^2(\Omega)} = \int_Q \psi_t\phi dxdt + \int_Q \psi_t\phi_t dxdt$$

And with the condition of the system (1.17) , we find

$$\int_Q \psi_{tt}\phi dxdt = -\int_Q \psi_t\phi_t dxdt - \langle\psi_t(0), \phi(0)\rangle_{L^2(\Omega)}$$

And we have

$$\int_Q \psi_t\phi_t dxdt = \langle\psi(T), \phi_t(T)\rangle_{L^2(\Omega)} - \langle\psi(0), \phi_t(0)\rangle_{L^2(\Omega)} - \int_Q \psi\phi_{tt} dxdt.$$

Then

$$\int_Q \psi_t\phi_t dxdt = -\langle\psi(0), \phi_t(0)\rangle_{L^2(\Omega)} - \int_Q \psi\phi_{tt} dxdt,$$

So

$$\int_Q \psi_{tt}\phi dxdt = \langle \psi(0),\phi_t(0)\rangle_{L^2(\Omega)} - \langle \psi_t(0),\phi(0)\rangle_{L^2(\Omega)} + \int_Q \psi\phi_{tt}dxdt \quad (1.18)$$
$$= \left\langle \psi(0),\phi^1\right\rangle_{L^2(\Omega)} - \left\langle \psi'(0),\phi^0\right\rangle_{L^2(\Omega)} + \int_Q \psi\phi'' dxdt.$$

For the second integral (II) , and according to the second identity of Green (3.4) we have

$$\int_Q \psi\Delta\phi dxdt - \int_Q \phi\Delta\psi dxdt = \int_\Sigma \left(\psi\frac{\partial\phi}{\partial\eta} - \phi\frac{\partial\psi}{\partial\eta}\right) d\Gamma dt,$$

Then

$$-\int_Q \phi\Delta\psi dxdt = -\int_Q \psi\Delta\phi dxdt + \int_\Sigma \psi\frac{\partial\phi}{\partial\eta} d\Gamma dt \quad (1.19)$$

We add (1.18) to (1.19), we get

$$\int_Q \phi(\psi_{tt}-\Delta\psi)dxdt = \int_Q \psi(\phi_{tt}-\Delta\phi)dxdt + \left\langle \psi(0),\phi^1\right\rangle_{L^2(\Omega)} - \left\langle \psi_t(0),\phi^0\right\rangle_{L^2(\Omega)} + \int_\Sigma \psi\frac{\partial\phi}{\partial\sigma}d\Gamma dt = 0,$$

And according to the systems (1.16) and (1.17) we have : $\phi_{tt} - \Delta\phi = 0$ and $\psi_t - \Delta\psi = 0$ in Q then

$$\left\langle \psi(0),\phi^1\right\rangle_{L^2(\Omega)} - \left\langle \psi_t(0),\phi^0\right\rangle_{L^2(\Omega)} + \int_\Sigma \psi\frac{\partial\phi}{\partial\eta}d\Gamma dt = 0.$$

And we have $\psi = \frac{\partial\phi}{\partial\eta}$ on Σ_0 and $\psi = 0$ on $\tilde{\Sigma}_0$. Then

$$\int_{\Sigma_0}\left(\frac{\partial\phi}{\partial\eta}\right)^2 d\Gamma dt = \left\langle \psi_t(0),\phi^0\right\rangle_{L^2(\Omega)} - \left\langle \psi(0),\phi^1\right\rangle_{L^2(\Omega)}, \quad (1.20)$$

We consider $\left(\left\langle \psi_t(0),\phi^0\right\rangle_{L^2(\Omega)} - \left\langle \psi(0),\phi^1\right\rangle_{L^2(\Omega)}\right)$ like a scaler product of $\{\psi_t(0), -\psi(0)\}$ and $\{\phi^0,\phi^1\}$, then we get

$$\left\langle \Lambda\left\{\phi^0,\phi^1\right\},\left\{\phi^0,\phi^1\right\}\right\rangle = \int_{\Sigma_0}\left(\frac{\partial\phi}{\partial\eta}\right)^2 d\Gamma dt,$$

We introduce the basic idea : we say that the uniqueness property holds whenever

Theorem 1.2 (Uniqueness theorem) *[2]*

We have the following

$$\frac{\partial\phi}{\partial\eta} = 0 \text{ on } \Sigma_0 \Rightarrow \left\{\phi^0,\phi^1\right\} = 0 \text{ in } \Omega.$$

When the uniqueness property holds,we can introduce the Hilbert space F completion of $D(\Omega)\times D(\Omega)$ with the norm

$$\left\|\left\{\phi^0,\phi^1\right\}\right\|_F^2=\left\langle\Lambda\left\{\phi^0,\phi^1\right\},\left\{\phi^0,\phi^1\right\}\right\rangle. \tag{1.21}$$

By construction, Λ is an isometry between F and F^* (where F^* the dual of F) .

Remark 1.6 *We consider $\theta=\theta(x,t)$ the solution of the problem (1.16) corresponding to the initial data $\left\{\theta^0,\theta^1\right\}$,then we get*

$$\left\langle\Lambda\left\{\phi^0,\phi^1\right\},\left\{\theta^0,\theta^1\right\}\right\rangle=\left\langle\left\{\phi^0,\phi^1\right\},\left\{\theta^0,\theta^1\right\}\right\rangle,\forall\left\{\phi^0,\phi^1\right\},\left\{\theta^0,\theta^1\right\}\in D(\Omega)\times D(\Omega),$$

Then from the Cauchy-Schwarz inequality (3.6), we get

$$\left|\left\langle\Lambda\left\{\phi^0,\phi^1\right\},\left\{\theta^0,\theta^1\right\}\right\rangle\right|\le\left\|\left\{\phi^0,\phi^1\right\}\right\|_F\left\|\left\{\theta^0,\theta^1\right\}\right\|_F\ ,\forall\left\{\phi^0,\phi^1\right\},\left\{\theta^0,\theta^1\right\}\in D(\Omega)\times D(\Omega),$$

This inequality shows the continuity of the bilinear form defined by Λ in $D(\Omega)\times D(\Omega)$ We define by F the completion of this space with respect to the norm (1.21), we thus obtain a Hilbert space .
The continuous bilinear form

$$\left(\left\{\phi^0,\phi^1\right\},\left\{\theta^0,\theta^1\right\}\right)\to\left\langle\Lambda\left\{\phi^0,\phi^1\right\},\left\{\theta^0,\theta^1\right\}\right\rangle,$$

Admits an extension by continuity at closure F. Then, we obtain a continuous bilinear form on the Hilbert space F which is coercive,then according to Lax-Milgram lemma (3.8) , for each $\left\{\theta^0,\theta^1\right\}\in F^*$, there is a unique $\left\{\phi^0,\phi^1\right\}\in F$ such that

$$\left\langle\Lambda\left\{\phi^0,\phi^1\right\},\left\{z^0,z^1\right\}\right\rangle=\left\langle\left\{z^0,z^1\right\},\left\{\theta^0,\theta^1\right\}\right\rangle_{F\times F^*},$$

Therefore, for every $\left\{\theta^0,\theta^1\right\}\in F$ and for each $\left\{z^0,z^1\right\}\in F^*$ there is a unique $\left\{\phi^0,\phi^1\right\}\in F$ who is the solution of the equation $\Lambda\left\{\phi^0,\phi^1\right\}=\left\{z^0,z^1\right\}$ in F^*.From all this, it follows that $\Lambda:F\to F^*$ is an isomorphism.

Therefore, if $\{-y^0,y^1\}\in F^*$ then the equation

$$\Lambda\left\{\phi^0,\phi^1\right\}=\left\{-y^0,y^1\right\}$$

Has a unique solution in F. This solution, we consider

$$\begin{cases} y_{tt}-\nu^2\Delta y=0 & \text{in } Q,\\ y=u & \text{on } \Sigma_0,\\ y=0 & \text{on } \tilde{\Sigma}_0,\\ y(x,0)=y^0\ ,\ y_t(x,0)=y^1 & \text{in } \Omega. \end{cases}$$

Then $y = \psi$ hence $y(T) = y_t(T) = 0$.
Finally, we built a control u which gives the null controllability.

Chapter 2

Averaged null controllability for a parameter-dependent wave equation

We are interested here in the averaged null controllability of the wave equation with a control of the Neumann type on the system boundary.
In this chapter we treat the problem of wave equation we introduce our problem , we also demonstrate an averaged inverse and direct inequalities giving some **coercivity** and **continuity** results respectively for the main introduced operator in **HUM** method.

2.1 Wave equation depending on a parameter

We consider the following wave equation

$$\begin{cases} y_{tt} - \nu^2 \Delta y = 0 & \text{in } Q, \\ y = u & \text{on } \Sigma_0, \\ y = 0 & \text{on } \tilde{\Sigma}_0, \\ y(x,0) = y^0 \ , \ y_t(x,0) = y^1 & \text{in } \Omega. \end{cases} \tag{2.1}$$

Where Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$ and $\Gamma = \partial\Omega$ and $\Sigma_0 = \Gamma_0 \times (0,T)$ where Γ_0 open part of Γ,and $\tilde{\Sigma}_0 = \tilde{\Gamma}_0 \times (0,T)$ where $\tilde{\Gamma}_0 = \Gamma - \Gamma_0$ and $Q = \Omega \times (0,T)$, and $\Sigma = \Gamma \times (0,T)$. The velocity of propagation parameter ν is unknown in $(0,1)$, and the function u presents a boundary control in $L^2(\Sigma_0)$, $y^0 \in H_0^1(\Omega)$ and $y^1 \in L^2(\Omega)$ are independent of ν.
This wave equation has a unique solution $y = y(x,t,\nu) \in X = [C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))]$.[8]

Definition 2.1 ***Null averaged controllability***
The system (2.1) is null averaged controllable on X if for initial conditions (y^0, y^1) and every final target $y_d \in X$,there exists a control $u \in L^2([0,T], L^2(\Sigma_0))$ (independent of the parameter ν) such

that

$$\left(\int_0^1 y(x,T,\nu)d\nu, \int_0^1 y_t(x,T,\nu)d\nu\right) = (0,0) \ .$$

2.2 The averaged energy

We consider the following homogeneous equation with $\phi^1, \phi^0 \in D(\Omega)$

$$\begin{cases} \phi_{tt} - \nu^2\Delta\phi = 0 & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \phi(x,0) = \phi^0 \ , \ \phi_t(x,0) = \phi^1 & \text{in } \Omega. \end{cases} \tag{2.2}$$

Let's multiply the homogeneous equation by the multiplicative ϕ' then we integrate on Ω and taking the average with respect to ν

$$\begin{aligned} \int_0^1 \int_\Omega \phi_t \ (\phi_{tt} - \nu^2\Delta\phi)dxd\nu &= \int_0^1 \int_\Omega \phi_t \ \phi_{tt}dxd\nu - \int_0^1 \int_\Omega \nu^2\phi_t\Delta\phi dxd\nu \\ &= \int_0^1 \int_\Omega \frac{1}{2}\frac{d}{dt}\left|\phi_t\right|^2 dxd\nu + \int_0^1 \underbrace{\nu^2\left(\int_\Omega \nabla\phi_t\nabla\phi dx - \int_\Sigma \phi_t\nabla\phi\eta d\Sigma\right)}_{Green\ formula\ 3.7}d\nu \\ &= \frac{1}{2}\frac{d}{dt}\int_0^1 \int_\Omega \left|\phi_t\right|^2 dxd\nu + \frac{1}{2}\frac{d}{dt}\int_0^1 \int_\Omega \nu^2\left|\nabla\phi\right|^2 dxd\nu \\ &= \frac{1}{2}\frac{d}{dt}\left(\int_0^1 \int_\Omega \left(\left|\phi_t\right|^2 + \nu^2\left|\nabla\phi\right|^2\right) dxd\nu\right) \end{aligned}$$

Definition 2.2 (The averaged energy) *[1]We define for all $t \in (0,T)$ the averaged energy with respect to ν associated to the solution of the wave homogeneous equation by the quadratic form*

$$E_a(t) = \frac{1}{2}\int_0^1 \int_\Omega \left(\left|\phi_t\right|^2 + \nu^2\left|\nabla\phi\right|^2\right) dxd\nu$$

Theorem 2.1 *Let $\phi = \phi(x,t,\nu)$ be a solution to the homogeneous wave equation . Then the averaged energy is conserved,i.e.*

$$E_a(t) = E_a(0) = \frac{1}{2}\int_0^1 \int_\Omega \left(\left|\phi^1\right|^2 + \nu^2\left|\nabla\phi^0\right|^2\right) dxd\nu.$$

Proof. Let's multiply the homogeneous equation by the multiplicative ϕ_t then we integrate on Ω

and taking the average with respect to ν

$$
\begin{aligned}
0 &= \int_0^1 \int_\Omega \phi_t \, (\phi_{tt} - \nu^2 \Delta\phi) dx d\nu \\
&= \frac{1}{2}\frac{d}{dt} \int_0^1 \int_\Omega |\phi_t|^2 \, dx d\nu + \int_0^1 \int_\Omega \nu^2 \nabla\phi_t \nabla\phi dx d\nu \\
&= \frac{1}{2}\frac{d}{dt} \left(\int_0^1 \int_\Omega \left(|\phi_t|^2 + \nu^2 |\nabla\phi|^2\right) dx d\nu \right) \\
&= \frac{dE_u(t)}{dt}
\end{aligned}
$$

Then the energy is constant so it is conserved for all $t \in (0, T)$. ■

Remark 2.1 *The previous definition of the energy associated to the system is attributed to J.-L. LIONS [8] with the following form*

$$
E(t) = \frac{1}{2} \int_\Omega \left(|\phi_t|^2 + |\nabla\phi|^2\right) dx.
$$

2.3 Averaged inverse and direct inequalities

We are interested here to prove some averaged inverse inequality , we use this inequalities to proving the coercivity and continuity of an operator Λ who plays an important role in the application of **HUM** method.

And for simplicity of notation we write for example

$$
q_i \mu_i = \sum_{i=1}^{n} q_i \mu_i \tag{2.3}
$$

Theorem 2.2 *Let $q = (q_i)$ be a vector field in $\left[C^1(\overline{\Omega})\right]^n$ independent of the parameter ν. Then, for every weak solution for the homogeneous equation , then we have the following identity*

$$
\begin{aligned}
\frac{1}{2} \int_0^1 \int_\Sigma \nu^2 q_i \left|\frac{\partial\phi}{\partial\eta}\right|^2 .\eta_i d\Sigma d\nu &= \int_0^1 \int_\Omega \phi_t q_i \frac{\partial\phi}{\partial x_i} dx \Big|_0^T d\nu \\
&+ \frac{1}{2} \int_0^1 \int_Q \frac{\partial q_i}{\partial x_i} \left(|\phi_t|^2 + \nu^2 |\nabla\phi|^2\right) dx dt d\nu \\
&- \int_0^1 \int_Q \nu^2 \frac{\partial\phi}{\partial x_k} \frac{\partial q_k}{\partial x_k} \frac{\partial\phi}{\partial x_i} dx dt d\nu.
\end{aligned} \tag{2.4}
$$

Proof. Let's multiply the homogeneous equation (2.2) by the multiplicative $q_i \frac{\partial \phi}{\partial x_i}$ and integrating over $(0,1)\times(0,T)\times\Omega$

$$\begin{aligned} 0 &= \int_0^1 \int_Q (\phi_{tt} - \nu^2 \Delta\phi) q_i \frac{\partial \phi}{\partial x_i} dxdtd\nu \\ &= \underbrace{\int_0^1 \int_Q \phi_{tt} q_i \frac{\partial \phi}{\partial x_i} dxdtd\nu}_{(I)} - \underbrace{\int_0^1 \int_Q \nu^2 \Delta\phi q_i \frac{\partial \phi}{\partial x_i} dxdtd\nu}_{(II)} \end{aligned}$$

Analyze of the first integral (I)

$$\begin{aligned} \int_0^1 \int_Q \phi_{tt} q_i \frac{\partial \phi}{\partial x_i} dxdtd\nu &= \int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \bigg|_0^T d\nu - \int_0^1 \int_Q \phi_t q_i \frac{\partial \phi_t}{\partial x_i} dxdtd\nu \\ &= \int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \bigg|_0^T d\nu - \frac{1}{2}\int_0^1 \int_Q q_i \frac{\partial}{\partial x_i} |\phi_t|^2 dxdtd\nu \\ &= \int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \bigg|_0^T d\nu + \frac{1}{2}\int_0^1 \int_Q \frac{\partial q_i}{\partial x_i} |\phi_t|^2 dxdtd\nu. \end{aligned} \tag{2.5}$$

Analyze of the second integral (II):

$$\int_0^1 \int_Q \nu^2 \Delta\phi q_i \frac{\partial \phi}{\partial x_i} dxdtd\nu = \underbrace{\int_0^1 \int_\Sigma \nu^2 q_i \frac{\partial \phi}{\partial x_i} \nabla\phi.\eta d\Sigma d\nu - \int_0^1 \int_Q \nu^2 \nabla(\phi)\nabla\left(q_i \frac{\partial \phi}{\partial x_i}\right) dxdtd\nu}_{Green\ formula\ (3.4)},$$

And we have:

$$\begin{aligned} \nabla(\phi).\nabla\left(q_i\frac{\partial\phi}{\partial x_i}\right) &= \begin{pmatrix} \frac{\partial\phi}{\partial x_1} \\ \vdots \\ \frac{\partial\phi}{\partial x_j} \end{pmatrix} . \begin{pmatrix} \frac{\partial q_i}{\partial x_1}\frac{\partial\phi}{\partial x_i} + q_i \frac{\partial^2\phi}{\partial x_1 \partial x_i} \\ \vdots \\ \frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i} + q_i \frac{\partial^2\phi}{\partial x_j \partial x_i} \end{pmatrix} \\ &= \frac{\partial\phi}{\partial x_1}\left(\frac{\partial q_i}{\partial x_1}\frac{\partial\phi}{\partial x_i} + q_i\frac{\partial^2\phi}{\partial x_1\partial x_i}\right) + \cdots + \frac{\partial\phi}{\partial x_j}\left(\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i} + q_i\frac{\partial^2\phi}{\partial x_j\partial x_i}\right) \\ &= \frac{\partial\phi}{\partial x_1}\frac{\partial q_i}{\partial x_1}\frac{\partial\phi}{\partial x_i} + \cdots + \frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i} + \frac{\partial\phi}{\partial x_1}q_i\frac{\partial^2\phi}{\partial x_1\partial x_i} + \cdots + \frac{\partial\phi}{\partial x_j}q_i\frac{\partial^2\phi}{\partial x_j\partial x_i}, \end{aligned}$$

And according to (2.3) for simplify we write for all $j = \overline{1\cdots n}$

$$\begin{aligned} \nabla(\phi).\nabla\left(q_i\frac{\partial\phi}{\partial x_i}\right) &= \frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i} + \frac{\partial\phi}{\partial x_j}q_i\frac{\partial^2\phi}{\partial x_j\partial x_i} \\ &= \frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i} + \frac{\partial\phi}{\partial x_j}q_i\frac{\partial}{\partial x_j}\frac{\partial\phi}{\partial x_i} \\ &= \frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i} + \frac{1}{2}q_i\frac{\partial}{\partial x_i}\left|\frac{\partial\phi}{\partial x_j}\right|^2 \\ &= \frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i} + \frac{1}{2}q_i\frac{\partial}{\partial x_i}|\nabla\phi|^2 \\ &= |\nabla\phi|^2 + \frac{1}{2}q_i\frac{\partial}{\partial x_i}|\nabla\phi|^2 \end{aligned}$$

Then we get

$$\begin{aligned}
\int_0^1\int_Q \nu^2\Delta\phi q_i\frac{\partial\phi}{\partial x_i}dxdtd\nu &= \int_0^1\int_\Sigma \nu^2 q_i\frac{\partial\phi}{\partial x_i}\nabla\phi.\eta d\Sigma d\nu - \frac{1}{2}\int_0^1\int_Q \nu^2 q_i\frac{\partial}{\partial x_i}\left|\nabla\phi\right|^2 dxdtd\nu \\
&\quad -\int_0^1\int_Q \nu^2\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}dxdtd\nu \\
&\overset{Green}{=} \int_0^1\int_\Sigma \nu^2 q_i\frac{\partial\phi}{\partial x_i}\nabla\phi.\eta d\Sigma d\nu - \frac{1}{2}\int_0^1\int_\Sigma \nu^2 q_i\left|\nabla\phi\right|^2.\eta_i d\Sigma d\nu \\
&\quad +\frac{1}{2}\int_0^1\int_Q \nu^2\frac{\partial q_i}{\partial x_i}\left|\nabla\phi\right|^2 dxdtd\nu - \int_0^1\int_Q \nu^2\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}dxdtd\nu \\
&= \int_0^1\int_\Sigma \nu^2 q_i\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial\eta_i}d\Sigma d\nu - \frac{1}{2}\int_0^1\int_\Sigma \nu^2 q_i\left|\nabla\phi\right|^2.\eta_i d\Sigma d\nu \\
&\quad +\frac{1}{2}\int_0^1\int_Q \nu^2\frac{\partial q_i}{\partial x_i}\left|\nabla\phi\right|^2 dxdtd\nu - \int_0^1\int_Q \nu^2\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}dxdtd\nu,
\end{aligned}$$

and we just have

$$\frac{\partial\phi}{\partial x_i}\nabla\phi.\eta = \frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial\eta_i} = \left|\frac{\partial\phi}{\partial\eta_i}\right|^2 = \left|\nabla\phi\right|^2.\eta_i,$$

then we result

$$\begin{aligned}
\int_0^1\int_Q \nu^2\Delta\phi q_i\frac{\partial\phi}{\partial x_i}dxdtd\nu &= \frac{1}{2}\int_0^1\int_\Sigma \nu^2 q_i\left|\nabla\phi\right|^2.\eta_i d\Sigma d\nu \\
&\quad +\frac{1}{2}\int_0^1\int_Q \nu^2\frac{\partial q_i}{\partial x_i}\left|\nabla\phi\right|^2 dxdtd\nu \\
&\quad -\int_0^1\int_Q \nu^2\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}dxdtd\nu.
\end{aligned} \tag{2.6}$$

Then according to (2.5) and (2.6) , we get

$$\begin{aligned}
0 &= -\frac{1}{2}\int_0^1\int_\Sigma \nu^2 q_i\left|\nabla\phi\right|^2.\eta_i d\Sigma d\nu - \frac{1}{2}\int_0^1\int_Q \nu^2\frac{\partial q_i}{\partial x_i}\left|\nabla\phi\right|^2 dxdtd\nu \\
&\quad +\int_0^1\int_Q \nu^2\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}dxdtd\nu + \int_0^1\int_\Omega \phi_t q_i\frac{\partial\phi}{\partial x_i}dx\Big|_0^T d\nu \\
&\quad +\frac{1}{2}\int_0^1\int_Q \frac{\partial q_i}{\partial x_i}\left|\phi_t\right|^2 dxdtd\nu \\
&= \int_0^1\int_\Omega \phi_t q_i\frac{\partial\phi}{\partial x_i}dx\Big|_0^T d\nu + \frac{1}{2}\int_0^1\int_Q \frac{\partial q_i}{\partial x_i}\left(\left|\phi_t\right|^2 - \nu^2\left|\nabla\phi\right|^2\right)dxdtd\nu \\
&\quad -\frac{1}{2}\int_0^1\int_\Sigma \nu^2 q_i\left|\frac{\partial\phi}{\partial\eta_i}\right|^2.\eta_i d\Sigma d\nu + \int_0^1\int_Q \nu^2\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}dxdtd\nu
\end{aligned}$$

and this is rewards the identity (2.4):

$$\begin{aligned}\frac{1}{2}\int_0^1\int_\Sigma \nu^2 q_i\left|\frac{\partial\phi}{\partial\eta}\right|^2 .\eta_i d\Sigma d\nu &= \int_0^1\int_\Omega \phi_t q_i\frac{\partial\phi}{\partial x_i}dx\Big|_0^T d\nu + \frac{1}{2}\int_0^1\int_Q \frac{\partial q_i}{\partial x_i}\left(|\phi_t|^2+\nu^2|\nabla\phi|^2\right)dxdtd\nu\\ &-\int_0^1\int_Q \nu^2\frac{\partial\phi}{\partial x_k}\frac{\partial q_k}{\partial x_k}\frac{\partial\phi}{\partial x_i}dxdtd\nu.\end{aligned}$$

■

Average inverse inequality

In order to arrive at a result of uniqueness of the type of Theorem (1.2) **Uniqueness theorem** and a fortiori to obtain additional information on the initial data space in which the controllability exact or null takes place.

We first introduce some notations :

Let $x^0\in\mathbb{R}^n$, then we define the vector :

$$m(x) = x - x^0$$

with components

$$m_i(x) = x_i - x_i^0, 1\le i\le n$$

and we introduce a partition of the boundary Σ as follows :

$$\Gamma(x^0) = \{x\in\Gamma, m(x).\eta(x) > 0\}$$

where $\eta(x)$ is a field of unit normal vectors.

And $\Sigma(x^0)=\Gamma(x^0)\times\,]0,T[$, and we also introduce :

$$R(x^0) = \|m(x)\|_{L^\infty(\Omega)} = \sup_{x\in\overline{\Omega}}\left\|x-x^0\right\| \text{ and } T(x^0)=2R(x^0).$$

Theorem 2.3 (Averaged inverse inequality) *Let $T>T(x^0)$ and for every ϕ weak solution of the homogeneous equation (2.2) , then we have the following averaged observability inequality:*

$$(T-T(x^0))E_a(0)\le\frac{R(x^0)}{2}\int_0^1\int_{\Sigma(x^0)}\nu^2\left|\frac{\partial\phi}{\partial\eta}\right|^2 d\Gamma dt d\nu.$$

Proof. [1]We have the (2.4) identity :

$$\begin{aligned}\frac{1}{2}\int_0^1\int_\Sigma \nu^2 q_i\left|\frac{\partial\phi}{\partial\eta}\right|^2 .\eta_i d\Sigma d\nu &= \int_0^1\int_\Omega \phi_t q_i\frac{\partial\phi}{\partial x_i}dx\Big|_0^T d\nu + \frac{1}{2}\int_0^1\int_Q \frac{\partial q_i}{\partial x_i}\left(|\phi_t|^2+\nu^2|\nabla\phi|^2\right)dxdtd\nu\\ &-\int_0^1\int_Q \nu^2\frac{\partial\phi}{\partial x_k}\frac{\partial q_k}{\partial x_k}\frac{\partial\phi}{\partial x_i}dxdtd\nu.\end{aligned}$$

With the choice of multiplicative:

$$q_i(x) = m_i(x) = x_i - x_i^0, 1 \leq i \leq n$$

then we can write according to (2.3):

$$\frac{\partial q_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial q_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial (x_i - x_i^0)}{\partial x_i} = n, \text{and } \frac{\partial \phi}{\partial x_k} \frac{\partial q_k}{\partial x_k} \frac{\partial \phi}{\partial x_i} = |\nabla \phi|^2$$

therefore, the identity (2.4) becomes:

$$\begin{aligned} \frac{1}{2} \int_0^1 \int_\Sigma \nu^2 m_i(x) \left| \frac{\partial \phi}{\partial \eta} \right|^2 .\eta_i d\Sigma d\nu &= \int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \Big|_0^T d\nu + \frac{n}{2} \int_0^1 \int_Q |\phi_t|^2 \, dxdtd\nu \\ &+ \frac{2-n}{2} \int_0^1 \int_Q \nu^2 |\nabla \phi|^2 \, dxdtd\nu, \end{aligned}$$

then on $\Sigma(x^0)$, thanks to the Cauchy-Schwarz inequality (3.6), we have:

$$0 < m(x).\eta(x) = \sum_{i=1}^{n} m_i(x).\eta_i(x) \leq \left(\sum_{i=1}^{n} m_i^2(x) \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \eta_i^2(x) \right)^{\frac{1}{2}} = \|m(x)\| \leq R(x^0)$$

so we can deduce that :

$$\frac{1}{2} \int_0^1 \int_\Sigma \nu^2 m_i(x) \left| \frac{\partial \phi}{\partial \eta} \right|^2 .\eta_i d\Sigma d\nu \leq \frac{1}{2} \int_0^1 \int_{\Sigma(x^0)} \nu^2 m_i(x) \left| \frac{\partial \phi}{\partial \eta} \right|^2 .\eta_i d\Sigma d\nu \leq \frac{R(x^0)}{2} \int_0^1 \int_{\Sigma(x^0)} \nu^2 \left| \frac{\partial \phi}{\partial \eta} \right|^2 d\Sigma d\nu,$$

Then we get the following inequality :

$$\begin{aligned} & \int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \Big|_0^T d\nu + \frac{n}{2} \int_0^1 \int_Q |\phi_t|^2 \, dxdtd\nu + \frac{2-n}{2} \int_0^1 \int_Q \nu^2 |\nabla \phi|^2 \, dxdtd\nu \\ \leq \; & \frac{R(x^0)}{2} \int_0^1 \int_{\Sigma(x^0)} \nu^2 \left| \frac{\partial \phi}{\partial \eta} \right|^2 d\Sigma d \end{aligned}$$

and we can write:

$$\begin{aligned} & \int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \Big|_0^T d\nu + \frac{n}{2} \int_0^1 \int_Q |\phi_t|^2 \, dxdtd\nu + \frac{2-n}{2} \int_0^1 \int_Q \nu^2 |\nabla \phi|^2 \, dxdtd\nu \\ & + \frac{1}{2} \int_0^1 \int_Q |\phi_t|^2 \, dxdtd\nu - \frac{1}{2} \int_0^1 \int_Q |\phi_t|^2 \, dxdtd\nu \\ = & \int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \Big|_0^T d\nu + \frac{n-1}{2} \int_0^1 \int_Q |\phi_t|^2 \, dxdtd\nu + \frac{2-n}{2} \int_0^1 \int_Q \nu^2 |\nabla \phi|^2 \, dxdtd\nu \\ & + \frac{1}{2} \int_0^1 \int_Q |\phi_t|^2 \, dxdtd\nu \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \Big|_0^T d\nu + \frac{n-1}{2}\int_0^1 \int_Q |\phi_t|^2 \, dx dt d\nu - \frac{n-1}{2}\int_0^1 \int_Q \nu^2 |\nabla \phi|^2 \, dx dt d\nu \\
&+ \frac{1}{2}\int_0^1 \int_Q |\phi_t|^2 \, dx dt d\nu + \frac{1}{2}\int_0^1 \int_Q \nu^2 |\nabla \phi|^2 \, dx dt d\nu \\
&= \int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \Big|_0^T d\nu + \frac{n-1}{2}\int_0^1 \int_Q \left(|\phi_t|^2 - \nu^2 |\nabla \phi|^2\right) dx dt d\nu \\
&+ \frac{1}{2}\int_0^1 \int_Q \left(|\phi_t|^2 + \nu^2 |\nabla \phi|^2\right) dx dt d\nu
\end{aligned}$$

then the conservation of the averaged energy (Definition 2.2) gives us:

$$\begin{aligned}
\frac{1}{2}\int_0^1 \int_Q \left(|\phi_t|^2 + \nu^2 |\nabla \phi|^2\right) dx dt d\nu &= \int_0^T \underbrace{\frac{1}{2}\int_0^1 \int_\Omega \left(|\phi_t|^2 + \nu^2 |\nabla \phi|^2\right) dx d\nu dt}_{average\ energy} \\
&= \int_0^T E_a(0) dt \\
&= T E_a(0)
\end{aligned}$$

so our inequality becomes :

$$\begin{aligned}
&\int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \Big|_0^T d\nu + \underbrace{\frac{n-1}{2}\int_0^1 \int_Q \left(|\phi_t|^2 - \nu^2 |\nabla \phi|^2\right) dx dt d\nu + T E_a(0)}_{(\bigstar)} \\
&\leq \frac{R(x^0)}{2}\int_0^1 \int_{\Sigma(x^0)} \nu^2 \left|\frac{\partial \phi}{\partial \eta}\right|^2 d\Sigma dt d\nu,
\end{aligned}$$

then now we simplify ($\bigstar$) first of all we multiply the homogeneous equation(2.2) by ϕ and integrate we get:

$$\underbrace{\int_0^1 \int_Q \phi \phi_{tt} dx dt d\nu}_{(I)} - \underbrace{\int_0^1 \int_Q \nu^2 \phi \Delta \phi dx dt d\nu}_{(II)} = 0,$$

then we analyze the first integral (I),we have:

$$\begin{aligned}
\int_0^1 \int_\Omega \phi \phi_t dx \Big|_0^T d\nu &= \int_0^1 \int_0^T \int_\Omega \frac{\partial}{\partial t}(\phi \phi_t) \, dx dt d\nu \\
&= \int_0^1 \int_Q \phi \phi_{tt} dx dt d\nu + \int_0^1 \int_Q |\phi_t|^2 \, dx dt d\nu,
\end{aligned}$$

then we get :

$$\int_0^1 \int_Q \phi \phi_{tt} dx dt d\nu = \int_0^1 \int_\Omega \phi \phi_t dx \Big|_0^T d\nu - \int_0^1 \int_Q |\phi_t|^2 \, dx dt d\nu. \tag{2.7}$$

Then we analyze the second integral (II),we have according to the green formula (3.4) and in the (2.2) equation $\phi = 0$ on Σ:

$$\int_0^1 \int_Q \nu^2 \phi \Delta \phi dxdtd\nu = -\int_0^1 \int_Q \nu^2 \left|\nabla \phi\right|^2 dxdtd\nu. \tag{2.8}$$

Then according to (2.7) and (2.8) we get :

$$\begin{aligned} 0 &= \int_0^1 \int_Q \phi\phi_{tt} dxdtd\nu - \int_0^1 \int_Q \nu^2 \phi \Delta \phi dxdtd\nu = \int_0^1 \int_\Omega \phi\phi_t dx \Big|_0^T d\nu - \int_0^1 \int_Q \left|\phi_t\right|^2 dxdtd\nu \\ &\quad + \int_0^1 \int_Q \nu^2 \left|\nabla\phi\right|^2 dxdtd\nu \\ &= \int_0^1 \int_\Omega \phi\phi_t dx \Big|_0^T d\nu - \int_0^1 \int_Q \left(\left|\phi_t\right|^2 - \nu^2 \left|\nabla \phi\right|^2\right) dxdtd\nu \end{aligned}$$

Then we get :

$$(\bigstar) \Leftrightarrow \int_0^1 \int_\Omega \phi\phi_t dx \Big|_0^T d\nu = \int_0^1 \int_Q \left(\left|\phi_t\right|^2 - \nu^2 \left|\nabla \phi\right|^2\right) dxdtd\nu,$$

so our inequality becomes :

$$\underbrace{\int_0^1 \int_\Omega \phi_t \left(m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dx \Big|_0^T d\nu}_{(\bigstar\bigstar)} + TE_a(0) \leq \frac{R(x^0)}{2} \int_0^1 \int_{\Sigma(x^0)} \nu^2 \left|\frac{\partial \phi}{\partial \eta}\right|^2 d\Sigma dtd\nu.$$

Now we simplify $(\bigstar\bigstar)$, and thanks to the Cauchy $\varepsilon-$inequality (3.7) so for all $\varepsilon > 0$, we can write:

$$\int_0^1 \int_\Omega \phi_t \left(m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dx \Big|_0^T d\nu \leq \frac{\varepsilon}{2} \int_0^1 \int_\Omega \left|\phi_t\right|^2 dxd\nu + \frac{1}{2\varepsilon} \int_0^1 \int_\Omega \left| m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right|^2 dxd\nu, \tag{2.9}$$

and on the other hand :

$$\begin{aligned} \int_0^1 \int_\Omega \left| m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right|^2 dxd\nu &= \int_0^1 \int_\Omega \left| m_i \frac{\partial \phi}{\partial x_i} \right|^2 dxd\nu + \frac{(n-1)^2}{4} \int_0^1 \int_\Omega \left|\phi\right|^2 dxd\nu \\ &\quad + (n-1) \int_0^1 \int_\Omega \left| m_i \frac{\partial \phi}{\partial x_i} \right| \left|\phi\right| dxd\nu. \end{aligned}$$

Where:

$$\int_0^1 \int_\Omega m_i \frac{\partial \phi}{\partial x_i} \phi dxd\nu = \int_0^1 \int_\Omega \sum_{i=1}^n m_i \frac{\partial \phi}{\partial x_i} \phi dxd\nu = \frac{1}{2} \int_0^1 \int_\Omega \sum_{i=1}^n m_i \frac{\partial}{\partial x_i} \phi^2 dxd\nu$$

Moreover, according to the Gauss divergence formula (3.4) and as ($\phi = 0$ on Σ) we have :

$$\begin{aligned} \frac{1}{2} \int_0^1 \int_\Omega \sum_{i=1}^n m_i \frac{\partial}{\partial x_i} \phi^2 dxd\nu &= -\frac{1}{2} \int_0^1 \int_\Omega \sum_{i=1}^n \frac{\partial m_i}{\partial x_i} \phi^2 dxd\nu \\ &= -\frac{n}{2} \int_0^1 \int_\Omega \phi^2 dxd\nu, \end{aligned}$$

then :

$$\begin{aligned}\int_0^1 \int_\Omega \left| m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right|^2 dxd\nu &= \int_0^1 \int_\Omega \left| m_i \frac{\partial \phi}{\partial x_i} \right|^2 dxd\nu + \left[\frac{(n-1)^2}{4} - \frac{n(n-1)}{2} \right] \int_0^1 \int_\Omega |\phi|^2 dxd\nu \\ &= \int_0^1 \int_\Omega \left| m_i \frac{\partial \phi}{\partial x_i} \right|^2 dxd\nu - \frac{(n^2-1)}{4} \int_0^1 \int_\Omega |\phi|^2 dxd\nu\end{aligned}$$

but we have according to Cauchy Schwarz inequality (3.6) the following :

$$\begin{aligned}\left(\sum_{i=1}^n m_i \frac{\partial \phi}{\partial x_i} \right)^2 &\leq \left(\sum_{i=1}^n (m_i)^2 \right) \left(\sum_{i=1}^n \left(\frac{\partial \phi}{\partial x_i} \right)^2 \right) \\ &\leq \|m(x)\|^2 |\nabla \phi(x)|^2 .\end{aligned}$$

Then we write :

$$\begin{aligned}\int_0^1 \int_\Omega \left| m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right|^2 dxd\nu &\leq \int_0^1 \int_\Omega \left| m_i \frac{\partial \phi}{\partial x_i} \right|^2 dxd\nu \\ &\leq R^2(x^0) \int_0^1 \int_\Omega |\nabla \phi(x)|^2 dxd\nu\end{aligned}$$

we can take $\varepsilon = R(x^0)$ in (2.9) we get :

$$\begin{aligned}\int_0^1 \int_\Omega \phi_t \left(m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dx \Big|_0^T d\nu &\leq \frac{R(x^0)}{2} \int_0^1 \int_\Omega |\phi_t|^2 dxd\nu + \frac{R(x^0)}{2} \int_0^1 \int_\Omega |\nabla \phi(x)|^2 dxd\nu \\ &= R(x^0)E_a(0),\end{aligned}$$

so we simplify the following :

$$\begin{aligned}\left| \int_0^1 \int_\Omega \phi_t \left(m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dx \Big|_0^T d\nu \right| &\leq 2 \sup_{t\in(0,T)} \left| \int_0^1 \int_\Omega \phi_t \left(m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dxd\nu \right| \\ &\leq 2 \left\| \int_0^1 \int_\Omega \phi_t \left(m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dxd\nu \right\|_{L^\infty(0,T)} \\ &\leq 2R(x^0)E_a(0),\end{aligned}$$

when we take $T(x^0) = 2R(x^0)$, we get:

$$\left| \int_0^1 \int_\Omega \phi_t \left(m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dx \Big|_0^T d\nu \right| \leq T(x^0)E_a(0),$$

but we have :

$$\int_0^1 \int_\Omega \phi_t \left(m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dx \Big|_0^T d\nu + TE_a(0) \leq \frac{R(x^0)}{2} \int_0^1 \int_{\Sigma(x^0)} \nu^2 \left| \frac{\partial \phi}{\partial \eta} \right|^2 d\Sigma dt d\nu$$

and on the other hand

$$\begin{aligned} TE_a(0) - T(x^0)E_a(0) &\leq TE_a(0) - \left| \int_0^1 \int_\Omega \phi_t \left(m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dx \Big|_0^T d\nu \right| \\ &\leq \int_0^1 \int_\Omega \phi_t \left(m_i \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dx \Big|_0^T d\nu + TE_a(0), \end{aligned}$$

so we get directly :

$$\left(T - T(x^0)\right) E_a(0) \leq \frac{R(x^0)}{2} \int_0^1 \int_{\Sigma(x^0)} \nu^2 \left| \frac{\partial \phi}{\partial \eta} \right|^2 d\Sigma dt d\nu.$$

■

Theorem 2.4 (Averaged direct inequality) *Let Ω be a bounded domain of $\mathbb{R}^d$ with frontier of class C^2. Then, there exists a constant $C > 0$ such that the solution of the homogeneous equation (2.2) verifies the following inequality:*

$$\int_0^1 \int_\Gamma \nu^2 \left| \frac{\partial \phi}{\partial \eta} \right|^2 d\Sigma dt d\nu \leq C \int_0^1 \int_\Omega \left(\left|\phi^1\right|^2 + \nu^2 \left|\nabla \phi^0\right|^2 \right) dx d\nu,$$

moreover this inequality implies the following averaged regularity property for the solution of (2.2):

$$\nu \frac{\partial \phi}{\partial \eta} \in L^2(\Sigma \times (0,1)).$$

where η is the unit normal vector.

Proof. See the section 2 of [1],and the chapter 1, lemma 3.1 in [8]. ■

2.4 Averaged null controllability(The Hilbert Uniqueness Method)

In this section we try to solve the problem of averaged null controllability of (2.1) , using the HUM method introduced in the last chapter.
So let's start by introducing the **backward equation**:

$$\begin{cases} \psi_{tt} - \nu^2 \Delta \psi = 0 & \text{in } Q, \\ \psi = \begin{cases} \int_0^1 \nu^2 \dfrac{\partial \phi}{\partial \eta} d\nu \\ 0 \end{cases} & \begin{matrix} \text{on } \Sigma_0, \\ \text{on } \tilde{\Sigma}_0, \end{matrix} \\ \psi(T) = 0, \; \psi_t(T) = 0 & \text{in } \Omega. \end{cases} \tag{2.10}$$

where Q , Σ_0, $\tilde{\Sigma}_0$ and Ω is the same one introduced in the subsection (1.3.1) , and like we do in section (1.3.2) for the HUM method we do here for define **The operator** Λ so first we just multiply the backward equation by ϕ and integrate on Q and taking average between (0,1) we just get according to (1.18) and (1.19) the following :

$$\int_0^1 \int_Q \phi \left(\psi_{tt} - \nu^2 \Delta \psi\right) dx dt d\nu = \int_\Omega \phi^1 \int_0^1 \psi(0) d\nu dx - \int_\Omega \phi^0 \int_0^1 \psi_t(0) d\nu dx + \int_0^1 \int_{\Sigma_0} \nu^2 \psi \nabla \phi . \eta d\Gamma dt d\nu,$$

the new here is we just taking the average between (0,1) , so the formula (1.20) becomes :

$$\int_\Omega \phi^1 \int_0^1 \psi(0) d\nu dx - \int_\Omega \phi^0 \int_0^1 \psi_t(0) d\nu dx = \int_{\Sigma_0} \left| \int_0^1 \nu^2 \frac{\partial \phi}{\partial \eta} d\nu \right|^2 d\Gamma dt,$$

and now we can define **The operator** Λ on $D(\Omega) \times D(\Omega)$ by:

$$\Lambda \left\{\phi^0, \phi^1\right\} = \left\langle \int_0^1 \psi(0) d\nu, -\int_0^1 \psi_t(0) d\nu \right\rangle,$$

so we get like we just do in the (page 14) and see [1] and [2]:

$$\left\langle \Lambda \left\{\phi^0, \phi^1\right\}, \left\{\phi^0, \phi^1\right\} \right\rangle = \int_{\Sigma_0} \left| \int_0^1 \nu^2 \frac{\partial \phi}{\partial \eta} d\nu \right|^2 d\Gamma dt,$$

and like we introduce before the space F we can here take this semi norm on F like that:

$$\left\| \left\{\phi^0, \phi^1\right\} \right\|_F^2 = \int_{\Sigma_0} \left| \int_0^1 \nu^2 \frac{\partial \phi}{\partial \eta} d\nu \right|^2 d\Gamma dt$$

in fact this semi norm defines a norm on space F is equivalent to verifying the following uniqueness theorem because its satisfies the first condition in the definition of the norm application.

Theorem 2.5 (uniqueness theorem) *[1]*
Let ϕ be a solution for the homogeneous equation (2.2) . Then, if $\frac{\partial \phi}{\partial \eta} = 0$ on $\Sigma_0 \times (0,1)$ we have $\phi = 0$ in $Q \times (0,1)$.

Proof. For the proof see the page 88 in [8] where we just need the averaged inverse inequality theorem (2.3). ■

Remark 2.2 *So like in the remark (1.6) **The operator** Λ is an isomorphism on space F which implies the averaged null controllability of the (2.1) with averaged control given by $u = \int_0^1 \nu^2 \frac{\partial \phi}{\partial \eta} d\nu$.*

Chapter 3

Averaged null controllability for a parameter-dependent vibrating plate equation

Plate vibrations are a subset of the more general issue of mechanical vibrations . Because one of the dimensions of a plate is substantially less than the other two, the equations regulating its motion are simpler than those guiding the motion of conventional three-dimensional objects. This implies that a two-dimensional plate theory will provide a good approximation to the actual three-dimensional motion of a plate-like object, which is confirmed , see [14].
We are interested here in the averaged null controllability of the vibrating plate equation with a control of the Neumann type on the system boundary.
In this chapter we treat the problem of the vibrating plate equation we introduce our problem,we also demonstrate an averaged inverse and direct inequalities giving some **coercivity** and **continuity** results respectively for the main introduced operator in **HUM** method like we do in the last chapter.

3.1 Vibrating plate equation depending on a parameter

Let us denote an open bounded subset Ω of $(\mathbb{R}^d, d \geq 1)$ with a regular border Γ and $T > 0$, and we use the term $Q = \Omega \times]0, T[$, $\Sigma = \Gamma \times]0, T[$. We consider the controlled system below, which describes plate vibrations:

$$\begin{cases} y_{tt} + \Delta(a(x,\nu)\Delta y) = 0 & \text{in } Q, \\ y = 0, \dfrac{\partial y}{\partial \eta} = u & \text{on } \Sigma, \\ y(x,0) = y^0,\ y_t(x,0) = y^1 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $a \in C^1(]0,1[\,, L^\infty(\Omega)), \mathcal{H} = H_0^2(\Omega) \times L^2(\Omega)$ is the state space[8], η is the unit normal vector on Γ, $U \subset L^2(0,T; L^\infty(\Omega))$ be set of admissible controls, For every $(y^1, y^0) \in H_0^2(\Omega) \times L^2(\Omega)$ and $u \in U$, our system has a unique weak solution :

$$(y, y_t) = \left(y_u, \frac{\partial y_u}{\partial t}\right) \in C(0,T;\mathcal{H}),$$

see the chapter IV in [8].

Definition 3.1 *[1]The system of vibrating plate is said to be averaged null controllable If there is a control u independent of the parameter ν such that*

$$\left(\int_0^1 y(x,T,\nu)\, d\nu, \int_0^1 y_t(x,T,\nu)\, d\nu\right) = (0,0)\ . \tag{3.2}$$

3.2 The averaged energy

We consider the following homogeneous plate equation, which has smooth beginning conditions:

$$\begin{cases} \phi_{tt} + \Delta(a(x,\nu)\Delta\phi) = 0 & \text{in } Q, \\ \phi = \dfrac{\partial \phi}{\partial \eta} = 0 & \text{on } \Sigma, \\ \phi(x,0) = \phi^0,\ \phi_t(x,0) = \phi^1 & \text{in } \Omega, \end{cases} \tag{3.3}$$

where $(\phi^0, \phi^1) \in H_0^2(\Omega) \times L^2(\Omega)$ don't depend on ν. The homogeneous plate equation is well-known for having a unique solution. $\phi = \phi(x,t,\nu)$ [15].

Definition 3.2 (The averaged energy) *[1]We define for all $t \in (0,T)$ the averaged energy with respect to ν associated to the solution of the vibrating plate homogeneous equation by the following quadratic form:*

$$E_a(t) = \frac{1}{2}\int_0^1 \int_\Omega \left[|\phi_t|^2 + a(x,\nu)\,|\Delta\phi|^2\right] dx d\nu\,. \tag{3.4}$$

Lemma 3.1 *[1]For all ϕ the solution of the homogeneous problem (3.3) the averaged energy (3.4) is conserved for all $t \in (0,T)$ i.e.:*

$$E_a(t) = E_a(0) = \frac{1}{2}\int_0^1 \int_\Omega \left[\left|\phi^1\right|^2 + a(x,\nu)\left|\Delta\phi^0\right|^2\right] dx d\nu.$$

Proof. [1]Let's multiplying the homogeneous equation with ϕ_t and we integrate on $(0,1)\times Q$ and by using the second identity of green (3.4) we get the following:

$$\begin{aligned}
0 &= \int_0^1 \int_\Omega \left(\phi_{tt} + \Delta(a(x,\nu)\Delta\phi)\right)\phi_t dx d\nu \\
&= \int_0^1 \int_\Omega \phi_t \phi_{tt} dx d\nu + \int_0^1 \int_\Omega \phi_t \Delta(a(x,\nu)\Delta\phi) dx d\nu \\
&= \frac{1}{2}\int_0^1 \frac{d}{dt}\int_\Omega |\phi_t|^2 dx d\nu + \underbrace{\frac{1}{2}\int_0^1 \frac{d}{dt}\int_\Omega a(x,\nu)\left|\Delta\phi_t\right|^2 dx d\nu}_{Green\ indentity\ II\ (3.4)} \\
&\quad \underbrace{-\int_0^1 \int_\Sigma \left[a(x,\nu)\Delta\phi\left(\frac{\partial\phi_t}{\partial\eta}\right) - \frac{\partial}{\partial\eta}\left(a(x,\nu)\Delta\phi\right)\phi_t\right] d\Sigma d\nu,}_{Green\ indentity\ II\ (3.4)}
\end{aligned}$$

but according to the boundary conditions in the homogeneous system we get :

$$\begin{aligned}
0 &= \int_0^1 \int_\Omega \left(\phi_{tt} + \Delta(a(x,\nu)\Delta\phi)\right)\phi_t dx d\nu \\
&= \frac{1}{2}\int_0^1 \frac{d}{dt}\int_\Omega |\phi_t|^2 dx d\nu + \frac{1}{2}\int_0^1 \frac{d}{dt}\int_\Omega a(x,\nu)\left|\Delta\phi_t\right|^2 dx d\nu \\
&= \frac{dE_a(t)}{dt},
\end{aligned}$$

then the energy is conserved.(i.e):

$$E_a(t) = E_a(0) = \frac{1}{2}\int_0^1 \int_\Omega \left[\left|\phi^1\right|^2 + a(x,\nu)\left|\Delta\phi^0\right|^2\right] dx d\nu.$$

∎

3.3 Averaged direct and inverse inequalities

The goal of this part is to create an identity for the problem's weak solutions (3.1) , from which we will show the estimations required for Hilbert Uniqueness Method (**HUM**) application and create observability theorems (the averaged inverse inequality).

In this chapter we treat the problem of vibrating plates equation we introduce our problem , we also demonstrate an averaged inverse and direct inequalities giving some **coercivity** and **continuity** results respectively for the main introduced operator in **HUM** method.

Let's begin by stating the following backward equation (2.10):

$$\begin{cases} \psi_{tt} - \nu^2 \Delta \psi = 0 & \text{in } Q, \\ \psi = \begin{cases} \int_0^1 \nu^2 \dfrac{\partial \phi}{\partial \eta} d\nu \\ 0 \end{cases} & \begin{matrix} \text{on } \Sigma_0, \\ \text{on } \tilde{\Sigma}_0, \end{matrix} \\ \psi(T) = 0, \ \psi_t(T) = 0 & \text{in } \Omega. \end{cases} \tag{3.5}$$

Lemma 3.2 *[1] Let $q = (q_k)$ be a vector field in $\left(C^1\left(\overline{\Omega}\right)\right)^n$ independent of the parameter ν, then for every weak solution ϕ for* (3.5)*, we have:*

$$\begin{aligned} & \frac{1}{2}\int_0^1 \int_{\Sigma} a(x,\nu) q_i \eta_i \left|\Delta\phi\right|^2 d\Sigma d\nu \\ = & \int_0^1 \int_{\Omega} \phi_t q_i \frac{\partial \phi}{\partial x_i} dx d\nu \bigg|_0^T + \frac{1}{2}\int_0^1 \int_Q \frac{\partial q_i}{\partial x_i} \left|\phi_t\right|^2 dxdtd\nu \\ & + \int_0^1 \int_Q a(x,\nu) \Delta q_i \Delta\phi \frac{\partial \phi}{\partial x_i} dxdtd\nu \\ & +2 \int_0^1 \int_Q a(x,\nu) \frac{\partial q_i}{\partial x_j} \Delta\phi \frac{\partial^2 \phi}{\partial x_j \partial x_i} dxdtd\nu \\ & -\frac{1}{2}\int_0^1 \int_Q \frac{\partial}{\partial x_i}\left(a(x,\nu) q_i\right) \left|\Delta\phi\right|^2 dxdtd\nu. \end{aligned} \tag{3.6}$$

Proof. Let's multiply the backward equation (3.5) by the mulltiplicative $q_i \dfrac{\partial \phi}{\partial x_i}$ and integrating over $(0,1) \times Q$:

$$\begin{aligned} 0 &= \int_0^1 \int_Q \left(\phi_{tt} + \Delta(a(x,\nu)\Delta\phi)\right) q_i \frac{\partial \phi}{\partial x_i} dxdtd\nu \\ &= \underbrace{\int_0^1 \int_Q \phi_{tt} q_i \frac{\partial \phi}{\partial x_i} dxdtd\nu}_{(I)} + \underbrace{\int_0^1 \int_Q \Delta(a(x,\nu)\Delta\phi) q_i \frac{\partial \phi}{\partial x_i} dxdtd\nu}_{(II)} \end{aligned}$$

Analyse of the first intgral (I):

$$\begin{aligned} \int_0^1 \int_Q \phi_{tt} q_i \frac{\partial \phi}{\partial x_i} dxdtd\nu &= \int_0^1 \int_{\Omega} \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \bigg|_0^T d\nu - \int_0^1 \int_Q \phi_t q_i \frac{\partial \phi_t}{\partial x_i} dxdtd\nu \\ &= \int_0^1 \int_{\Omega} \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \bigg|_0^T d\nu - \frac{1}{2}\int_0^1 \int_Q q_i \frac{\partial}{\partial x_i} \left|\phi_t\right|^2 dxdtd\nu \end{aligned} \tag{3.7}$$

and we observe that :

$$\begin{aligned}
&\int_0^1 \int_Q \phi_t q_i \frac{\partial \phi_t}{\partial x_i} dx dt d\nu \\
&= \frac{1}{2} \int_0^1 \int_Q q_i \frac{\partial}{\partial x_i} |\phi_t|^2 dx dt d\nu \\
&= -\frac{1}{2} \int_0^1 \int_Q \frac{\partial q_i}{\partial x_i} |\phi_t|^2 dx dt d\nu,
\end{aligned} \tag{3.8}$$

then we just get:

$$\int_0^1 \int_Q \phi_{tt} q_i \frac{\partial \phi}{\partial x_i} dx dt d\nu = \int_0^1 \int_\Omega \phi_t q_i \frac{\partial \phi}{\partial x_i} dx \Big|_0^T d\nu + \frac{1}{2} \int_0^1 \int_Q \frac{\partial q_i}{\partial x_i} |\phi_t|^2 dx dt d\nu. \tag{3.9}$$

Analyse of the second intgral (II) , first of all we apply the second identity of green (3.4) we get :

$$\begin{aligned}
&\int_0^1 \int_Q \Delta(a(x,\nu)\Delta\phi) q_i \frac{\partial \phi}{\partial x_i} dx dt d\nu \\
= &\int_0^1 \int_Q a(x,\nu)\Delta\phi \Delta \left(q_i \frac{\partial \phi}{\partial x_i} \right) dx dt d\nu \\
&+ \int_0^1 \int_\Sigma \frac{\partial}{\partial \eta} (a(x,\nu)\Delta\phi)\, q_i \frac{\partial \phi}{\partial x_i} d\Sigma d\nu \\
&- \int_0^1 \int_\Sigma a(x,\nu)\Delta\phi \frac{\partial}{\partial \eta} \left(q_i \frac{\partial \phi}{\partial x_i} \right) d\Sigma d\nu,
\end{aligned}$$

and because $q_i \dfrac{\partial \phi}{\partial x_i} = 0$ on Σ we get :

$$\begin{aligned}
\int_0^1 \int_Q \Delta(a(x,\nu)\Delta\phi) q_i \frac{\partial \phi}{\partial x_i} dx dt d\nu &= \int_0^1 \int_Q a(x,\nu)\Delta\phi \Delta \left(q_i \frac{\partial \phi}{\partial x_i} \right) dx dt d\nu \\
&- \int_0^1 \int_\Sigma a(x,\nu)\Delta\phi \frac{\partial}{\partial \eta} \left(q_i \frac{\partial \phi}{\partial x_i} \right) d\Sigma d\nu.
\end{aligned}$$

And we have :

$$\Delta \left(q_i \frac{\partial \phi}{\partial x_i} \right) = \left(\Delta q_i \frac{\partial \phi}{\partial x_i} + 2 \frac{\partial q_i}{\partial x_j} \frac{\partial^2 \phi}{\partial x_j \partial x_i} + q_i \frac{\partial \Delta \phi}{\partial x_i} \right),$$

and

$$\frac{\partial}{\partial \eta} \left(q_i \frac{\partial \phi}{\partial x_i} \right) = \left(\frac{\partial q_i}{\partial \eta} \frac{\partial \phi}{\partial x_i} + q_i \frac{\partial^2 \phi}{\partial \eta \partial x_i} \right) = q_i \frac{\partial^2 \phi}{\partial \eta \partial x_i},$$

because of $\dfrac{\partial \phi}{\partial x_i} = 0$ on Σ.

Then we write :

$$
\begin{aligned}
&\int_0^1\int_Q \Delta(a(x,\nu)\Delta\phi)q_i\frac{\partial\phi}{\partial x_i}dxdtd\nu\\
=\ &\int_0^1\int_Q a(x,\nu)\Delta\phi\Delta q_i\frac{\partial\phi}{\partial x_i}dxdtd\nu\\
&+2\int_0^1\int_Q a(x,\nu)\Delta\phi\frac{\partial q_i}{\partial x_j}\frac{\partial^2\phi}{\partial x_j\partial x_i}dxdtd\nu\\
&+\int_0^1\int_Q a(x,\nu)\Delta\phi q_i\frac{\partial\Delta\phi}{\partial x_i}dxdtd\nu\\
&-\int_0^1\int_\Sigma a(x,\nu)\Delta\phi q_i\frac{\partial^2\phi}{\partial\eta\partial x_i}d\Sigma d\nu.
\end{aligned}
$$

But we have:

$$
\begin{aligned}
\int_0^1\int_Q a(x,\nu)\Delta\phi q_i\frac{\partial\Delta\phi}{\partial x_i}dxdtd\nu &= \frac{1}{2}\int_0^1\int_Q a(x,\nu)q_i\frac{\partial}{\partial x_i}\left|\Delta\phi\right|^2dxdtd\nu\\
&= -\frac{1}{2}\int_0^1\int_Q\frac{\partial}{\partial x_i}\left(a(x,\nu)q_i\right)\left|\Delta\phi\right|^2dxdtd\nu\\
&\quad+\frac{1}{2}\int_0^1\int_\Sigma a(x,\nu)q_i\eta_i\left|\Delta\phi\right|^2dxdtd\nu,
\end{aligned}
$$

so we get:

$$
\begin{aligned}
&\int_0^1\int_Q \Delta(a(x,\nu)\Delta\phi)q_i\frac{\partial\phi}{\partial x_i}dxdtd\nu\\
=\ &\int_0^1\int_Q a(x,\nu)\Delta\phi\Delta q_i\frac{\partial\phi}{\partial x_i}dxdtd\nu\\
&+2\int_0^1\int_Q a(x,\nu)\Delta\phi\frac{\partial q_i}{\partial x_j}\frac{\partial^2\phi}{\partial x_j\partial x_i}dxdtd\nu\\
&-\frac{1}{2}\int_0^1\int_Q\frac{\partial}{\partial x_i}\left(a(x,\nu)q_i\right)\left|\Delta\phi\right|^2dxdtd\nu\\
&-\int_0^1\int_\Sigma\left(a(x,\nu)\Delta\phi q_i\frac{\partial^2\phi}{\partial\eta\partial x_i}-\frac{1}{2}a(x,\nu)q_i\eta_i\left|\Delta\phi\right|^2\right)d\Sigma d\nu.
\end{aligned}
$$

in other hand , because of $\phi\in H_0^2(\Omega)$, we have:

$$
\frac{\partial^2\phi}{\partial\eta\partial x_i}=\frac{\partial^2\phi}{\partial\eta^2}\eta_i \text{ and } \frac{\partial^2\phi}{\partial^2 x_i}=\frac{\partial^2\phi}{\partial\eta^2}\eta_i^2 \text{ on } \Sigma,
$$

and we have :

$$
\begin{aligned}
&\int_0^1\int_\Sigma\left(-a(x,\nu)\Delta\phi q_i\frac{\partial^2\phi}{\partial\eta\partial x_i}+\frac{1}{2}a(x,\nu)q_i\eta_i\left|\Delta\phi\right|^2\right)d\Sigma d\nu\\
=\ &-\frac{1}{2}\int_0^1\int_\Sigma a(x,\nu)q_i\eta_i\left|\Delta\phi\right|^2d\Sigma d\nu.
\end{aligned}
$$

So we just get the identity (3.5):

$$
\begin{aligned}
& \frac{1}{2}\int_0^1\int_\Sigma a(x,\nu)q_i\eta_i\left|\Delta\phi\right|^2 d\Sigma d\nu \\
= \quad & \int_0^1\int_\Omega \phi_t q_i\frac{\partial\phi}{\partial x_i}dxd\nu\Big|_0^T+\frac{1}{2}\int_0^1\int_Q\frac{\partial q_i}{\partial x_i}\left|\phi_t\right|^2 dxdtd\nu \\
& +\int_0^1\int_Q a(x,\nu)\Delta q_i\Delta\phi\frac{\partial\phi}{\partial x_i}dxdtd\nu \\
& +2\int_0^1\int_Q a(x,\nu)\frac{\partial q_i}{\partial x_j}\Delta\phi\frac{\partial^2\phi}{\partial x_j\partial x_i}dxdtd\nu \\
& \quad -\frac{1}{2}\int_0^1\int_Q\frac{\partial}{\partial x_i}\left(a(x,\nu)q_i\right)\left|\Delta\phi\right|^2 dxdtd\nu\,.
\end{aligned}
$$

■

Theorem 3.1 (The direct inequality) *[1] Let $T>0$ to be arbitrarily, and for every $\left(\phi^0,\phi^1\right)$ there exists a constant $c=c(T)>0$ such that the solution of the adjoint equation (3.5) verifies the following inequality:*

$$
\frac{1}{2}\int_0^1\int_\Sigma a(x,\nu)\left|\Delta\phi\right|^2 d\Sigma d\nu\le c\int_0^1\int_\Omega\left[\left|\phi^1\right|^2+a(x,\nu)\left|\Delta\phi^0\right|^2\right]dxd\nu\,. \tag{3.10}
$$

Proof. For the proof see [1] and the chapter 1, lemma (3.2) of [8] where we find the main idea of the proof. ■

Lemma 3.3 *[1]From the inequality (3.10) we deduce that for any weak solution ϕ of the homogeneous plate equation we have $\Delta\phi\in L^2(\Sigma)$.*
Let's start with the notation below , for $x^0\in\mathbb{R}^n$ we set

$$
\begin{aligned}
m(x) &= x-x^0 \qquad (x\in\mathbb{R}^n)\,, \\
\Gamma\left(x^0\right) &= \left\{x\in\Gamma; m(x).\eta(x)>0\right\}, \\
\tilde{\Gamma}\left(x^0\right) &= \Gamma\backslash\Gamma\left(x^0\right), \\
R(x^0) &= \sup_{x\in\overline{\Omega}}\left|m(x)\right|.
\end{aligned}
$$

And let λ_0 to be the first eigenvalue of the problem

$$
\Delta(a(x,\nu)\Delta\phi)=-\lambda_0^2 a(x,\nu)\Delta\phi,\ \ \phi\in H_0^2\left(\Omega\right).
$$

Where the eigenvalue λ_0^2 is characterized by

$$\lambda_0^2 = \min_{\phi \in H_0^2(\Omega)\setminus\{0\}} \int_0^1 \frac{\int_\Omega |\nabla \phi|^2 \, dx}{\int_\Omega a(x,\nu) \, |\Delta \phi|^2 \, dx} d\nu,$$

then we have:

$$\lambda_0^2 \leq \int_0^1 \frac{\int_\Omega |\nabla \phi|^2 \, dx}{\int_\Omega a(x,\nu) \, |\Delta \phi|^2 \, dx} d\nu \quad , \quad \forall \phi \in H_0^2 (\Omega) \ .$$

Theorem 3.2 (The inverse inequality) *[1] Suppose that Γ is of class C^3, so for every T we have*

$$T > T\left(x^0\right) = \frac{R(x^0)}{\lambda_0},$$

and any every solution ϕ of homogeneous problem (3.3), so we have the following inequality

$$\left(T - T(x^0)\right) E_a(0) \leq \frac{R(x^0)}{4} \int_0^1 \int_{\Sigma(x^0)} a(x,\nu) \, |\Delta \phi|^2 \, d\Sigma d\nu \, . \tag{3.11}$$

Proof. We just need the notation of lemma (3.3) and see [1]. ■

3.4 Averaged null controllability(The Hilbert Uniqueness Method)

The key procedures for calculating the control function u that guides the averaged state of the system (3.1) to the null state are presented in this section . Lions [8] established the Hilbert Uniqueness Technique (HUM), which is the basis for this method.

Theorem 3.3 *Assume that Theorem (3.2) assumptions are correct .Then, for any given set of initial data $(y^0, y^1) \in L^2(\Omega) \times H^{-2}(\Omega)$ there exists $u \in L^2(\Sigma(x^0))$ such that the solution of (3.1) satisfies (3.2).*

Proof. [1]To begin, consider the following backward equation

$$\begin{cases} \psi_{tt} + \Delta(a(x,\nu)\Delta\psi) = 0 & \text{in } Q, \\ \psi = 0 & \text{on } \Sigma, \\ \dfrac{\partial \psi}{\partial \eta} = \begin{cases} \int_0^1 a(x,\nu)\Delta\phi d\nu & \text{on } \Sigma\left(x_0\right), \\ 0 & \text{on } \tilde{\Sigma}\left(x_0\right), \end{cases} \\ \psi(x,T) = \psi_t(x,T) = 0 & \text{in } \Omega. \end{cases}$$

And we define the operator Λ on $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ by

$$\Lambda\left(\phi^0, \phi^1\right) = \left(\int_0^1 \psi_t(x,0,\nu) d\nu, - \int_0^1 \psi(x,0,\nu) d\nu \right).$$

And we have by mulitipling the backward equation by ϕ and integrate on $(0.1)\times Q$:

$$\begin{aligned}
&\int_0^1\int_Q(\psi_{tt}+\Delta(a(x,\nu)\Delta\psi))\,\phi dxdtd\nu\\
=\;&\int_0^1\int_Q\psi_{tt}\phi dxdtd\nu+\int_0^1\int_Q\phi\Delta(a(x,\nu)\Delta\psi dxdtd\nu\\
=\;&\int_0^1\int_\Omega\psi(T,x,\nu)\phi_t(T,x,\nu)dxdtd\nu-\int_0^1\int_\Omega\psi(0,x,\nu)\phi_t(0,x,\nu)dxdtd\nu\\
&-\int_0^1\int_\Omega\psi_t(T,x,\nu)\phi(T,x,\nu)dxdtd\nu+\int_0^1\int_\Omega\psi_t(0,x,\nu)\phi(0,x,\nu)dxdtd\nu\\
&+\int_0^1\int_\Sigma\psi\frac{\partial}{\partial\eta}(a(x,\nu)\Delta\phi)d\Sigma d\nu-\int_0^1\int_\Sigma a(x,\nu)\Delta\phi\frac{\partial\psi}{\partial\eta}d\Sigma d\nu\\
=\;&0\,.
\end{aligned}$$

Then according to the boundary condition, we get

$$\int_\Omega\phi^0(x)\int_0^1\psi_t(x,0,\nu)d\nu dx-\int_\Omega\phi^1(x)\int_0^1\psi(x,0,\nu)d\nu dx=\int_\Sigma\left|\int_0^1a(x,\nu)\Delta\phi d\nu\right|^2d\Sigma,$$

then we set:

$$\left(\Lambda\left(\phi^0,\phi^1\right),\left(\phi^0,\phi^1\right)\right)=\int_\Sigma\left|\int_0^1a(x,\nu)\Delta\phi d\nu\right|^2d\Sigma.$$

Then we set the Hilbert space $F=H_0^2(\Omega)\times L^2(\Omega)$ (see [11]) completed by $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ for the norm

$$\left\|\left(\phi^0,\phi^1\right)\right\|_F=\left\|\int_0^1a(x,\nu)\Delta\phi d\nu\right\|_{L^2(\Sigma)}.$$

From (3.11) and the Lax-Milgram lemma (3.8) results that Λ defines an isomorphism from $H_0^2(\Omega)\times L^2(\Omega)$ to $H_0^{-2}(\Omega)\times L^2(\Omega)$. From HUM we deduce the null controllability of the initial data $(y^1,-y^0)\in H_0^{-2}(\Omega)\times L^2(\Omega)$, the equation $\Lambda\left(\phi^0,\phi^1\right)=(y^1,-y^0)$ has a unique solution $\left(\phi^0,\phi^1\right)$. Then the control $u\in L^2(\Sigma_0)$ given by

$$u=\int_0^1a(x,\nu)\Delta\phi d\nu,$$

Where ϕ the solution of homogeneous plate equation (3.3) associated with the data $\left(\phi^0,\phi^1\right)\in H_0^2(\Omega)\times L^2(\Omega)$. ■

Conclusion & perspectives

Our work has led us to characterized the average control for the wave equation, and the vibrating plates equation where we controlled the displacement to be compatible with unknown physical proprieties.

We have study the HUM method to contol our systems in average. We have avoided the missing velocity of propagation parameter by controlling the average of the state with respect to this parameter. Then, we get coontrolling systems characterizing the averaged control.

As well as, we have characterized the control for an wave equation by thr HUM argument, and we do the same with vibrating plates equation.

We note that in both studied problems, the control has characterized by an adjoint system which has a simple structure.

In the future, the notion of averaged control and HUM method could be applied to control other distributed systems depending on an unknown parameter on the boundary.

Appendices

Definition 3.3 *We call a vector field an application $f:\mathbb{R}^n \to \mathbb{R}^n$ who has $x=(x_1,\cdots,x_n)$ associate $f(x)=(f_1(x),\cdots,f_n(x))$ for a function $g:\mathbb{R}^n\to\mathbb{R}$,its **Gradient** is the vector field defined by $\nabla g(x)=(\frac{\partial g(x)}{\partial x_1},\cdots,\frac{\partial g(x)}{\partial x_n})$.*
*For a vector field $f:\mathbb{R}^n\to\mathbb{R}^n$ we call **Divergence** the function $\operatorname{div} f(x)=\frac{\partial f_1(x)}{\partial x_1}+\cdots+\frac{\partial f_n(x)}{\partial x_n}$.*
*We call **Laplacian** of a function $g:\mathbb{R}^n\to\mathbb{R}$ the function $\Delta g(x)=\operatorname{div}(\nabla g)=\frac{\partial^2 g(x)}{\partial^2 x_1}+\cdots+\frac{\partial^2 g(x)}{\partial^2 x_n}$.*

Let Ω be a bounded subset of $\mathbb{R}^n$, the border of which is regular.

Definition 3.4 *We call normal to the domain Ω a vector field $\eta(x)$ defined on the bord $\partial\Omega$ of Ω and such that at any point $x\in\partial\Omega$ where the bord is regular , $\eta(x)$ either orthogonal to the edge and unitary $(\|\eta(x)\|=1)$.We calls **external normal** a normal which points towards the outside of the field at any point.*
*We call **normal derivative** of a regular function g on the bord of a domain Ω the function defined on the regular points of $\partial\Omega$ by $\frac{\partial g(x)}{\partial\eta}=\nabla g(x).\eta(x)$,(where $\nabla g(x).\eta(x)$ is the scalar product of the vector $\nabla g(x)$ with the vector $\eta(x)$).*

Meaning of surface or contour integrals

In dimension two

Let Γ be a regular parameterized curve of $\mathbb{R}^2$, $\{x(t)=(x_1(t),x_2(t),t\in[a,b]\}$.We call an integral of a function u on Γ :

$$\int_\Gamma u d\sigma=\int_a^b u(x_1(t),x_2(t))\,\|x'(t)\|\,dt,$$

where $\|x'(t)\|=\sqrt{x_1'(t)^2+x_2'(t)^2}$.

Theorem 3.4 (Divergence and Green formula) *[11]*
Let Ω be a domain of $\mathbb{R}^n$,and $\eta(x)$ its exterior normal. Let u and v be two regular functions , w a field of vectors defined on Ω. So

$$\int_\Omega \operatorname{div} w dx=\int_{\partial\Omega} w.\eta d\sigma \quad (\textit{divergence formula}) \tag{3.12}$$

$$\int_\Omega (\Delta u)v dx=-\int_\Omega \nabla u.\nabla v dx+\int_{\partial\Omega}\frac{\partial u}{\partial\eta}v d\sigma \quad (\textit{green formula I}) \tag{3.13}$$

$$\int_\Omega (v\Delta u-u\Delta v)\,dx=\int_\Omega\left(v\frac{\partial u}{\partial\eta}-u\frac{\partial v}{\partial\eta}\right)dx \quad (\textit{green formula II}) \tag{3.14}$$

Theorem 3.5 (classicl Kalman rank condition) *[13]*

Consider the following finite dimensional linear control system where A is a real $n \times n$ matrix, B is a real $n \times m$ matrix and y^0 a vector in $\mathbb{R}^n$, the function $y : [0,T] \to \mathbb{R}^n$ represents the state and $u : [0,T] \to \mathbb{R}^n$ the control with the form

$$\begin{cases} y'(t) = Ay(t) + Bu(t), 0 < t < T, \\ y(0) = y^0. \end{cases}$$

This System is controllable in some time T if and only if

$$Rank\left[B, AB, A^2B, ..., A^{n-1}B\right] = n.$$

Consequently, if our system is controllable in some time $T > 0$ it is controllable in any time.

Theorem 3.6 (Cauchy-Schwarz inequality) *[11]*

$$\forall u, v \in L^2(\Omega); \left|\int_\Omega uvdx\right| \leq \left(\int_\Omega |u|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2\, dx\right)^{\frac{1}{2}}.$$

Theorem 3.7 (Cauchy inequality with ε) *[11]*
Also called $\varepsilon-$inequality the following :

$$|ab| \leq \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2.$$

For all $\varepsilon > 0$, $a, b \in \mathbb{R}$.

Bilinear forms and the Lax-Milgram Theorem:

Definition 3.5 *[11]In the variational formulation of boundary value problems a key role is played by bilinear forms. Given two linear spaces H_1, H_2 , a bilinear form in $H_1 \times H_2$ is a function :*

$$a : H_1 \times H_2 \longrightarrow \mathbb{R}$$

satisfying the following properties:
i) *$\forall y \in H_2$, the function:*

$$x \mapsto a(x,y)$$

is linear in H_1 .
ii) *$\forall x \in H_1$, the function:*

$$y \mapsto a(x,y)$$

is linear in H_2 .
*When $H_1 = H_2$, we simply say that a is a **bilinear form** in the hilbert space H .*

Definition 3.6 (abstract variational problem) *[11]*

Let H be a Hilbert space a be a bilinear form in H and $F \in H^*$. Consider the following problem, called abstract variational problem:

$$\begin{cases} \text{Find } u \in H \text{ such that :} \\ a(u,v) = \langle F, v \rangle \ , \ \forall v \in H \ . \end{cases} \tag{3.15}$$

Theorem 3.8 (Lax - Milgram) *[11]*

Let H be a real Hilbert space . Let $a = a(u,v)$ be a bilinear form in H. If:

i)** a is **continuous.

ii)** a is **coercive.

Then there exists a unique solution $\hat{u} \in H$ of the problem (2.13).

Proof. For the proof see the page 336 in [11]. ■

Bibliography

[1] Abdelli, M., & Hafdallah, A. (2021). Averaged null controllability for some hyperbolic equations depending on a parameter. Journal of Mathematical Analysis and Applications, 495(1), 124697.

[2] Bensoussan, A. (1993). An introduction to the hilbert uniqueness method. In Analysis and Optimization of Systems: State and Frequency Domain Approaches for Infinite-Dimensional Systems (pp. 184-198). Springer, Berlin, Heidelberg.

[3] Biccari, U., Navarro-Quiles, A., & Zuazua, E. (2020). Stochastic optimization methods for the simultaneous control of parameter-dependent systems. arXiv preprint arXiv:2005.04116.

[4] Kalman, R. E. (1960). Contributions to the theory of optimal control. Bol. soc. mat. mexicana, 5(2), 102-119.

[5] Lebeau, G., & Zuazua, E. (1998). Null-Controllability of a System of Linear Thermoelasticity. Archive for rational mechanics and analysis, 141(4), 297-329.

[6] Lohéac, J., & Zuazua, E. (2015). Averaged controllability of parameter dependent conservative semigroups.

[7] Lohéac, J., & Zuazua, E. (2016). From averaged to simultaneous controllability. In Annales de la Faculté des sciences de Toulouse: Mathématiques (Vol. 25, No. 4, pp. 785-828).

[8] Lions, J. L. (1988). Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1. RMA, 8.

[9] Méthode HUM pour la contrôlabilité exacte des EDP hyperboliques de type ondes.

[10] Siradjuddin, I., Amalia, Z., Setiawan, B., Wicaksono, R. P., & Yudaningtyas, E. (2017, July). Stabilising a cart inverted pendulum system using pole placement control method. In 2017 15th International Conference on Quality in Research (QiR): International Symposium on Electrical and Computer Engineering (pp. 197-203). IEEE.

[11] Salsa, S. (2016). Partial differential equations in action: from modelling to theory (Vol. 99).Springer.

[12] Zuazua, E. (2014). Averaged control. Automatica, 50(12), 3077-3087.

[13] Zuazua, E. (2006). Controllability of partial differential equations (Doctoral dissertation, Optimization and Control).

[14] Reddy, J. N. (2006). Theory and analysis of elastic plates and shells. CRC press.

[15] Zuazua, E. (1988). Contrôlabilité exacte en un temps arbitrairement petit de quelques modeles de plaques. Appendix, 1, 465-491.