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Thème

About the dynamics of Zeraoulia-Sprott mapping

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بسم الله الرحمن الرحيم

شكر و عرفان

نحمد الله و نشكره شكرا جزيلا إذ هو خالقنا، و معيننا

فهو الأولى بالشكر في كل الأوقات و الظروف.

نحمد الله عز و جل و نثني عليه الخير كله الذي وفقنا

لإتمام هذا العمل، و نسأله ان يجعل هذا كله

خالصا لوجهه الكريم و أن ينفعنا به و ينتفع به من بعدنا.

نتقدم بكل إحترام و تقدير بشكرنا و عرفاننا للأستاذ و البروفيسور

الفاضل الذي كان موجهنا في البحث العلمي

"زراولية الحاج"، الذي كان له الفضل الكبير في شق الطريق نحو النجاح

و على كل النصائح و التوجيهات.

كما نتقدم بالشكر لكل الأساتذة في تكويننا عبر مسيرتنا الدراسية

من الإبتدائية إلى الجامعة،

و إلى كل من قدم لنا يد المساعدة من قريب أو من بعيد

فله منا خالص الإحترام و التقدير،

نسأل الله أن يجازي الجميع كل الخير.

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اللهم أنفعني بما علمتني و بارك لي و زدني علما، اللهم اجعل القرآن ربيع قلبي و نور صدري و جلاء حزني و ذهاب همي –آمين-

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إلى من ساندتني منذ ولادتي و لم تبخل عني بشيء، إلى الشمعة التي تشتعل من أجل الجميع، إلى أمي الثانية خالتي فطوم شفاك الله و حفظك من كل شر.

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وفي الأخير قالوا المرحيل قلت لست براحلة فمازلت على الدرب أسير إلى آخر رمق في حياتي إن شاء الله.

بثينة

إهداء

الحمد لله وكفى والصلاة على الحبيب المصطفى وأهله ومن وفى أما بعد:

الحمد لله دائما وأبدا الذي وفقنا لتثمين هذه الخطوة في مسيرتنا الدراسية بمذكرتنا ثمرة الجهد و النجاح بفضله تعالى مهداة إلى:

جدتي:

مربيتي و مربية الأجيال-طراد الزهرة-عبارات الشكر تخجل منكي فأنتي أكبر منها,لكي مني كل التقدير و العرفان بعدد قطرات المطر وعدد من حج واعتمر.

والدي

:تتسابق الكلمات وتتزاحم العبارات لتنظم عقد الشكر الذي لا يستحقه إلا أنت,إليك يامن بذلت ولم تنتظر العطاء، إليك أهدي عبارات الشكر والتقدير شفاك الله يا أبي الغالي وأطال عمرك واداملك تاجا فوق رؤوسنا.

والدتي:

من وضع المولى سبحانه وتعالى الجنة تحت قدميها ووقرها في كتابه العزيز-أمي الحبيبة-حفظك الله يا غاليتي ألف شكر على ما قدمتي لي.

أخي العزيز يوسف

: نبراس البيت ونوره حفظك الله وأنار دربك وكتب لك كل الخير.

أخواتي:

إكرام، رحاب، إيناس، زعرورة يا مصدر سعادتي وفقكم الله وأنار دربكم وسهل أعمالكم.

أخوالي

: أبي الثاني عبد الغاني، سليم،رشيد،أحمد وكل زوجاتهم ،أبنائهم وبناتهم إن قلت شكرا فشكري لن يوفيكم حقكم، حقا سعيتم فكان السعي مشكورا، وإن جف حبري عن التعبير يكتبكم قلبي يالحب تعبيرا.

خالاتي:

_وهيبة,أمال وأزواجهم ،وأبنائهم وبناتهم يا سندي وياسعادتي أدامكم الله فرحة لقلبي شكرا لوجودكم في حياتي وألف إمتنان لدعكم لي.

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جداتي:يمينة، عائشة ،أعمامي وعمتي،أخوالي وخالاتي وجميع زوجاتهم وأزواجهم، أبنائهم وبناتهم حفظكم الله وبارك لكم ،لكم مني فائق الشكر و التقدير.

_ولا أنسى أبدا عائلتي الثانية -سعودي- أتقدم لكم بأجمل عبارات الشكر و الإمتنان من قلب فاض بالإحترام والتقدير لكم داعية الله عز
وجل أن يحفظكم ويحميكم .

صديقاتي

:إكرام، أميرة، هاجر،صفاء، مروة، نور مصدر الفرح والسعادة شكرا لكم لكل ثانية أمضيتها معكم أدام الله عشرتنا و محبتنا وفقكم الله وأنار دربكم.

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:بثينة لكي مني جزيل الشكر أدام الله عشرتنا ووفقكي يا صديقتي الغالية.

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Abstract

The objective of this thesis is to study the dynamical behaviors of the **Zeraoulia- Sprott mapping**. In particular, this map is the first simple rational map whose fraction has no vanishing denominator and gives chaotic attractors .

- In the first chapter, we mentioned some important and comprehensive concepts of dynamical system theory.
- In the second chapter, we introduced a two-dimentional smooth discrete bounded map capable of generating **multi-fold strange attractors** .
- In the third chapter we studied periodic 2-orbits of the **Zeraoulia-Sprott mapping** and we investigete also the bounded and unbounded orbits.

Resumé

L'objectif de cette thèse est d'étudier le comportement dynamique **d’application de Zeraoulia-Sprott**. En particulier, l'application étudiée est la première application rationnelle simple dont la dénominateur est non zéro et donne des attracteurs chaotiques.

- Dans le premier chapitre, nous avons mentionné quelques concepts importants de la théorie des systèmes dynamique.
- Dans le deuxième chapitre, nous avons introduit une application lisse, bornée, discrète et bidimensionnelle capable de générer des **attracteurs étranges multi-plis**.
- Dans le dernier chapitre nous avons étudié les périodique de période 2 de **l’applications de Zeraoulia-Sprott** et nous avons également étudié leur orbites bornées et non bornées.

ملخص

الهدف من هذه الرسالة هو دراسة السلوك الديناميكي لتطبيقات **زراولية-سبروت**.
على وجه الخصوص التطبيق المدروس هو أول تطبيق كسري بسيط لا يحتوي كسره على مقام صفري يعطي جاذبات فوضوية.

- ذكرنا في الفصل الاول بعض المفاهيم الهامة والشاملة للنظام الديناميكي.
- في الفصل الثاني قدمنا تطبيقا ثنائي الابعاد قابل للاشتقاق محدود منفصل قادر على توليد **جاذبات غريبة متعددة-الطيات.**
- في الفصل الاخير درسنا الدور 2 لتطبيقات **زراولية-سبروت** و درسنا ايضا مداراتها المحدودة و غير المحدودة.

Contents

General Introduction

Dynamical systems are developed during the 19^{th} century. Indeed in the end of this century, the French mathematician, physicist and philosopher **Henri Poincaré** had already put highlights the phenomenon of *sensitivity to initial conditions.* He showed in his study of solar system that there were *stable and unstable orbits* and that sometimes, a very weak disruption in the system could cause an orbit to change the state. He surrendered take into account that perfectly similar causes may not have the same effects.

In our work, we focus on the simplest two dimensional rational discrete chaotic mapping called the **Zeraoulia- Sprott map** which describes different random evolutionary processes. It produces several new chaotic attractors obtained via the *quasi-periodic route to chaos*.

In this thesis, we will provide a basic analysis of this mapping and give a detailed study of its dynamics. Our thesis is composed of three chapters: The first chapter presents some definitions and concepts that we will used later. In the second chapter we will introduce a two-dimensional, C^{∞}-discrete bounded map capable of generating *multi- fold strange attractors* via period-doubling bifurcation routes to chaos. The third chapter is devoted to the analysis of **Zeraoulia- Sprott map** and its dynamics.

Chapter 1

Notions of dynamical systems and prelimanery concepts

1.1 Dynamical systems

A dynamical system is a model describing the evolution over time of a set of interacting objects, it is defined by a *triplet* (X, T, f) made up of the state space X, the time domain T, and a state transition application $f : X \times T \to X$ which makes it possible to define from a vector of initial conditions the state of the system at any time.

Definition 1.1 *A discrete dynamical system is described by a system of equations with finite diferences, in other words, by a following recurrence:*

$$\begin{cases} x_{k+1} = f(x_k, c), k \in \mathbb{N} \\ \\ \qquad x_0 \;\; \textit{given} \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a function,* $c \in \mathbb{R}$ *is the vector of parameters,* $x_0 \subset \mathbb{R}^n$ *is the initial value,* $x_k \in \mathbb{R}^n$ *is the vector of the system states at iteration* t_k.

For an initial value x_0 we obtain $x_1 = f(x_0), x_2 = f(x_1) = f(f(x_0)) = f^2(x_0)$. In general, we have

$$x_n = f^n(x_0), \text{ where } f^n = f \underbrace{\circ \ldots\ldots \circ f}_{n \text{ fois}}$$

Here $f(x_0)$ is called the *first iteration* of x_0 by the function f. $f^2(x) = f(f(x_0))$ is called the *second iteration* of x_0 by the function f. $f^n(x_0)$ is called n^{th} *iteration* of x_0 by the function f.

Definition 1.2 *The phase space is a structure corresponding to the set of all possible states of the considered system. It can be a vector space, or a measurable space.*

Definition 1.3 *A discrete-time dynamical system is a pair (x, f) where x is a compact metric space and f is a map from x into itself.*

1.2 Fixed points

Definition 1.4 *A point x^* is called a fixed point of the map $x_{t+1} = f(x_t)$ if $f(x^*) = x^*$.*

Geometrically: The fixed point is an intersection of the curve of our function $y = f(x)$ with the line $y = x$.

1.2.1 Stability of fixed points

Finding solutions for nonlinear systems is not easy. Usually these solutions do not provide enough information to control systematic stability. Therefore, we need to finding the approximated linear system to study the stability of nonlinear systems.

Definition 1.5 *A fixed point x^* of $f : I \to I$, $I \subset \mathbb{R}^n$, is said to be attractive, if there is a neighborhood of x^* such that for any v_0 in this neighborhood, the sequence $(v_n)_{n\in\mathbb{N}}$ defined by v_0 and $v_{n+1} = f(v_n)$, converges to x^*, i.e.,*

$$\forall x_0 \in I,\ \exists\delta > 0, \|x^* - x_0\| < \delta \Rightarrow \|x^* - f(x_0)\| < 0$$

If

$$\forall x_0 \in I, \exists\delta > 0, \|x^* - x_0\| < \delta \Rightarrow \lim_{k\to\infty} f(x_k) = x^*$$

then the point x^* is asymptotically stable.

Definition 1.6 *A fixed point x^* of $f : I \to I$, $I \subset \mathbb{R}^n$, is unstable if:*

$$\forall x \in I, \exists\epsilon > 0, \|x^* - x_0\| < \delta \Rightarrow \|x^* - f(x)\| > 0$$

Theorem 1.1 *Let x^* be a fixed point of f and suppose $f \in C^1$.*
(i) If $|\lambda| < 1$ for every eigenvalue λ of $Df(x^*)$, then x^* is an asymptotically stable fixed point of f.
(ii) If $|\lambda| > 1$ for some eigenvalue λ of $Df(x^*)$, then x^* is not a Lyapunov stable fixed point of f.

• The eigenvalues of $Df(x^*)$ are called the *stability multipliers* of x^*.

• Here we consider the linear map in two dimensions and write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The stability multipliers of the fixed point 0 are the roots of $\lambda^2 - \tau\lambda + \delta = 0$ where $\tau = a + d$, $\delta = ad - cb$. Solving $|\lambda| = 1$ produces three cases: (i) $\lambda = 1$, here $\delta = \tau - 1$. (ii) $\lambda = -1$, here $\delta = -\tau - 1$. (iii) $\lambda = e^{i\Phi}$ for some $\Phi \in (0, \pi)$, here $\delta = 1$ and $\tau \in (-2, 2)$. The origin 0 is stable in the triangle of the (τ, δ)-plane bounded by the lines (i), (ii), and (iii).

Definition 1.7 *Two maps $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ are said to be conjugate if there exists a homeomorphism $h : X_1 \to X_2$ such that $h(f_1(y)) = f_2(h(y))$, for all $y \in X_1$.*

Definition 1.8 *A family of maps $x \to f(x, \mu)$, where $f : X \times \mathbb{R}^m \to X$, is structurally stable at a given value of μ if $x_{i+1} \to f\left(x_i, \widetilde{\mu}\right)$ is conjugate to $x_{i+1} \to f(x_i, \mu)$ for all $\widetilde{\mu}$ in some neighbourhood of μ.*

In order to describe *structural stability* more generally we need to think about spaces of functions. For simplicity we consider only phase spaces X that are compact and we begin in one dimension:

Definition 1.9 *A forward invariant region of $f(x)$ is a set $\Omega \subset X$ for which $f(\Omega) \subset \Omega$. A trapping region is a non-empty, compact set $\Omega \subset X$ for which $f(\Omega) \subset int(\Omega)$.*

Definition 1.10 *A set $\Lambda \subset X$ is said to be an attracting set of $f(x)$ if there exists a trapping region Ω such that:*

$$\Lambda = \bigcap_{n=0}^{\infty} f^n(\Omega)$$

An attractor is an attracting set that contains a dense orbit [1].

Definition 1.11 *A C^1-map f on a smooth manifold X is said to be Axiom A if the non-wandering set of f is compact and hyperbolic and the the set of periodic points of f is dense in the non-wandering set.*

Theorem 1.2 *A C^1-map f on a smooth compact manifold X is C^1 structurally stable if and only if it is Axiom A.*

Definition 1.12 *A continuous map $f : [0, 1] \to [0, 1]$ is said to be unimodal if there exists $c \in (0, 1)$ such that either $f(x) < c$ for all $x \neq c$ or $f(x) > c$ for all $x \neq c$. Furthermore, f is said to be S-unimodal if it is C^3 and the Schwarzian derivative*

$$s(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is non-positive in $[0,c)\cup(c,1[$.

Theorem 1.3 *Let* $f:[0,1]\to[0,1]$ *be a* S*-unimodal map with extremum* $c\in(0,1)$. *Suppose* $f(c)=1$, $f(1)=0$ *and* $f''(c)\neq 0$, *then there exists a unique attractor* $\Lambda\subset[0,1]$ *such that* $\Lambda=\omega(x)$ *for almost all* $x\in[0,1]$ *and either* (i) Λ *is periodic solution,* (ii) Λ *is a cycle of disjoint intervals,or* (iii) Λ *is a Feigenbaum-like attractor.*

1.2.2 Nature of fixed points

Definition 1.13 *Let* $f:\mathbb{R}\to\mathbb{R}$, *we define the multiplier* f *by:* $m=f'(x^*)$ *as the tangent of the fixed point* x^**of* f *which determines the type* (*or nature*) *of the fixed point.*

Theorem 1.4 *Suppose* x^* *is a fixed point of* $x_{k+1}=f(x_k)$, *then the fixed point* x^* *is: (1) Attractive if* $|m|<1$. *(2) Repulsive if* $|m|>1$. *(3) Indifferent if* $|m|=1$ *we cannot conclude. (4) Super stable if* $m=0$. *Here* m *is called the multuplicator of* f *at point* x^*. $[2]$.

Proof. We use Taylor's formula in the neighborhood of x^* with $f(x^*)=x^*$and $f'(x^*)=m$ to get

$$\begin{aligned} f(x) &= f(x^*)+f'(x^*)(x-x^*)+O\left((x-x^*)^2\right) \\ &= x^*+m(x-x^*)+O(x-x^*)^2 \end{aligned}$$

$$\begin{aligned} f^2(x) &= f(f(x))=f(f(x^*))+m(m(x-x^*))+O(x-x^*)^2 \\ &= x^*+m^2(x-x^*)+O(x-x^*)^2 \end{aligned}$$

then

$$f^p(x)=x^*+m^p(x-x^*)+O(x-x^*)^2$$

Thus, we get the following results:
(1) x^* is attractive if $|m|<1$ because

$$f^p(x)=x^*+m^p(x-x^*)+O(x-x^*)^2$$

when $p\to\infty\Rightarrow f^p(x)\to x^*$.
(2) x^* is repulsive if $|m|>1$: $f^p(x)$ away from x^*, i.e., $|f^p(x)-x^*|\to\infty$.
(3) x^* is indifferent if $m=\pm1$: The nature of x^* depends on terms of order greater than 1 of one of the development of Taylor.
(4) x^* is super stable if $m=0$: The attraction is the strongest because the first order term in $(x-x^*)$ is completely disappears. ■

Example 1.1 *Stability of the fixed points for the following map:* $x_{n+1} = ax_n(1-x_n)$, $0 \leq a \leq 4$. *We have*

$$f(x) = x \Longrightarrow ax(1-x) = x \Longrightarrow x(-ax + (a-1)) = 0$$

$$\begin{cases} x = 0 \\ -ax + (a-1) = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = 0 \\ x_2 = \frac{a-1}{a},\ a \neq 0 \end{cases}$$

We have $f'(x) = a - 2ax$:

i) $m_1 = f'(x_1) = f'(0) = a$:

$m_1 < 1 \Longleftrightarrow a < 1$, x_1 *is attractive.*

$m_1 = 1 \Longleftrightarrow a = 1$, *case of doubt.*

$m_1 > 1 \Longleftrightarrow a > 1$, x_1 *is repulsive.*

$m_1 = 0 \Longleftrightarrow a = 0$, *super stable.*

ii) $m_2 = f'(x_2) = f'(\frac{a-1}{a}) = 2 - a$.

$m_2 < 1 \Longleftrightarrow 2 - a < 1 \Longleftrightarrow a > 1$, x_2 *is attractive.*

$m_2 = 1 \Longleftrightarrow 2 - a = 1 \Longleftrightarrow a = 1$, *case of doubt.*

$m_2 > 1 \Longleftrightarrow 2 - a > 1 \Longleftrightarrow a < 1$, x_2 *is repulsive.*

$m_2 = 0 \Longleftrightarrow 2 - a = 0 \Longleftrightarrow a = 2$, *super stable.*

1.2.3 Some definitions of chaos

Definition 1.14 ***Larousse definition****: General confusion of elements, of matter, before the creation of the world.*

Definition 1.15 ***Definition of E. Lorenz****: A system agitated by forces where only exist three independent frequencies, can become destabilized, its movements then becoming totally irregular and erratic.*

Definition 1.16 ***R. L. Devaney****: Let* (I, d) *be a compact metric space* (d *is a distance*) *and let* f *be a function such that* $f : I \to I, x_{k+1} = f(x_k)$ *.This discrete dynamic system is said to be chaotic if the following conditions are not verified:*
(1) f *has a sensitivity to the initial conditions: There exists a real number* $\epsilon > 0$ *such that, for all* $x_0 \in I$ *and for all* $\rho > 0$*, there exists a point* $y_0 \in I$ *and an integer* $k > 0$*, satisfying:*

$$d(x_0, y_0) < \rho \Rightarrow d(x_k, y_k) > 0$$

(2) f *is topologically transitive: If there exists* $x_k \in I$ *such that the orbit* $O = \left\{f^k(x_k), k \in \mathbb{N}\right\}$ *is dense in* I.

(3) The set of periodic points of f is dense in I : $\{x_0 \in I, \exists k > 0, x_k = x_0\}$ is dense in I.

Characterization of chaos: 1. Sensitivity to initial conditions. 2. Phase space. 3. Fractal dimension. 4. Lyapunov dimension. 5.Capacity dimension (Kolmogorov). 6. Strange attractors. 7. Positive Lyapunov exponents.

Definition 1.17 *We call the sequence $\{f(x_0)\}_{k=0}^{n}$ an orbit (or a trajectory) of the point x_0 and we note it by $O(x_0)$, i.e.,*

$$\begin{aligned} x_1 &= f(x_0), x_2 = f(x_1) = f(f(x_0)) = f^2(x_0), \\ x_3 &= f(x_2) = f^3(x_0), x_k = f^k(x_0) \end{aligned}$$

The orbits is:

$$O(x) = \{x_0, x_1, x_2, ..., x_k\} = \left\{x_0, f(x_0), f^2(x_0), ..., f^k(x_0)\right\}$$

Definition 1.18 *The chaotic attractors is the set of all limits for the sequence (x_n, y_n) when $n \to +\infty$ for different values of (x_0, y_0), or it becomes an accumulation set of all limits.*

Definition 1.19 *Lyapunov exponent is a mathematical quantity calculated from the application, and classified into three parts to see the nature of the solution, and we have an algorithm that calculates it: (i) If $LE < 0$, then the solution is a fixed point. (ii) If $LE = 0$, then the solution is a cyclical solution. (iii) If $LE > 0$, then the solution is chaotic and bounded.*

Definition 1.20 *A quasi-periodic solution is described by the following function $x_{t+T} = \lambda x_t$ where t is the period of cycle.*

Definition 1.21 *A bifurcation is a quantitative or qualitative change in the solution of a dynamical system with a modification of the parameters on which it depends, and in a manner more precise the disappearance or change of stability or the appearance of new solutions.*

Types of bifurcation: There are two types of bifurcation (local and global) each of these bifurcations is characterized by a normal form. For example, we recall the following types of bifurcations: 1. Fold bifurcation. 2. Transcritical bifurcation. 3. Fourch bifurcation. 4. Flip bifurcation. 5. Neimark-Sacker bifurcation. 6. Hopf bifurcation. 7. Doubling of period bifurcation...etc.
We are interested here in the so-called *local bifurcations*, i.e., **relative to a fixed point of a discrete system**.

Definition 1.22 *A fixed point x^* of a map f is said to be hyperbolic if no eigenvalue of $Df(x^*)$ has modulus 1.*

Non-hyperbolicity occurs if $Df(x^*)$ has (i) aneigenvalue 1, (ii) an eigenvalue -1, or (iii) eigenvalues $e^{\pm i\phi}$, where $\phi \in (0,\pi)$. These correspond to (i) saddle-node bifurcations, (ii) period-doubling (or flip) bifurcations, and (iii) Neimark-Sacker bifurcations.
We now describe this in these theories:

Theorem 1.5 *Consider the one-dimensional map*

$$x \to f(x,\mu) \tag{1.1}$$

where $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is C^k $(k \geq 2)$. Suppose:
(i) $f(0,0) = 0$, ($x = 0$ is a fixed point when $\mu = 0$).
(ii) $\frac{\partial f}{\partial x}(0,0) = 1$, (the associated stability multiplier is 1).
(iii) $\frac{\partial f}{\partial \mu}(0,0) \neq 0$, (transversality condition).
(iv) $\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$, (non-degeneracy condition).
Then there exists $\delta > 0$ and a unique C^k function $\xi : [-\delta,\delta] \to \mathbb{R}$ with

$$\xi(0) = 0, \xi'(0) = 0, \xi''(0) = \frac{\frac{\partial^2 f}{\partial x^2}(0;0)}{\frac{\partial f}{\partial \mu}(0;0)}$$

such that $f(x,\xi(x)) = x$ for all $x \in [-\delta,\delta]$.

If (1.1) satisfies the conditions of this Theorem, we say that (1.1) has a **saddle-node bifurcation** at $\mu = 0$. Here two fixed points (one stable, one unstable) **collide** and **annihilate**.

Theorem 1.6 *Consider (1.1) where f is C^k $(k \geq 3)$. Suppose:*
(i) $f(0,0) = 0$ ($x = 0$ is a fixed point when $\mu = 0$).
(ii) $\frac{\partial f}{\partial x}(0,0) = -1$ (the associated stability multiplier is -1).
(iii) $\alpha = \left(\frac{\partial^2 f}{\partial\mu\partial x} + \frac{1}{2}\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}\right)\big|_{(x,\mu)=(0,0)} \neq 0$ (transversality condition).
(iv) $\beta = \left(\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + \frac{1}{3}\frac{\partial^3 f}{\partial x^3}\right)\big|_{(x,\mu)=(0,0)} \neq 0$ (non-degeneracy condition).
Then there exists $\delta > 0$ and a unique C^{k-1} function $\xi : [-\delta,\delta] \to \mathbb{R}$ with $\xi(0) = 0$, $\xi'(0) = 0$, $\xi''(0) = -\frac{\beta}{\alpha}$, such that $f^2(x,\xi(x)) = x$ for all $x \in [-\delta,\delta]$.

Here f^2 refers to the second iterate of f (not the square of f). If (1.1) satisfies the conditions of this theorem, we say that (1.1) has a **period-doubling bifurcation** at $\mu = 0$. Here a fixed point changes stability and a period-2 solution is created.

Theorem 1.7 *Consider (1.1) where f is C^k $(k \geq 4)$. Suppose: $i)$ $f(0,\mu) = 0$ for all μ in a neighbourhood of 0 ($x = 0$ is a fixed point for small μ), $ii)$ $D\ f(0,\mu)$ has eigenvalues $r(\mu)e^{\pm\Phi(\mu)}$ with $r(0) = 1$ and $e^{in\Phi(0)} \neq 1$ for $n = 1,2,3,4$ (at $\mu = 0$ the associated stability multipliers have modulus*

1 *and are not strongly resonant*), $iii)$ $r^{'}(0) \neq 0$ (*transversality condition*), $iv)$ $\alpha \neq 0$ *where* α *is the first Lyapunov coefficient* (*non-degeneracy condition*). *Then* (1.1) *has an invariant topological circle, of size asymptotically proportional to* $\sqrt{|\mu|}$, *emanating from* $x = 0$ *for either* $\mu < 0$ *or* $\mu > 0$.

If (1.1) satisfies the conditions of this Theorem, then (1.1) has a **Neimark-Sacker bifurcation** at $\mu = 0$ and a fixed point changes stability and an invariant circle is created on which the dynamics may be quasiperiodic or weakly resonant.

1.3 Basins of attraction

An **attractor's basin of attraction** is the region of the phace space, over which iterations are defined, such that any point (any initial condition) in that region will asymptotically be iterated into the attractor. For a stable linear system every point in the phase space is in the basin of attraction. However, in nonlinear systems, some points may mapped directly or asymptotically to infinity, while other points may lie in different basin of attraction and mapped asymptotically into a different attractor, other initial conditions may be in or mapped directly to a non-attracting point or cycle.

Chapter 2

About C^{∞}-multifold Zeraoulia-Sprott chaotic attractors

Discrete mathematical models are usually derived from theory or experimental observation, or as an approximation to the Poincaré section for some continuous-time models. Many papers have described chaotic systems, one of the most famous being a two-dimensional discrete map suggested by Hénon and studied in detail by others $[4, 5, 6, 7]$. It is possible to change the form of this map to obtain other chaotic attractors or to make some C^1-modifications to obtain **multifold strange chaotic attractors** with possible applications in secure communications because of their chaotic properties $[9, 10]$. The Hénon map $[4]$ is a prototypical two-dimensional invertible iterated map with a chaotic attractor and is a **simplified model of the Poincaré map for the Lorenz equation** proposed by M. Hénon in 1976 and given by:

$$H\left(x_n, y_n\right) = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - ax_n^2 + by_n \\ x_n \end{pmatrix} \tag{2.1}$$

1. For $b = 0$, the Hénon map reduces to the quadratic map $[11]$, which is **conjugate** to the logistic map.

2. Bounded solutions exist for the Hénon map over a range of a and b values and a portion of this range yields chaotic attractors.

3. The Hénon map does not have attractors with *multifolds*.

However, a C^1-modifications can result in such attractors $[20]$:

$$F\left(x_n, y_n\right) = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - a\sin x_n + by_n \\ x_n \end{pmatrix} \tag{2.2}$$

or equivalently:

$$x_{n+1} = 1 - a \sin x_n - x_{n-1} \tag{2.3}$$

where the quadratic term x^2 in the Hénon map is replaced by the nonlinear term $\sin x$. The essential motivation for this work is to develop a C^{∞} mapping that is capable of generating chaotic attractors with multifolds via a period-doubling bifurcation route to chaos. The fact that this map is C^{∞} in some ways simplifies the study of the map and avoids some problems related to the lack of continuity or differentiability of the map. The choice of the term $\sin x$ has an important role since it makes the solutions bounded for values of b such that $|b| \leq 1$, and all values of a, while they are unbounded for $|b| > 1$.

2.1 Analytical results

In all proofs given here, we use the following standard results:

Theorem 2.1 *Let $(x_n)_n$ and $(z_n)_n$ be two real sequences if $|x_n| \leq |z_n|$ and $\lim\limits_{n \to +\infty} |z_n| = A < +\infty$ then $\lim\limits_{n \to +\infty} |x_n| \leq A$, or if $|z_n| \leq |x_n|$ and $\lim\limits_{n \to +\infty} |z_n| = +\infty$ then $\lim\limits_{n \to +\infty} |x_n| = +\infty$. See [20] .*

We use this result to construct a sequence $(z_n)_n$ that satisfies the above conditions for determining whether the difference equation (2.3) has bounded or unbounded orbits.

Theorem 2.2 *For all values of a and b the sequence $(x_n)_n$ given in (2.3) satisfies the following inequality:*

$$|1 - x_n + b x_{n-2}| \leq |a|$$

See [20] .

Proof. We have for every $n > 1$: $\quad x_n = 1 - a \sin x_{n-1} + b x_{n-2}$,then, one has:

$$|-x_n + 1 + b x_{n-2}| = |a \sin x_{n-1}| \leq |a|$$

since $\sup_{x \in R} |\sin x| = 1$. ∎

Theorem 2.3 *For every $n > 1$, and all values of a and b, and for all values of the initial conditions $(x_0, x_1) \in \mathbb{R}^2$, the sequence $(x_n)_n$ satisfies the following equalities:*

(a) *If* $b \neq 1$, *then:*

$$x_n = \begin{cases} \frac{b^{\frac{n-1}{2}}-1}{b-1} + b^{\frac{n-1}{2}} x_1 - a \sum_{P=1}^{P=\frac{n-1}{2}} b^{p-1} \sin x_{n-(2p-1)} \text{ if } n \text{ is odd} \\ \frac{b^{\frac{n}{2}}-1}{b-1} + b^{\frac{n}{2}} x_0 - a \sum_{P=1}^{P=\frac{n}{2}} b^{p-1} \sin x_{n-(2p-1)} \text{ if } n \text{ is even} \end{cases} \tag{2.4}$$

(b) *If* $b = 1$, *then*

$$x_n = \begin{cases} \frac{n-1}{2} + x_1 - a \sum_{P=1}^{P=\frac{n-1}{2}} \sin x_{n-(2p-1)} \text{ if } n \text{ is odd} \\ \frac{n}{2} + x_0 - a \sum_{P=1}^{P=\frac{n}{2}} \sin x_{n-(2p-1)} \text{ if } n \text{ is even} \end{cases} \tag{2.5}$$

See [20].

Proof. Assume that n is odd, then we have for every $n > 1$, the following equalities:

$$x_n = 1 - a \sin x_{n-1} + b x_{n-2} \tag{2.6}$$

$$x_{n-2} = 1 - a \sin x_{n-3} + b x_{n-4} \tag{2.7}$$

$$x_{n-4} = 1 - a \sin x_{n-5} + b x_{n-6} \tag{2.8}$$

Then the results in (2.4) and (2.5) are obtained by successive substitutions of (2.7), (2.8),... into (2.6) for all $k = n-2, n-4, ..., 2$. The other cases can be obtained using the same logic. ■

Theorem 2.4 *The fixed points* (l, l) *of the map* (2.3) *exist if one of the following conditions holds:*
(i) *If* $a \neq 0$, *and* $b \neq 1$, *then* l *satisfies the following conditions:*

$$\begin{cases} 1 - a \sin l + (b-1) l = 0 \text{ and } l \leq \frac{1+|a|}{1-b} \text{ if } b > 1 \\ \frac{1+|a|}{1-b} \leq l, \text{ if } b < 1 \end{cases}$$

(ii) *If* $b = 1$, *and* $|a| \geq 1$, *then,* l *is given by* $l = \arcsin(\frac{1}{a})$.
(iii) *If* $b \neq 1$, *and* $a = 0$, *then,* l *is given by* $l = \frac{1}{1-b}$.
(iv) *If* $a = 0$, *and* $b = 1$, *there are no fixed points for the map* (2.3). *See* [20].

Proof. The proof is direct except for the case (i) where we apply Theorem 2.2, and therefore one concludes that all fixed points of the map (2.3) are confined to the interval $\left]-\infty, \frac{1+|a|}{1-b}\right]$ if

$b > 1$ and to $\left[\frac{1+|a|}{1-b}, +\infty\right[$ if $b < 1$. On the other hand, case (iii) gives a simple linear second-order difference equation $x_n = 1 + bx_{n-2}$, for which the situation is standard. Since the location of the fixed points for map (2.3) cannot be calculated analytically, their stability will be studied numerically. ■

2.2 Existence of bounded orbits

In this section, we determine sufficient conditions for the map (2.3) to have bounded solutions. This is the interesting case since it includes the periodic, quasi-periodic, and chaotic orbits.

Theorem 2.5 *The orbits of the map* (2.3) *are bounded for all* $a \in \mathbb{R}$*, and* $|b| < 1$*, and all initial conditions* $(x_0, x_1) \in \mathbb{R}^2$*. See* [20] .

Proof. From equation (2.3) and the fact that $\sin x$ is a bounded function for all $x \in \mathbb{R}$, one has the followings inequalities for all $n > 1$:

$$|x_n| \leq 1 + |a| + |bx_{n-2}| \tag{2.9}$$

$$|x_{n-2}| \leq 1 + |a| + |bx_{n-4}| \tag{2.10}$$

$$|x_{n-4}| \leq 1 + |a| + |bx_{n-6}| \tag{2.11}$$

This implies from (2.9), (2.10), (2.11),... that

$$|x_n| \leq 1 + |a| + |bx_{n-2}| \tag{2.12}$$

$$|x_n| \leq \ (1 + |a|) + |b| (1 + |a| + |bx_{n-4}|) \tag{2.13}$$

$$|x_n| \leq \ (1 + |a|) + (1 + |a|) |b| + |b|^2 |x_{n-4}| , \ldots \tag{2.14}$$

Hence, from (2.10) and (2.14) one has:

$$|x_n| \leq (1 + |a|) + (1 + |a|) |b| + |b|^2 \ (1 + |a|) + |b|^3 |x_{n-6}| , \ldots \tag{2.15}$$

Since $|b| < 1$, then the use of (2.15) and induction about some integer k using the sum of a geometric growth formula permits us to obtain the following inequalities for every $n > 1$, $k \geq 0$

$$|x_n| \leq (1 + |a|) \left(\frac{1 - |b|^k}{1 - |b|}\right) + |b|^k |x_{n-2k}| \tag{2.16}$$

where k is the biggest integer j such that $j \leq \frac{n}{2}$. Thus one has the following two cases:

(1) if n is odd, i.e., $\exists m \in N$, such that $n = 2m+1$, then the biggest integer $k \leq \frac{n}{2}$ is $k = \frac{n-1}{2}$, for which $(x_n)_n$satisfies the following inequalities:

$$|x_{2m+1}| \leq (1+|a|)\left(\frac{1-|b|^m}{1-|b|}\right) + |b|^m |x_1| \ = z_m \tag{2.17}$$

(2) if n is even, i.e., $\exists m \in N$, such that $n = 2m$, then, the biggest integer $k \leq \frac{n}{2}$ is $k = \frac{n}{2}$, for which (x_n) satisfies the following inequalities:

$$|x_{2m}| \leq (1+|a|)\left(\frac{1-|b|^m}{1-|b|}\right) + |b|^m |x_0| \ = u_m \tag{2.18}$$

Thus, since $|b| < 1$, the sequences $(z_m)_m$ and $(u_m)_m$ are bounded, and one has:

$$\begin{cases} z_m \leq \frac{(1+|a|)}{1-|b|} + \left| |x_1| - \frac{(1+|a|)}{1-|b|} \right|, \text{ for all } m \in \mathbb{N} \\ \\ u_m \leq \frac{(1+|a|)}{1-|b|} + \left| |x_0| - \frac{(1+|a|)}{1-|b|} \right|, \text{ for all } m \in \mathbb{N} \end{cases} \tag{2.19}$$

Thus Formulas (2.17), (2.18), and inequalities (2.19) give the following bounds for the sequence $(x_n)_n$:

$$|x_n| \leq \max\left(\frac{(1+|a|)}{1-|b|} + \left| |x_0| - \frac{(1+|a|\,\delta)}{1-|b|} \right|, \frac{(1+|a|)}{1-|b|} + \left| |x_1| - \frac{(1+|a|)}{1-|b|} \right| \right) \tag{2.20}$$

Finally, for all values of a and all values of b satisfying $|b| < 1$ and all initial conditions $(x_0, x_1) \in \mathbb{R}^2$, one concludes that all orbits of the map (2.3) are bounded, i.e., in the subregion of $\mathbb{R}^4$:

$$\Omega_1 = \left\{(a, b, x_0, x_1) \in \mathbb{R}^4 : |b| < 1\right\} \tag{2.21}$$

■

2.3 Existence of unbounded orbits

In this section, we determine sufficient conditions for which the orbits of the map (2.3) are unbounded. First we prove the following theorem:

Theorem 2.6 *The map* (2.3) *possesses unbounded orbits in the following subregions of* $\mathbb{R}^4$*:*

$$\Omega_2 = \left\{(a, b, x_0, x_1) \in \mathbb{R}^4 : |b| > 1 \textit{ and both } |x_0|, |x_1| > \frac{|a|+1}{|b|-1}\right\} \tag{2.22}$$

and

$$\Omega_3 = \left\{(a, b, x_0, x_1) \in \mathbb{R}^4 : |b| = 1 \textit{ and } |a| < 1\right\} \tag{2.23}$$

See $[12]$.

Proof. (a) For every $n>1$, we have: $x_n = 1 - a\sin x_{n-1} + bx_{n-2}$ then $|bx_{n-2} - a\sin x_{n-1}| = |x_n - 1|$ and $||bx_{n-2}| - |a\sin x_{n-1}|| \leq |x_n - 1|$. (We use the inequalities: $|x| - |y| \leq ||x| - |y|| \leq |x - y|$). This implies that:

$$|bx_{n-2}| - |a\sin x_{n-1}| \leq |x_n| + 1 \tag{2.24}$$

Since $|\sin x_{n-1}| \leq 1$, this implies $-|a\sin x_{n-1}| \geq -|a|$, and $|bx_{n-2}| - |a\sin x_{n-1}| \geq |bx_{n-2}| - |a|$. Finally, one has from (2.24) that:

$$|bx_{n-2}| - (|a|+1) \leq |x_n|$$

Then, by induction as in the previous section, one has:

$$|x_n| \geq \begin{cases} \left(\frac{-(|a|+1)}{|b|-1} + |x_1|\right)|b|^{\frac{n-1}{2}} + \frac{|a|+1}{|b|-1}, \text{ if } n \text{ is odd} \\ \\ \left(\frac{-(|a|+1)}{|b|-1} + |x_0|\right)|b|^{\frac{n}{2}} + \frac{|a|+1}{|b|-1}, \text{if } n \text{ is even} \end{cases}$$

Thus, if $|b|>1$, and both $|x_0|, |x_1| > \frac{|a|+1}{|b|-1}$, one has $\lim_{n\to+\infty}|x_n| = +\infty$.
(b) for $b=1$, one has:

$$|x_n| \geq \begin{cases} (1-|a|)(\frac{n-1}{2}) + x_1, \text{ if } n \text{ is odd} \\ \\ (1-|a|)(\frac{n}{2}) + x_0, \text{ if } n \text{ is even} \end{cases}$$

Hence, if $|a|<1$, then one has $\lim_{n\to+\infty}|x_n| = +\infty$.
For $b=-1$, one has from Theorem 2.3 the inequalities:

$$|x_n| \leq \begin{cases} -(\frac{n-1}{2}) + x_1 + \left|\sum_{p=1}^{p=\frac{n-1}{2}} a(-1)^{p-1}\sin x_{n-(2p-1)}\right|, \text{ if } n \text{ is odd} \\ \\ -(\frac{n}{2}) + x_0 + \left|\sum_{p=1}^{p=\frac{n}{2}} a(-1)^{p-1}\sin x_{n-(2p-1)}\right|, \text{ if } n \text{ is even} \end{cases}$$

Because $\left|a(-1)^{p-1}\sin x_{n-(2p-1)}\right| \leq |a|$, then one has:

$$|x_n| \leq \begin{cases} (1-|a|)(\frac{n-1}{2}) + x_1, \text{ if } n \text{ is odd} \\ \\ (1-|a|)(\frac{n}{2}) + x_0, \text{ if } n \text{ is even} \end{cases}$$

Thus, if $|a|<1$, then one has $\lim_{n\to+\infty}|x_n| = -\infty$. Note that there is no similar proof for the following subregions of $\mathbb{R}^4$:

$$\Omega_4 = \left\{(a,b,x_0,x_1) \in \mathbb{R}^4 : |b|>1, \text{ and both } |x_0|, |x_1| \leq \frac{|a|+1}{|b|-1}\right\}$$

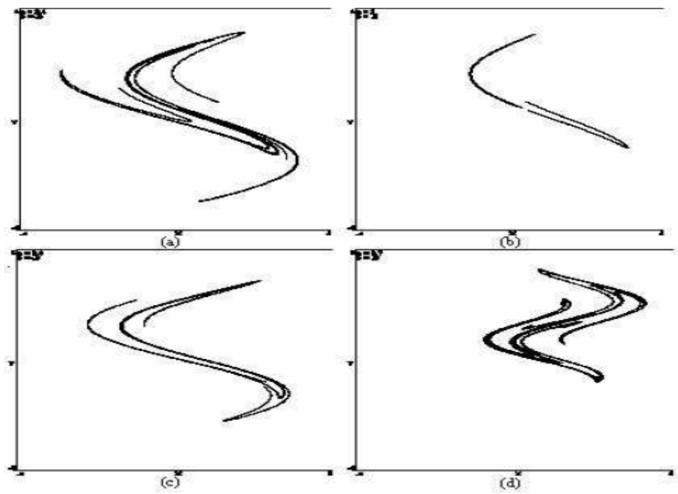

Figure 2.1: Chaotic multifold attractors of the map (2.3) obtained for (a) $a = 2.4$,$b = -0.5$. (b) $a = 2$, $b = 0.2$. (c) $a = 2.8$, $b = 0.3$. (d) $a = 2.7$, $b = 0.6$.

$$\Omega_5 = \left\{(a, b, x_0, x_1) \in \mathbb{R}^4 : |b| = 1, \text{ and } |a| \geq 1\right\}.$$

■

2.4 Some observed multifold attractors

In this section, we present some observed mutifold chaotic attractors obtained by an appropriate choice of the parameters a and b.

2.5 Route to Chaos

It is well known that the Hénon map typically undergoes a **period-doubling route to chaos** as the parameters are varied. By contrast, the Lozi map [17] has no period-doubling route, but rather it goes directly from a **border-collision bifurcation** developed from a stable periodic orbit. Similarly, the chaotic attractor given in [12] is obtained from a **border-collision period-doubling bifurcation scenario**. The **Zeraoulia-Sprott map** (2.3) is obtained from a **quasi-periodic route to chaos**. Thus, the four chaotic systems go via different and distinguishable routes to chaos. Furthermore, the multifold chaotic attractors presented in Fig. 2.1 are obtained from the map (2.3) via a **period-doubling bifurcation route to chaos** as shown in Fig. 2.5(a).

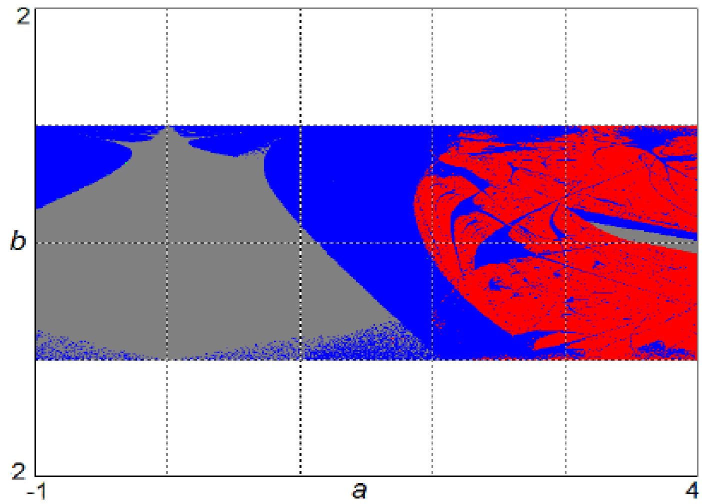

Figure 2.2: Regions of dynamical behaviors in ab-space for the map (2.3).

2.6 Dynamical behaviors with parameter variation

In this section, the dynamical behaviors of the map (2.3) are investigated numerically.

1. Figure 2.2 shows regions of unbounded (white), fixed point (gray), periodic (blue), and chaotic (red) solutions in the ab-plane for the map (2.3).

2. If we fix parameter $b = 0.3$ and vary $-1 \leq a \leq 4$, the map (2.3) exhibits the dynamical behaviors as shown in Fig. 2.5.

3. In the interval $-1 \leq a \leq 0.76$, the map (2.3) converges to a fixed point.

4. For $0.76 < a \leq 1.86$, there is a series of period-doubling bifurcations as shown in Fig. 2.5 (a).

5. In the interval $1.86 < a \leq 2.16$, the orbit converges to a chaotic attractor.

6. For $2.16 < a \leq 2.27$, it converges to a fixed point.

7. For $2.27 < a \leq 2.39$, there are periodic windows.

8. For $2.39 < a \leq 2.92$, it converges to a chaotic attractor.

9. For $a > 2.92$, the map (2.3) is chaotic. For example, the Lyapunov exponents for $a = 3$ and $b = 0.3$ are $\lambda_1 = 0.56186$ and $\lambda_2 = -1.76583$, giving a Kaplan-Yorke dimension of $D_{KY} = 1.31818$. There are also fixed points and periodic orbits.

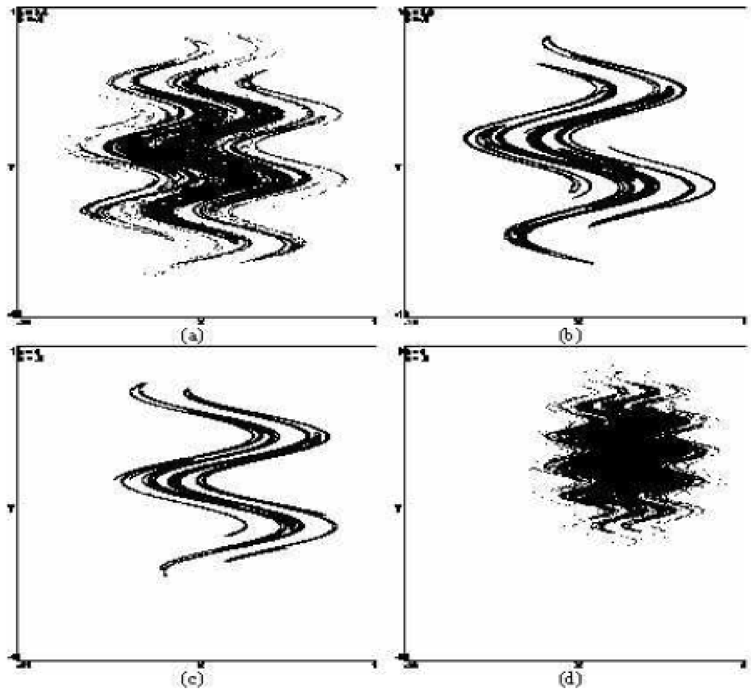

Figure 2.3: Chaotic multifold attractors of the map (2.3) obtained for (a) $a = 3.4$, $b = -0.8$. (b) $a = 3.6$, $b = -0.8$. (c) $a = 4$, $b = 0.5$. (d) $a = 4$, $b = 0.9$.

10. This map is invertible for all $b = 0$, especially for $|b| < 1$,and there is no hyperchaos since the sum of the Lyapunov exponents $\lambda_1 + \lambda_2 = \ln |b|$ is never positive.

11. Generally, if we fix $b = 0.3$ and $-150 \leq a \leq 200$, map (2.3) is chaotic over all the range as shown in Fig. 2.6, except for the small intervals mentioned above and shown in Fig. 2.5.

12. If we fix parameter $a = 3$ and vary $b \in \mathbb{R}$, the map (2.3) exhibits very complicated dynamical behaviors as shown in Fig. 2.7, which shows some fixed points and some periodic windows.

13. Finally, for $|b| > 1$, the map (2.3) does not converge as shown in the previous section analytically.

14. There are regions of ab-space where **two coexisting attractors occur** as shown in black in Fig. 2.8, both in the regular and chaotic regimes. For example, with $a = 2$ and $b = -0.6$, a two-cycle $(1.314326, -0.584114)$ coexists with a period-3 strange attractor. Similarly, for $a = 2.2$ and $b = -0.36$, there is a strange attractor surrounded by a second period-3 strange attractor as shown in black in Fig. 2.9 with their corresponding basins of attraction shown in yellow and magenta, respectively.

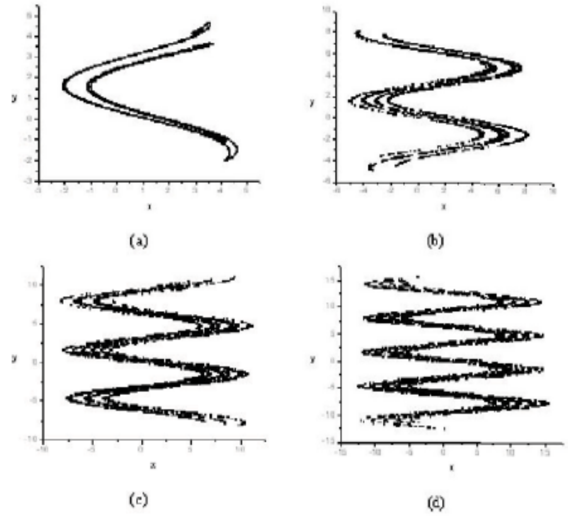

Figure 2.4: Multifold chaotic attractors of the map (2.3) obtained for $b = 0.3$ and (a) $a = 3$. (b) $a = 5$. (c) $a = 7$. (d) $a = 10$.

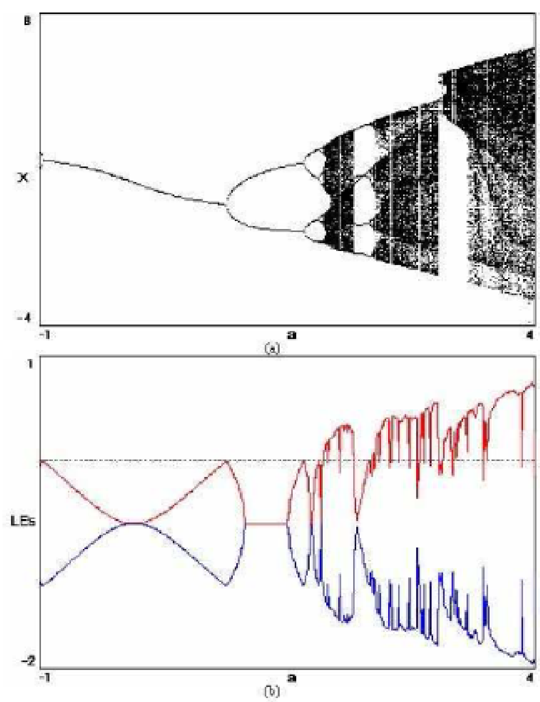

Figure 2.5: (a) Bifurcation diagram for the map (2.3) obtained for $b = 0.3$ and $-1 \leq a \leq 4$. (b) Variation of the Lyapunov exponents of map (2.3) over the same range of a.

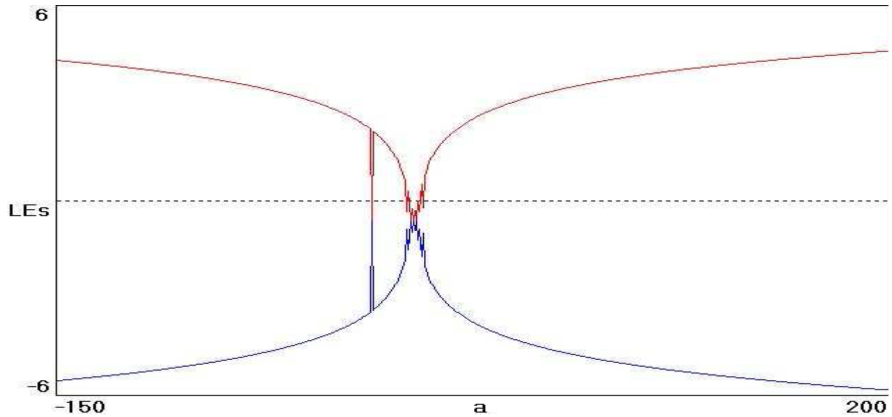

Figure 2.6: Variation of the Lyapunov exponents of map (2.3) over the range $-150 \leq a \leq 200$ with $b = 0.3$.

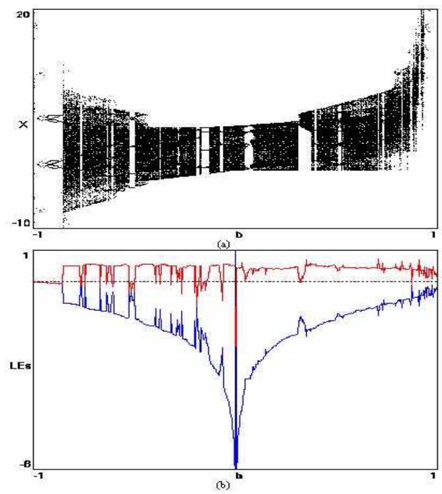

Figure 2.7: (a) Bifurcation diagram for the map (2.3) obtained for $a = 3$ and $-1 \leq b \leq 1$. (b) Variation of the Lyapunov exponents of map (2.3) for the same range of b.

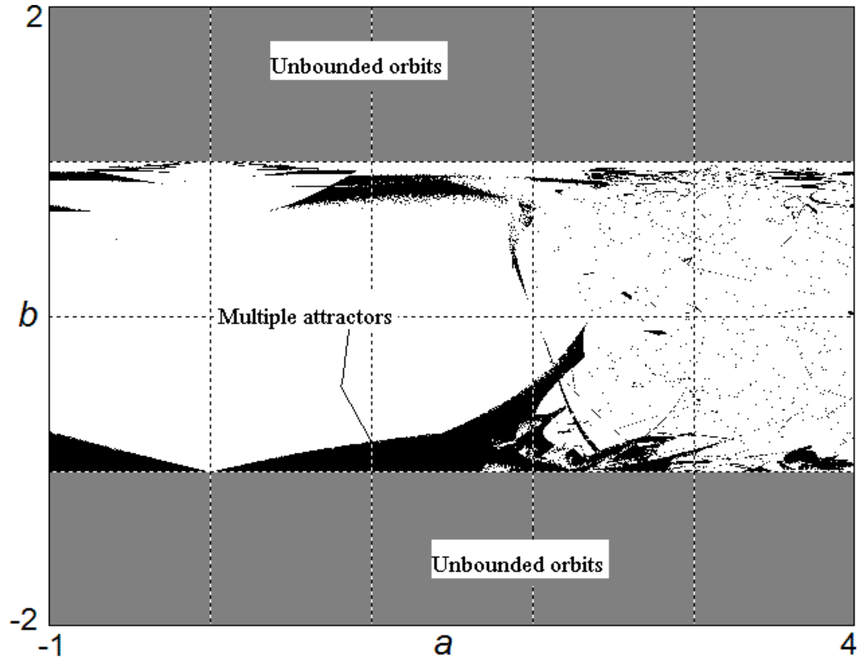

Figure 2.8: The regions of ab-space where multiple attractors are found (shown in black).

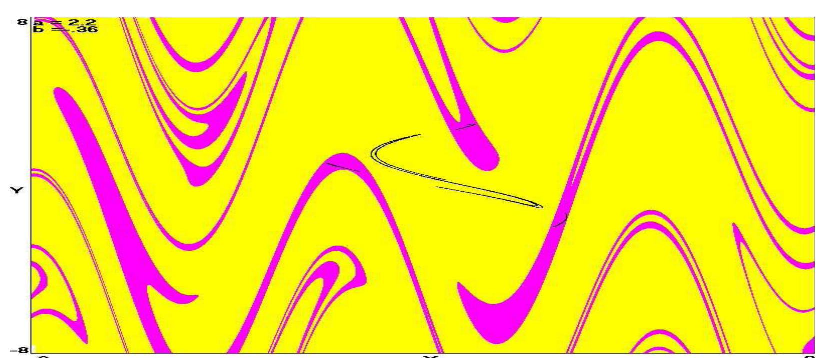

Figure 2.9: Two coexisting attractors occur for $a = 2.2$ and $b = -0.36$, where a strange attractor is surrounded by a second period-3 strange attractor with their corresponding basins of attraction shown in yellow and magenta, respectively.

Chapter 3

Periodic 2-orbits of the Zeraoulia-Sprott mapping

In this chaptre, we present general study on the cycles, especially cycles of order 2 and their stability.

3.1 The p-cycles

A cycles of order p (or periodic orbit of order p or a p-cycle) is a set of p points$\left(x_0^*, x_1^*, ..., x_{p-1}^*\right)$ verifying:

$$\begin{aligned}
\begin{pmatrix} x_{i+1}^* \\ y_{i+1}^* \end{pmatrix} &= f\begin{pmatrix} x_i^* \\ y_i^* \end{pmatrix}, \ i = 0...p-2 \\
\begin{pmatrix} x_P^* \\ y_p^* \end{pmatrix} &= f\begin{pmatrix} x_{P-1}^* \\ y_{p-1}^* \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\
\begin{pmatrix} x_i^* \\ y_i^* \end{pmatrix} &= f^P\begin{pmatrix} x_i^* \\ y_i^* \end{pmatrix}, \ i = 0...p-1 \\
\begin{pmatrix} x_i^* \\ y_i^* \end{pmatrix} &\neq f^h\begin{pmatrix} x_i^* \\ y_i^* \end{pmatrix}, \ i = 0...p-1 \quad i \leq h \leq p
\end{aligned}$$

p is the minimal integer that $f^p\begin{pmatrix} x_0^* \\ y_0^* \end{pmatrix} = \begin{pmatrix} x_0^* \\ y_0^* \end{pmatrix}$. An orbit of ordre 2 is defined by

$$\forall n \geq 1 : x_{n+2} = x_n \text{ and } y_{n+2} = y_n$$

$$(f \circ f)(x, y) = \begin{cases} f^2(x, y) = (x, y) \to A \\ f(x, y) = (x, y) \to B, \ \text{ the fixed points} \end{cases}$$

The periodic solution is: $\{A/B\}$. We have the following results: (x, y) is a fixed point of $f \Rightarrow (x, y)$ is a fixed point of $f^2 \Rightarrow (x, y)$ is a fixed point of f^n, $\forall n \geq 1$.

Theorem 3.1 *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $f(x^*) = x^*$, we calculate the jacobian matrix $Df(x^*)$. If the two eigenvalues λ_1, λ_2 are real:*
$\forall i = 1, 2 : |\lambda_i| < 1$, x^* is an attractive saddle.
$\forall i = 1, 2 : |\lambda_i| > 1$, x^* is an repulsive saddle.
If the two eigenvalues λ_1, λ_2 are complex:
$\forall i = 1, 2 : |\lambda_i| < 1$, x^* is an attractive facus.
$\forall i = 1, 2 : |\lambda_i| > 1$, x^* is an repulsive facus.
If the two eigenvalues λ_1, λ_2 are real and $|\lambda_1| < 1$, $|\lambda_2| > 1$, then x^* is a Node point.

3.2 Cardan method

This method [13] makes it possible to obtain formulas, called **Cardan formulas**, giving as a function of p and q for the solutions of the equation:

$$z^3 + pz + q = 0$$

It allows to prove that the equations of degree 3 are solvable by radicals. Only the equations of degree 1, 2, 3, 4 are solvable by radicals in all cases, that is to say that only these equations have general methods of resolution giving the solutions according to the coefficients of the polynomial using only the four usual operations on rational numbers, and the extraction of n^{-th} roots.

Theorem 3.2 *The complex solutions z_k $(0 \leq k \leq 2)$ of the third degree equation $z^3 + pz + q = 0$, where the coefficients p and q are real, are given by:*

$$z_k = u_k + v_k$$

with

$$u_k = j^k \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{\frac{-\Delta}{27}}\right)}$$

and $3u_k v_k = -p$ *from where*

$$v_k = j^{-k} \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{\frac{-\Delta}{27}}\right)}$$

where $\Delta = -\left(4p^3 + 27q^2\right)$ is the discriminant of the equation and where $j = e^{\frac{2i\pi}{3}}$.
- $\Delta > 0$, *then there are three distinct real solutions.*

- $\Delta = 0$, *then a solution is multiple and all are real.*
- $\Delta < 0$, *then one solution is real and the other two are complex conjugates.*

Remark 3.1 *By asking* $p = 3p', q = 2q'$*and* $\Delta = 4 \times 27\Delta'$ *we obtain:*

$$\Delta' = -\left(q'^2 + p'^2\right), u_n = j^k \sqrt[3]{-q' + \sqrt{-\Delta'}}, v_n = j^{-k} \sqrt[3]{-q' - \sqrt{-\Delta'}}$$

If we start from the general equation $ax^3 + bx^2 + cx + d = 0$, $a \neq 0$ *we come back to the reduced form by puting:*

$$x = z - \frac{b}{3a}, p = \frac{-b^2}{3a^3} + \frac{c}{a} \text{ and } q = \frac{b}{27a}\left(\frac{2b^2}{a^2} - \frac{9c}{a}\right) + \frac{d}{a}$$

• *if* Δ *is negative: The equation then has a real solution and two complexes roots. We pose:*

$$u = \sqrt[3]{\frac{-q + \frac{\sqrt{-\Delta}}{27}}{2}}, \; v = \sqrt[3]{\frac{-q - \frac{\sqrt{-\Delta}}{27}}{2}}, \; u = \sqrt[3]{-q' + \sqrt{-\Delta'}} \text{ and } v = \sqrt[3]{-q' + \sqrt{-\Delta'}}$$

The only real solution is then $z_0 = u + v$, *there are also two complex solutions combined with each other:*

$$\begin{cases} z_1 = ju + \bar{j}v \\ \\ z_2 = j^2 u + \overline{j^2} v \end{cases} \text{where } j = \frac{-1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \text{and } j^2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2} = e^{i\frac{4\pi}{3}}$$

• *if* Δ *is null: if* $p = q = 0$, *the equation has* 0 *as a triple solution. Otherwise,* p *and* q *are non-nulls.*

The equation then has two real solutions, a single and a double:

$$\begin{cases} z_0 = 2\sqrt[3]{\frac{-q}{2}} = \frac{3q}{p} \\ \\ z_1 = z_2 = -\sqrt[3]{\frac{-q}{2}} = \frac{-3q}{2p} \end{cases}$$

• *if* Δ *is positive: The equation then has three real solutions. Solutions are the sums of two complexes conjugués* $j^k u$ *and* $\overline{j^k u}$ *where* $u = \sqrt[3]{\frac{-q + \frac{\sqrt{-\Delta}}{27}}{2}}$ *and* $k \in \{0, 1, 2\}$ *either the following set:*

$$\begin{cases} z_0 = u + \overline{u} \\ z_0 = ju + \overline{ju} \\ z_0 = j^2 u + \overline{j^2 u} \end{cases}$$

The real form of the solutions is obtained by writing $j^k u$ *in the trigonometric form, which gives:*

$$z_k = 2\sqrt{\frac{-p}{3}} \cos\left(\frac{1}{3} \arccos\left(\frac{3q}{2p}\sqrt{\frac{2}{-p}}\right) + \frac{2k\pi}{3}\right)$$

with $k \in \{0, 1, 2\}$.

3.2.1 Principle of the method

Consider the following general equation of the third degree: $ax^3 + bx^2 + cx + d = 0$. By puting $x = z - \frac{b}{3a}$, we come back to an equation of the form:

$$z^3 + pz + q = 0$$

We are now going to put $z = u + v$ with u and v complex, so as to have two unknowns instead of one and thus give oneself the possibility of subsequently setting a condition on u and v allowing the problem to be simplified. The equation $z^3 + pz + q = 0$ becomes so:

$$(u + v)^3 + p(u + v) + q = 0$$

this equation turns into the following form:

$$\begin{aligned} u^3 + v^3 + 3uv^2 + 3vu^2 + p(u + v) + q &= 0 \\ u^3 + v^3 + (3uv + p)(u + v) + q &= 0 \end{aligned}$$

The announced simplification condition will then be $3uv + p$ what gives us on the one hand $u^3 + v^3 + q = 0$ and on the other hand $uv = \frac{-p}{3}$ which, by raising the two limbs to the power of 3, gives $u^3v^3 = -\frac{p^3}{27}$, we finally obtain the sum-product system of the two following unknowns u^3 and v^3:

$$\begin{cases} u^3 + v^3 = -q \\ \\ u^3v^3 = -\frac{p^3}{27} \end{cases}$$

The unknowns u^3 and v^3 being two complexes of which we know the sum and the product, they are therefore the solutions of the quadratic equation:

$$X^2 + qX - \frac{p^3}{27} = 0$$

The discriminant of this quadratic equation is $\delta = q^2 - 4 \times 1 \times \frac{-p^3}{27} = q^2 + \frac{4}{27}p^3$ and the roots are:

$$\begin{cases} u^3 = \frac{-q+\sqrt{\delta}}{2} \text{ and } v^3 = \frac{-q-\sqrt{\delta}}{2}, \text{ if } \delta \text{ is positive} \\ \\ u^3 = \frac{-q+i\sqrt{\delta}}{2} \text{ and } v^3 = \frac{-q-i\sqrt{\delta}}{2}, \text{ if } \delta \text{ is negative} \\ \\ u^3 = v^3 = \frac{-q}{2}, \text{ if } \delta \text{ is null} \end{cases}$$

Note that the discriminant Δ of the third degree equation $z^3 + pz + q = 0$ is related to the discriminant δ above by the relation $\Delta = -27\delta$.

3.3 Quartic equation

Elimination of degree 3 term: The equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

is reduced, after division by a and change of variable $x = y - \frac{b}{4a}$ to an equation of the form:

$$y^4 + py^2 + qy + r = 0$$

with

$$\begin{cases} p = \frac{c}{a} - \frac{3b^2}{8a^2} \\ q = \frac{d}{a} - \frac{bc}{2a^2} + \frac{b^3}{8a^3} \\ r = \frac{e}{a} - \frac{bd}{4a^2} + \frac{cb^2}{16a^3} - \frac{3b^4}{256a^2} \end{cases}$$

One can then solve the equation by the method of Ferrari, that of Descartes, or that below of Lagrange. All three provide, under different appearances, the same formula for the four solutions.

3.3.1 Lagrange method

It is a question of finding an expression involving the 4 roots y_1, y_2, y_3, y_4 of: $y^4 + py^2 + qy + r = 0$, and allowing to obtain, by permutations, only 3 distinct values. This is the case for example of: $-(y_1 + y_2)(y_3 + y_4)$ which, by permutations, only gives the values:

$$\begin{cases} z_1 = -(y_1 + y_2)(y_3 + y_4) \\ z_2 = -(y_1 + y_3)(y_2 + y_4) \\ z_3 = -(y_1 + y_4)(y_2 + y_3) \end{cases}$$

Any symmetric polynomial in z_1, z_2, z_3 can be expressed as a symmetric polynomial of y_1, y_2, y_3, y_4. The coefficients of the polynomial $R(z) = (z - z_1)(z - z_2)(z - z_3)$ can be expressed as a function of p, q et r. It is certain that the property $y_1 + y_2 + y_3 + y_4 = 0$ facilitates calculations. We show in fact that then:

$$\begin{cases} z_1 + z_2 + z_3 = -2p \\ \sum_{i<j} z_i z_j = p^2 - 4r \\ z_1 z_2 z_3 = q^2 \end{cases}$$

The three reals z_1, z_2, z_3 are then solutions of the equation:

$$z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0$$

It remains now to find y_1, y_2, y_3, y_4 in terms of z_1, z_2, z_3 knowing that $y_1 + y_2 + y_3 + y_4 = 0$. We then notice that:

$$\begin{cases} z_1 = (y_1 + y_2)^2 = (y_3 + y_4)^2 \\ z_2 = (y_1 + y_3)^2 = (y_2 + y_4)^2 \\ z_3 = (y_1 + y_4)^2 = (y_2 + y_3)^2 \end{cases}$$

so that:

$$\begin{cases} y_1 + y_2 = \sqrt{z_1} \text{ et } y_3 + y_4 = -\sqrt{z_1} \\ y_1 + y_3 = \sqrt{z_2} \text{ et } y_2 + y_4 = -\sqrt{z_2} \\ y_1 + y_4 = \sqrt{z_3} \text{ et } y_2 + y_3 = -\sqrt{z_3} \end{cases}$$

(the notation$\sqrt{z_i}$ as one of the square roots of z_i). The values of y_i are then found by simple addition.

Results: The solutions of $y^4 + py^2 + qy + r = 0$ are:

$$\begin{cases} y_1 = \frac{1}{2}(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}) \\ y_2 = \frac{1}{2}(\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}) \\ y_3 = \frac{1}{2}(-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}) \\ y_4 = \frac{1}{2}(-\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}) \end{cases}$$

Case inventory: In the case where the coefficients p, q and r are real, we notice that the product of the roots of the polynomial R is q^2, we are therefore limited to the shape of the roots of the polynomial R and on the solutions of the quartic equation.

• If the three roots of R are real positive, we get four real values.

• If the three roots of R are real and two are negative, we get two pairs of conjugate complexes.

• If R has a real root and two conjugate complex roots, the real root is positive and we get two real values and two conjugate complexes.

3.3.2 Special equations

Among the equations of degree four some particular, can be solved only using the quadratic equations, this is the case for bicarled equations and symmetric equations or, more generally equations $ax^4 + bx^3 + cx^2 + dx + e = 0$ as $ad^2 = eb^2$

Quadruple equations: They are written in the form:

$$ax^4 + bx^2 + c = 0$$

and are resolved by changing the variable $y = x^2$ and the resolution of $ay^2+by+c = 0$. Quadruple equations, as well as some other equations of degree 4, can also be solved by circular or hyperbolic trigonometry.

Symmetric equations: They are written in the form:

$$ax^4 + bx^3 + cx^2 + dx + a = 0$$

and are resolved by changing the variable $z = x + \frac{1}{x}$ and the resolution of $az^2 + bz + c - 2a = 0$. This process is generalized to equations of the form $ax^4 + bx^3 + cx^2 + kbx + k^2a = 0$ (with $k \neq 0$), which are resolved by posing $z = x + \frac{k}{x}$.

3.4 Some basic properties of the Zeraoulia-Sprott mapping

In the following new 1-D discrete iterative system with a rational fraction was discovered in a study of evolutionary algorithms:

$$g(x) = \frac{1}{0.1 + x^2} - ax \tag{3.1}$$

where a is a parameter [15]. The map (3.1) describes different random evolutionary processes, and it is much more complicated than the logistic system. In [16] an extended version of the former one-dimensional discrete chaotic system given in [15] to two-dimensions is given as follows:

$$h(x, y) = \begin{pmatrix} \frac{1}{0.1+x^2} - ay \\ \frac{1}{0.1+y^2} + bx \end{pmatrix} \tag{3.2}$$

where a and b are parameters. The map (3.2) has more complicated dynamical behavior than the one-dimensional map (3.1). Based on these studies in [15, 16] a new and very simple 2-D map, characterized by the existence of only one rational fraction **with no vanishing denominator** is constructed and it is given by:

$$f(x, y) = \begin{pmatrix} \frac{-ax}{1+y^2} \\ x + by \end{pmatrix} \tag{3.3}$$

where a and b are bifurcation parameters.

1. The new map (3.3) is algebraically simpler but with more complicated behavior than map (3.2).
2. It produces several new chaotic attractors obtained via the quasi-periodic route to chaos.
3. The map (3.3) is defined for all points in the plane.
4. The associated function $f(x, y)$ of the map (3.3) is of class $C^\infty(\mathbb{R}^2)$, and it has no vanishing denominator.

5. The new chaotic map (3.3) is symmetric under the coordinate transformation $(x, y) \longrightarrow (-x, -y)$, and this transformation persists for all values of the map parameters.

6. The fixed points of map (3.3) are the real solutions of the equations $\frac{-ax}{1+y^2} = x$ and $x + by = y$. Hence, one may easily obtain the equations $(a + 1 + y^2)\, x = 0$ and $(1 - b)\, y = x$. Assume that $-1 \leq a \leq 4$. Then if $b \neq 1$, the only fixed point of the map (3.3) is $p = (0, 0)$, and if $b = 1$,then the y-axis is invariant by iteration of the map f.

7. The Jacobian matrix of map (3.3) evaluated at a point (x, y) is given by:

$$Df\,(x, y) = \begin{pmatrix} \frac{-a}{1+y^2} & \frac{2axy}{(1+y^2)^2} \\ 1 & b \end{pmatrix}$$

   and at the fixed point $p = (0, 0)$, the Jacobian matrix is given by $Df\,(0, 0) = \begin{pmatrix} -a & 0 \\ 1 & b \end{pmatrix}$.

8. The local stability of p is studied by evaluating the eigenvalues of the Jacobian $Df\,(p)$. The eigenvalues of $Df\,(p)$ are: $\lambda_1 = -a$ and $\lambda_2 = b$. Then one has the following results:

   (1) If $|a| < 1$ and $|b| < 1$, then p is asymptotically stable.

   (2) If $|a| > 1$ or $|b| > 1$, then p is an unstable fixed point.

   (3) If $|a| < 1$ and $|b| > 1$, or $|a| > 1$ and $|b| < 1$, then p is a saddle point.

   (4) If $|a| = 1$ or $|b| = 1$, then p is a non-hyperbolic fixed point.

3.5 Observation of chaotic attractors

In this section, we will illustrate some observed chaotic attractors, along with some other dynamical phenomena.

1. In (3.3) the chaotic attractors are obtained via a **period-doubling bifurcation route to chaos** as shown in Fig. 3.3(a). Possibly, the map (3.3) is the first simple rational map whose fraction has no vanishing denominator that gives chaotic attractors via a quasi-periodic route to chaos.

2. Figure 3.3 (b) shows regions of unbounded (white), fixed point (gray), periodic (blue), quasi-periodic (green), and chaotic (red) solutions in the abplane for the map (3.3).

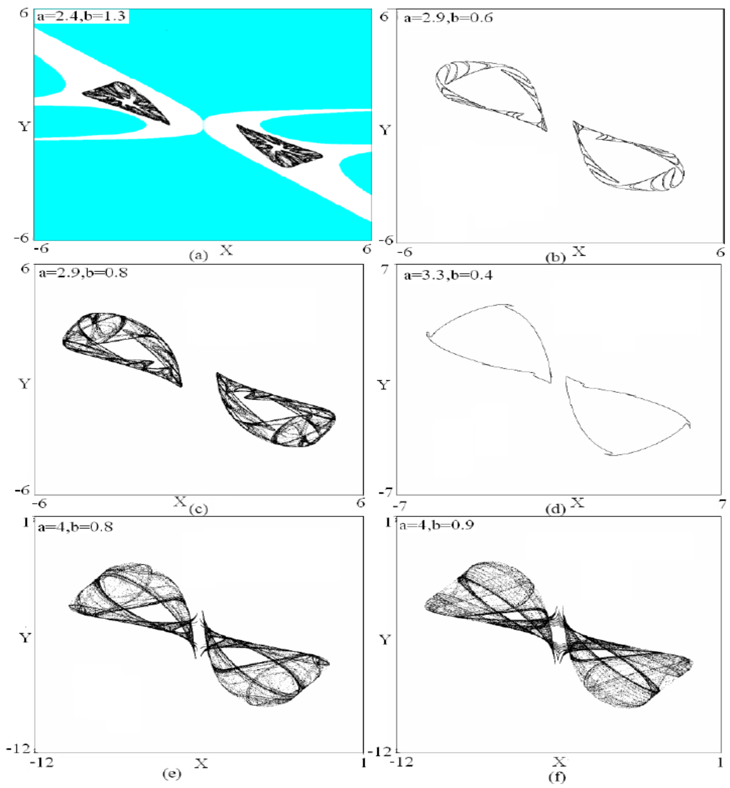

Figure 3.1: Attractors of the map (3.3) with (a) $a = 2.4$, $b = 1.3$, (b) $a = 2.9$,$b = 0.6$, (c) a= 2.9, $b = 0.8$, (d) $a = 3.3$, $b = 0.4$, (e) $a = 4$, $b = 0.8$, (f) $a = 4$, $b = 0.9$.

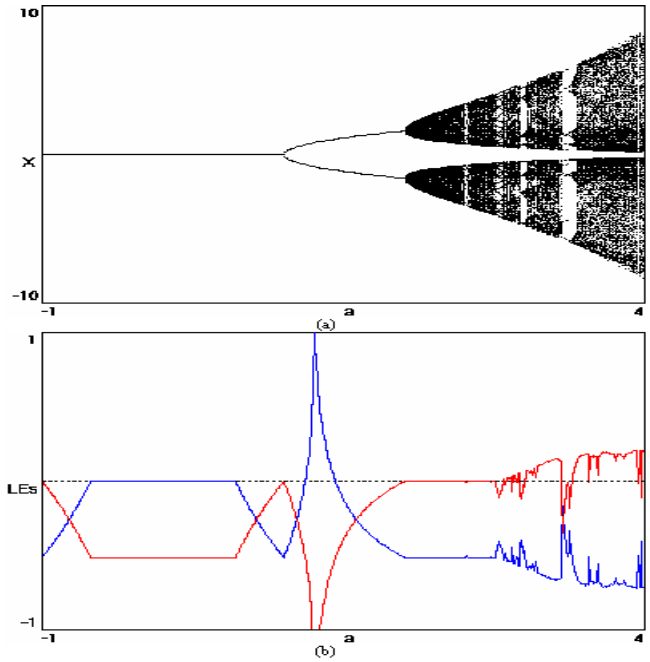

Figure 3.2: (a) The quasi-periodic route to chaos for the map (3.3) obtained for $b = 0.6$ and $-1 < a \leq 4$. (b) Variation of the Lyapunov exponents of map (3.3) versus the parameter $-1 < a \leq 4$ with $b = 0.6$.

3. If we fix parameter $b = 0.6$ and vary $-1 \leq a \leq 4$, the map (3.3) exhibits the following dynamical behaviors as shown in Fig. 3.2(a):

   In the interval $-1 \leq a \leq 1$, the map (3.3) converges to the fixed point $(0, 0)$.

   For $1 < a \leq 2$, it converges to a period-2 attractor followed by a quasi-periodic orbit for $2 < a \leq 3$ as shown in Fig. 3.4(a).

   In the interval $3 < a \leq 4$, it converges to a chaotic attractor shown in Fig. 3.4 (b) via a quasi-periodic route to chaos except for a number of periodic windows. See [17] .

3.6 Periodic 2-orbits of the Zeraoulia-Sprott mpaping

In order to calculate the cycle 2 for the map $f(x, y)$. First we must found the fixed points:

$$f(x,y) = \begin{pmatrix} \frac{-ax}{1+y^2} \\ x + by \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} \frac{-ax}{1+y^2} = x \\ x + by = y \end{cases} \Longrightarrow \begin{cases} (a + 1 + y^2)x = 0 \\ (1 - b)y = x \end{cases}$$

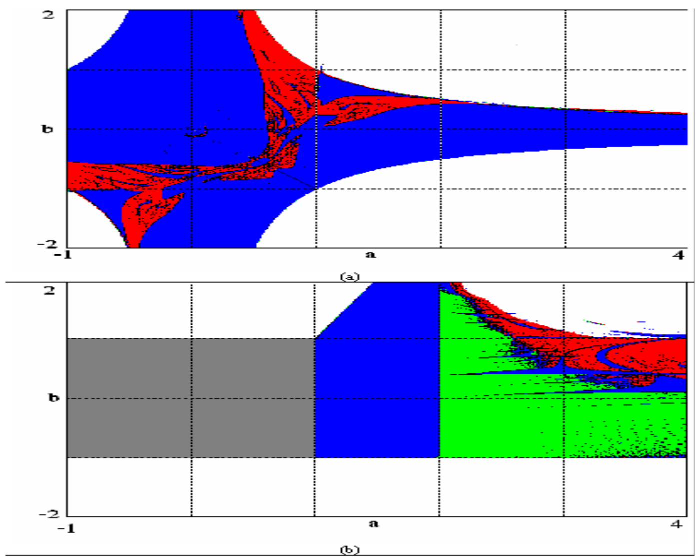

Figure 3.3: (a) Regions of dynamical behaviors in the ab-plane for the rational map (3.2). (b) Regions of dynamical behaviors in the ab-plane for the rational map (3.3).

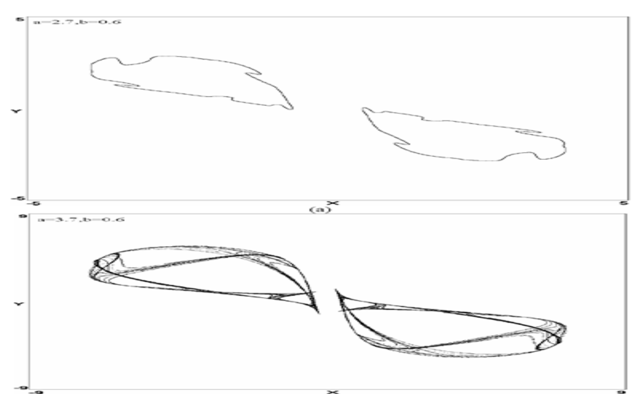

Figure 3.4: Attractors of the map (3.3) (a) Quasi-periodic orbit for $a = 2.7$,$b = 0.6$. (b) Chaotic orbit for $a = 3.7$, $b = 0.6$.

Assume that $-1 \leq a \leq 4$

$$\begin{cases} (a+1+y^2)=0 \text{ or } x=0 \\ \text{and} \\ (1-b)y=x \end{cases} \Longrightarrow \begin{cases} a+1+y^2=0 \\ \text{and} \\ (1-b)y=x \end{cases} \text{or} \begin{cases} x=0 \\ \text{and} \\ (1-b)y=x \end{cases}$$

For the case $a+1+y^2=0$ the solution is refused. For $x=0$ and $(1-b)y=x$, we have two options: If $b \neq 1$, $x=0$ and $(1-b)y=x \Longrightarrow (1-b)y=0 \Longrightarrow y=0$, then the only fixed point is $p(x,y)=p(0,0)$. If $b=1$, $x=0$ and $(1-b)y=x, (1-b)y=0 \Longrightarrow (1-b)=0$ or $y=0 \Longrightarrow y \neq 0$, then the y-axis is invariant by iteration of the map f.
We will calculate the cycle 2 for the map f:

$$f^2(x,y)=\begin{pmatrix} \frac{-a(\frac{-ax}{1+y^2})}{1+(x+by)^2} \\ (\frac{-ax}{1+y^2})+b(x+by) \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$$

By compensation we find:

$$\begin{cases} (x+by)^2=\frac{a^2-(1+y^2)}{1+y^2} \\ \\ x=\frac{(y-b^2y)(1+y^2)}{-a+b(1+y^2)} \end{cases}$$

Then we get:

$$y^8+(3+b^2-2ab)y^6+(3+3b^2-6ab)y^4+(1+a^2b^2+2a^3b+a^2+3b^2-6ab)y^2+(a^2+b^2+2a^3b-2ab-a^2b^2-a^4)=0$$

To find the roots of the above polynomial, we must change it into a fourth degree polynomial, then use **Quartic equation method**. First let's put:

$$\begin{aligned} k_1 &= 1 \\ k_2 &= 3+b^2-2ab \\ k_3 &= 3+3b^2-6ab \\ k_4 &= 1+a^2b^2+2a^3b+a^2+3b^2-6ab \\ k_5 &= a^2+b^2+2a^3b-2ab-a^2b^2-a^4 \end{aligned}$$

and $y^2=z$, we get:

$$k_1z^4+k_2z^3+k_3z^2+k_4z+k_5=0$$

Secondly, let's put $z=\alpha-\frac{k_2}{4k_1}$, then we find:

$$\alpha^4+p\alpha^2+q\alpha+r=0$$

Such that:

$$\begin{aligned}
p &= \frac{k_3}{k_1} - \frac{3k_2^2}{8k_1^2} = \frac{3+3b^2-6ab}{1} - \frac{3(3+b^2-2ab)^2}{8} \\
&= -\frac{3}{2}a^2b^2 + \frac{3}{2}ab^3 - \frac{3}{2}ab - \frac{3}{8}b^4 + \frac{3}{4}b^2 - \frac{3}{8}
\end{aligned}$$

$$\begin{aligned}
q &= \frac{k_4}{k_1} - \frac{k_2k_3}{2k_1^2} + \frac{k_2^3}{8k_1^3} \\
&= \frac{1-a^2b^2+2a^3b+a^2+3b^2-6ab}{1} - \frac{(3+b^2-2ab)*(3+3b^2-6ab)}{2} \\
&\quad + \frac{(3+3b^2-6ab)^3}{8} \\
&= -27a^3b^3 + 2a^3b + \frac{81}{2}a^2b^4 + \frac{67}{2}a^2b^2 + a^2 - \frac{81}{4}ab^5 - \frac{69}{2}ab^3 \\
&\quad - \frac{57}{4}ab + \frac{27}{8}b^6 + \frac{69}{8}b^4 + \frac{57}{8}b^2 - \frac{1}{8}
\end{aligned}$$

$$\begin{aligned}
r &= \frac{k_5}{k_1} - \frac{k_2k_4}{4k_1^2} + \frac{k_3k_2^2}{16k_1^2} - \frac{3k_2^4}{256k_1^4} \\
&= \frac{a^2+b^2+2a^3b-2ab-a^2b^2-a^4}{1} \\
&\quad - \frac{(3+b^2-2ab)*(1-a^2b^2+2a^3b+a^2+3b^2-6ab)}{4} \\
&\quad + \frac{(3+3b^2-6ab)*(3+b^2-2ab)^2}{16} - \frac{3(3+b^2-2ab)^4}{256} \\
&= -\frac{3}{16}a^4b^4 + a^4b^2 - a^4 + \frac{3}{8}a^3b^5 - \frac{11}{8}a^3b^3 + a^3b - \frac{9}{32}a^2b^6 \\
&\quad + \frac{13}{16}a^2b^4 - \frac{25}{32}a^2b^2 + \frac{1}{4}a^2 + \frac{3}{32}ab^7 - \frac{9}{32}ab^5 + \frac{9}{32}ab^3 \\
&\quad - \frac{3}{32}ab - \frac{3}{256}b^8 + \frac{3}{64}b^6 - \frac{9}{128}b^4 + \frac{3}{64}b^2 - \frac{3}{256}
\end{aligned}$$

The roots of $\alpha^4 + p\alpha^2 + q\alpha + r = 0$ are:

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(\sqrt{\beta_1} + \sqrt{\beta_2} + \sqrt{\beta_3}) \\
\alpha_2 &= \frac{1}{2}(\sqrt{\beta_1} - \sqrt{\beta_2} - \sqrt{\beta_3}) \\
\alpha_3 &= \frac{1}{2}(-\sqrt{\beta_1} + \sqrt{\beta_2} - \sqrt{\beta_3}) \\
\alpha_4 &= \frac{1}{2}(-\sqrt{\beta_1} - \sqrt{\beta_2} + \sqrt{\beta_3})
\end{aligned}$$

Since β_1, β_2, β_3 the roots of the polynomial R:

$$R(\beta) = \beta^3 + 2p\beta^2 + (p^2-4r)\beta - q^2$$

We solve R by using Cardon method, let's put $\beta = \gamma - \frac{2p}{3}$, then we get:

$$\gamma^3 + H_1\gamma + H_2 = 0$$

where

$$\begin{aligned} H_1 &= -\frac{4p^2}{3} + p^2 - 4r \\ H_2 &= \frac{2p}{27}(8p^2 - 9p^2 + 36r) - q^2 \end{aligned}$$

$$\Delta = -(4H_1^3 + 27H_2^2)$$

- If $\Delta < 0$, then one solution is real and the other two are complex conjugates:

$$\begin{aligned} \gamma_1 &= u + v \\ \gamma_2 &= ju + \overline{j}v \\ \gamma_3 &= j^2u + \overline{j^2} \end{aligned}$$

Such that: $j = e^{\frac{2i\pi}{3}}$, $u = \sqrt[3]{\frac{-H_2+\sqrt{\frac{-\Delta}{27}}}{2}}$, $v = \sqrt[3]{\frac{-H_2-\sqrt{\frac{-\Delta}{27}}}{2}}$.

- If $\Delta = 0$, then a solution is multiple and all are real:

$$\begin{aligned} \gamma_1 &= 2\sqrt[3]{\frac{-H_2}{2}} = \frac{3H_2}{H_1} \\ \gamma_2 &= \gamma_3 = -\sqrt[3]{\frac{-H_2}{2}} = \frac{-3H_2}{2H_1} \end{aligned}$$

- If $\Delta > 0$, then there are three distinct real solutions:

$$\begin{aligned} \gamma_1 &= u + \overline{u} \\ \gamma_2 &= ju + \overline{ju} \\ \gamma_3 &= j^2u + \overline{j^2u} \end{aligned}$$

Since $u = \sqrt[3]{\frac{-H_2+i\sqrt{\frac{-\Delta}{27}}}{2}}$. **The rest of the calculation must be numerical.**

3.7 Bounded and unbounded orbits of the Zeraoulia-Sprott mapping

In [23], the following problem was formulated: *Find regions in the a-b plane in which the map* (3.3) *is bounded and chaotic in the rigorous mathematical definition of chaos and boundedness of* *attractors*. The boundedness of the attractors of the map (3.3) is studied and the corresponding analytical estimation of absorbing set is obtained, thereby giving an answer to the above problem. Introduce the notation

$$h\left(y_k\right) = \frac{-a}{1+y_k^2},\ k \in \mathbb{N}_0$$

The eigenvalues and eigenvectors of the map are defined by the following relation:

$$\binom{x_{k+1}}{y_{k+1}} = A\binom{x_k}{y_k} + \left(\begin{array}{c} \frac{ax_k y_k^2}{1+y_k^2} \\ 0 \end{array}\right), A = \left(\begin{array}{cc} -a & 0 \\ 1 & b \end{array}\right)$$

$$\begin{aligned} \det\left(A - PI\right) &= \det\left(\begin{array}{cc} -a-p & 0 \\ 1 & b-p \end{array}\right) \\ &= \left(-a-p\right)\left(b-p\right) \\ &= p^2 + p\left(a-b\right) - ab \end{aligned}$$

Thus, the eigenvalues are $p = b$, $p = -a$. For the eigenvalue $p = -a$, one has

$$\left(\begin{array}{cc} -a & 0 \\ 1 & b \end{array}\right)\binom{x}{y} = \binom{-ax}{-ay}$$

Hence, the corresponding eigenvectors are proportional to $\left(1, -\frac{1}{a+b}\right)$. For the eigenvalue $p = b$, one has

$$\left(\begin{array}{cc} -a & 0 \\ 1 & b \end{array}\right)\binom{x}{y} = \binom{bx}{by}$$

and, thus, $x = 0$. Next, the following four cases are considered, with the associated properties proved:

- $|a| < 1$, $|b| < 1$: Global asymptotic stability,
- $|a| < 1$, $|b| > 1$: Existence of unbounded solutions,
- $|a| > 1$, $|b| < 1$: Localization of nontrivial global attractor,
- $|a| > 1$, $|b| > 1$: Existence of unbounded solutions.

The cases of $|a| = 1$ or $|b| = 1$ are not considered here, which require special consideration.

3.8 Case 1: $|a| < 1, |b| < 1$. Global asymptotic stability

There exists $\delta > 0$ such that $\max(|a|, |b|) < \delta < 1$. Then, $|h(y_k)| < \delta < 1$, $\forall k$, and $|x_k| \le \delta^k |x_0|$, $\forall k \ge 0$. Since $\delta < 1$, one has $x_k \underset{k \longrightarrow \infty}{\longrightarrow} 0$. For y_k, one has

$$\begin{aligned} |y_{k+1}| &= |by_k + x_k| = \left|b^2 y_{k-1} + bx_{k-1} + x_k\right| \\ &\le |b|^{k+1}|y_0| + |b|^k|x_0| + |b|^{k-1}|x_1| + |b|^{k-2}|x_2| + \ldots + |b||x_{k-1}| + |x_k| \\ &\le |b|^{k+1}|y_0| + (k+1)\delta^k|x_0| \end{aligned}$$

Taking into account the fact that $|b| < \delta < 1$, one gets $y_{k+1} \underset{k \to \infty}{\longrightarrow} 0$. Thus, for $|a| < 1, |b| < 1$, the global asymptotic stability is confirmed. (See Fig. 3.7).

3.9 Case 2 $|a| < 1, |b| > 1$. The existence of unbounded solutions

It will be shown that if $|a| < 1, |b| > 1$, then there exists (x_0, y_0) such that $|x_k(x_0, y_0)| \underset{k \to \infty}{\longrightarrow} 0$, $|y_k(x_0, y_0)| \underset{k \to \infty}{\longrightarrow} \infty$. For $|a| < 1$, there exists $\delta > 0$ such that $|a| < \delta < 1$. Then, $|h(y_k)| < \delta < 1$, $\forall k \ge 0$, and $|x_k| \le \delta^k |x_0|$, $\forall k \ge 0$. For y_k, one has

$$\begin{aligned} y_{k+1} &= by_k + x_k = b^2 y_{k-1} + bx_{k-1} + x_k \\ &= b^{k+1}y_0 + b^k x_0 + b^{k-1}x_1 + b^{k-2}x_2 + \ldots + bx_{k-1} + x_k \\ &= b^{k+1}y_0 + b^k\left(x_0 + \frac{x_1}{b} + \frac{x_2}{b^2} + \ldots + \frac{x_{k-1}}{b^{k-1}} + \frac{x_k}{b^k}\right) \end{aligned}$$

Taking into account the fact that $|x_k| \le \delta^k |x_0|$, where $|a| < \delta < 1$, one has

$$\begin{aligned} |y_{k+1}| &\ge |b|^{k+1}|y_0| - |b|^k\left(|x_0| + \frac{\delta^1|x_0|}{|b|} + \frac{\delta^2|x_0|}{|b|^2} + \ldots + \frac{\delta^{k-1}|x_0|}{|b|^{k-1}} + \frac{\delta^k|x_0|}{|b|^k}\right) \\ &\ge b^{k+1}|y_0| - b^k\left(|x_0| + \frac{\delta^1|x_0|}{|b|} + \frac{\delta^2|x_0|}{|b|^2} + \ldots + \frac{\delta^{k-1}|x_0|}{|b|^{k-1}} + \frac{\delta^k|x_0|}{|b|^k}\right) \end{aligned}$$

For the sum of the geometric (infinitely decreasing) series, one has

$$\begin{aligned} S_k\left(|x_0|, \frac{\delta}{|b|}\right) &= |x_0| + \frac{\delta^1|x_0|}{|b|} + \frac{\delta^2|x_0|}{|b|^2} + \ldots + \frac{\delta^{k-1}|x_0|}{|b|^{k-1}} + \frac{\delta^k|x_0|}{|b|^k} \\ &\le \frac{|x_0|}{1 - \frac{\delta}{|b|}} = \frac{|b||x_0|}{|b| - \delta} \end{aligned}$$

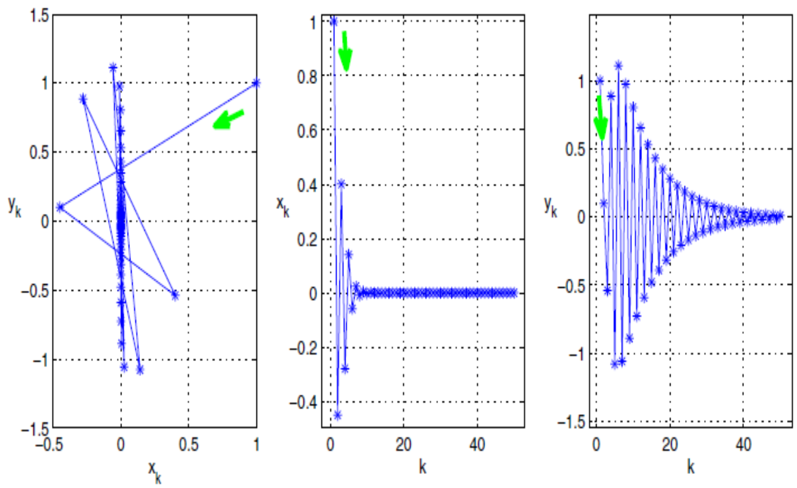

Figure 3.5: Case 1 ($|a| < 1$, $|b| < 1$): $a = 0.9$, $b = -0.9$,$x_1 = 1$, $y_1 = 1$, $k = [1, 50]$.

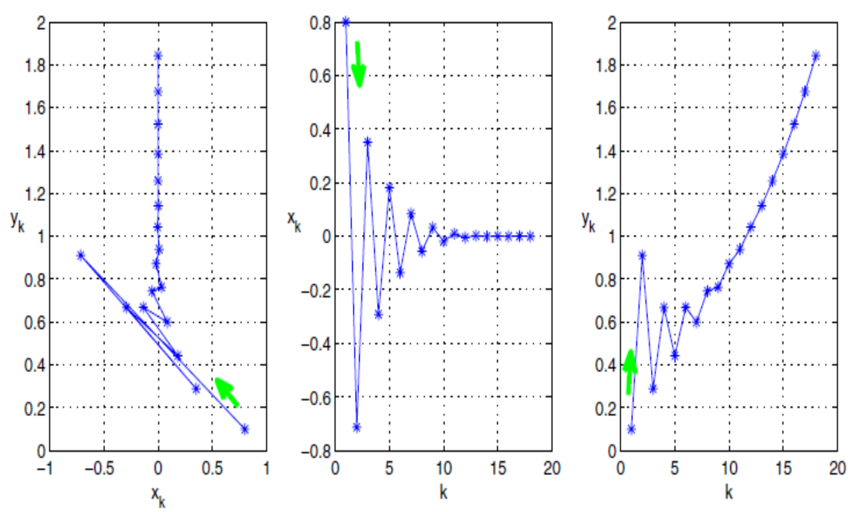

Figure 3.6: Case 2a: ($|a| < 1$,$|b| > 1$, $b > 0$): $a = 0.9$, $b = 1.1$, $x_1 = 0.8$, $y_1 = 0.1$, $k = [1, 18]$.

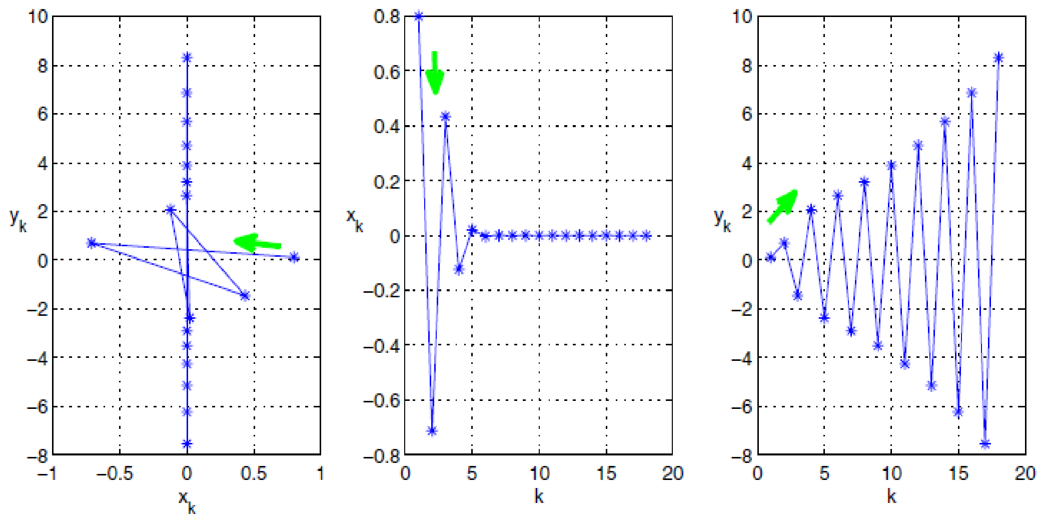

Figure 3.7: Case 2b: ($|a| < 1$, $|b| > 1$,$b < 0$): $a = 0.9$, $b = -1.1$, $x_1 = 0.8$, $y_1 = 0.1$, $k = [1, 18]$.

Therefore

$$\begin{aligned} |y_{k+1}| &\geq |b|^{k+1} |y_0| - |b|^k S_k \left(|x_0| , \frac{\delta}{|b|} \right) \\ &\geq |b|^{k+1} |y_0| - |b|^{k+1} \frac{|x_0|}{|b| - \delta} \\ &\geq |b|^{k+1} \left(|y_0| - \frac{|x_0|}{|b| - 1} \right) \end{aligned}$$

If $|y_0| \geq \frac{|x_0|}{|b|-1}$, then $|y_{k+1} (x_0, y_0)| \underset{k \to \infty}{\longrightarrow} \infty$. (See Figs. 3.8 and 3.9).

3.10 Case 3: $|a| > 1$, $|b| < 1$. The localization of global attractor

Let $0 < \delta < 1$ and introduce the notations

$$\begin{aligned} R_y &= \sqrt{|a| - 1 + \delta |a|} \\ R_x &= |b| R_y + \sqrt{a^2 - 1 + \delta a^2} \end{aligned}$$

Lemma 3.1 *If* $|a| > 1$ *and* $|b| < 1$*, then for any* $|x_0|$ *and* $|y_0| > R_y$*, one has*

$$|x_1| < |x_0| \frac{1}{1 + \delta}$$

Proof. *Since*

$$\begin{aligned} |x_0| - |x_1| &= |x_0| \left(1 - \frac{|a|}{1+y_0^2}\right) \\ &\geq |x_0| \left(1 - \frac{|a|}{1+R_y^2}\right) \\ &= |x_0| \left(1 - \frac{|a|}{1+|a|-1+\delta\,|a|}\right) \\ &= |x_0| \left(1 - \frac{1}{1+\delta}\right) \end{aligned}$$

one has

$$|x_1| \leq |x_0| \frac{1}{1+\delta}$$

Lemma 3.2 *If $|a| > 1$ and $|b| < 1$, then for $|y_0| \leq R_y$ and $|x_0| > R_x$, one has $|x_2| \leq |x_0| \frac{1}{(1+\delta)} < |x_0|$.*

■

Proof. *For y_1, one has*

$$\begin{aligned} |y_1| &= |x_0 + by_0| \geq |x_0| - |by_0| \\ &\geq |x_0| - |b|\,R_y > R_x - |b|\,R_y \\ &= |b|\,R_y + \sqrt{a^2-1+\delta a^2} - |b|\,R_y \\ &= \sqrt{a^2-1+\delta a^2} \end{aligned}$$

Therefore,

$$\frac{a^2}{1+y_1^2} < \frac{a^2}{1+a^2-1+\delta a^2} = \frac{1}{1+\delta} < 1$$

and

$$\begin{aligned} |x_0| - |x_2| &= |x_0| - \frac{a^2\,|x_0|}{(1+y_0^2)\,(1+y_1^2)} \\ &= |x_0| \left(1 - \frac{1}{(1+y_0^2)} \frac{a^2}{(1+y_1^2)}\right) \\ &\geq |x_0| \left(1 - \frac{1}{1+\delta}\right) \end{aligned}$$

Consequently, $|x_2| \leq \frac{|x_0|}{(1+\delta)} < |x_0|$. By Lemmas 3.1 and 3.2, one gets the following results. ■

Corollary 3.1 *For any x_0, y_0, there exists $n \in \mathbb{N}_0$ such that*

$$|x_n| \leq R_x$$

Corollary 3.2 *If* $|x_0| \leq R_x$, *then* $|x_m| \leq a^2 R_x$, $\forall m \geq 0$.

Proof. Let $|x_0| > R_x$. If $|y_0| > R_y$, then $|x_2| \leq |x_1|$.If $|y_1| \leq R_x$, then $|x_3| < |x_1|$. Therefore,

$$|x_m| \leq \max(|x_1|, |x_2|) \leq |ax_1| \leq a^2 |x_0| \leq a^2 R_x, \quad \forall m > 0.$$

■

Lemma 3.3 *If* $|a| > 1$ *and* $|b| < 1$, *then for* $|x_0| \leq M$ *and* $|y_0| > \frac{M+\delta}{1-|b|}$, *where* $M > 0$ *and* $\delta > 0$, *one has* $|y_1| < |y_0| - \delta < |y_0|$.

Proof. Since,

$$\begin{aligned} |y_0| - |y_1| &= |y_0| - |by_0 + x_0| \geq |y_0| - |b|\,|y_0| - |x_0| \geq |y_0|(1-|b|) - |x_0| \\ &> \frac{M+\delta}{1-|b|}(1-|b|) - M = \delta \end{aligned}$$

one has

$$|y_1| < |y_0| - \delta < |y_0|.$$

■

Corollary 3.3 *For* $|x_0| \leq R_x$ *and* $|y_0| > \frac{a^2 R_x + \delta}{1-|b|}$, *there exists* $n \in \mathbb{N}_0$ *such that*

$$|x_n| \leq a^2 R_x, |y_n| \leq \frac{a^2 R_x + \delta}{1-|b|}$$

Proof. *By Corollary* 3.2, *one has* $|x_n| \leq R_x$, $\forall n \geq 0$. *By Lemma* 3.3, *there exists* n *such that* $|y_n| \leq \frac{a^2 R_x + \delta}{1-|b|}$. ■

Lemma 3.4 *If* $|a| > 1$ *and* $|b| < 1$, *then for* $|x_0| \leq M$ *and* $|y_0| \leq \frac{M+\delta}{1-|b|}$, *where* $M > 0$ *and* $\delta > 0$, *one has*

$$|y_1| \leq \frac{M+\delta}{1-|b|}$$

Proof. We have

$$|y_1| = |by_0 + x_0| \leq |b|\,|y_0| + |x_0| \leq |b| \frac{M+\delta}{1-|b|} + M = \frac{M + |b|\,\delta}{1-|b|} < \frac{M+\delta}{1-|b|}$$

■

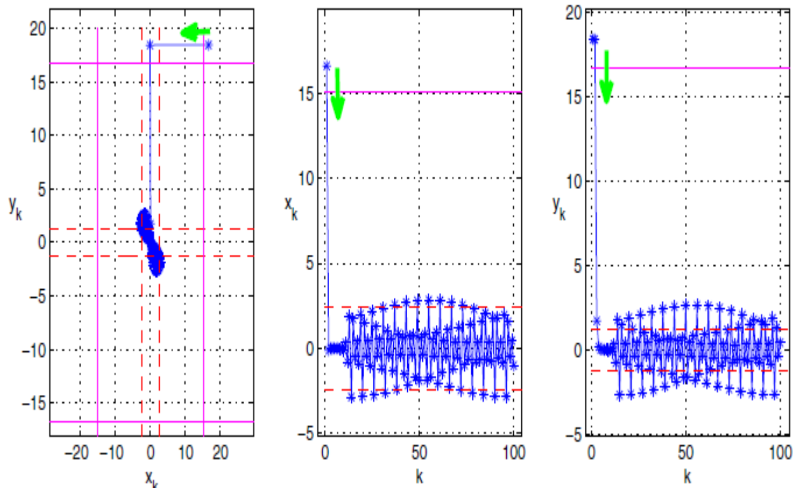

Figure 3.8: Case 3: ($|a| > 1$, $|b| < 1$): $a = 2.5$,$b = 0.1$, $\delta = 0.00001$, $x_1 = 16.59$,$y_1 = 18.44$, $k = [1, 100]$.

Corollary 3.4 *If* $|x_0| \leq R_x$*, then there exists* $N > 0$ *such that*

$$|y_n| \leq \frac{a^2 R_x + \delta}{1 - |b|}, \forall n > N$$

See [21] .

Proof. *By Corollaries* 3.2 *and* 3.3*, there exists* $n > 0$ *such that*

$$|x_n| \leq a^2 R_x, |y_n| \leq \frac{a^2 R_x + \delta}{1 - |b|}$$

where $|x_m| \leq a^2 R_x$*,* $\forall m \geq 0$*. It follows from Lemma* 3.4 *with* $M = a^2 R_x$ *that*

$$\forall k > n, |y_k| \leq \frac{a^2 R_x + \delta}{1 - |b|}$$

Therefore, by Corollaries 3.1*,* 3.2 *and* 3.4 *for any* $x_0 = 0$*,* $y_0 = 0$*, there exists* $n > 0$ *such that,* $\forall k > n$*,*

$$\begin{aligned}
|x_k| &\leq a^2 \left(|b| \sqrt{|a| - 1 + |a|\,\delta} + \sqrt{a^2 - 1 + a^2\delta} \right) \\
&= a^2 R_x \\
|y_k| &\leq \frac{a^2 \left(|b| \sqrt{|a| - 1 + |a|\,\delta} + \sqrt{a^2 - 1 + a^2\delta} \right) + \delta}{1 - |b|} \\
&= \frac{a^2 R_x + \delta}{1 - |b|}
\end{aligned}$$

■

Thus, all possible attractors are placed in the above absorbing set. Figure 3.10 shows the absorbing set and a self-excited attractor.

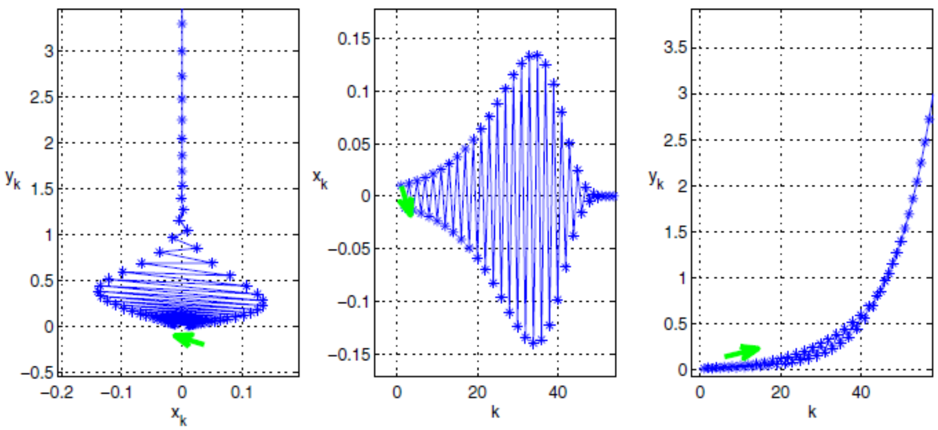

Figure 3.9: Case 4 : ($|a| > 1$,$|b| > 1$): $a = 1.1$, $b = 1.1$, $x_1 = 0.01$, $y_1 = 0.01$, $k = [1, 60]$.

3.11 Case 4: $|a| > 1$, $|b| > 1$.The existence of unbounded solutions

Consider a certain δ satisfying $|b|^{-1} < \delta < 1$. Let $|y_0| \geq \sqrt{|a| - 1}$ and $|x_0| \leq |y_0| \left(|b| - \delta^{-1}\right)$. Then,

$$|x_1| = \left|\frac{-ax_\circ}{1 + y_0^2}\right| \leq |x_0|, |y_1| = |by_0 + x_0| \geq |by_0| - |x_0| \geq \delta^{-1} |y_0| > |y_0|$$

and $|x_1| \leq |y_1| \left(|b| - \delta^{-1}\right)$. Therefore, $y_k(y_0) \underset{k\to\infty}{\longrightarrow} \infty$. Figure 3.11 shows a solution, which tends to infinity. See [21] .

3.12 Conclusion

The concept of chaos is related to all areas of life and it became possible to predict the behavior of any evolutionary phenomenon, and each of these phenomena is translated into equations and thus it is possible to study its dynamical behaviors, including the so called **Zeraoulia- Sprott map.** In this work, we present a more comprehensive study of the dynamical behaviors of **Zeraoulia-Sprott map**, based on theoretical analysis and numerical simulations. We also presented results on its ability to generate smooth multifold strange attractors via period-doubling bifurcations.

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