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Stability conditions for fractional reaction-diffusion systems

Presented by:

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Dedication

I dedicate this work to my family, with the hope that they perceive it as a testament to my profound gratitude and appreciation,

as well as to all my acquaintances.

I would like to convey my sincere appreciation to all individuals who contributed, whether directly or indirectly, to the successful culmination of this endeavor.

To all my friends who have always encouraged me, and to whom I wish more success.

Thanks!

Salim Zidi

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I extend my heartfelt appreciation to my entire family for their unwavering support and devotion, which have enabled me to attain this modest position. I am profoundly appreciative of the trust they have bestowed upon me.

In conclusion, I wish to extend my sincere appreciation to all individuals who have, in various capacities, contributed to the advancement of this work.

O Allah, bestow Your blessings upon Your esteemed Messenger, his family, and his companions, and grant us Your bounties in our lives.

Abstract

This thesis aims to investigate a space-fractional variant of the conventional Lengyel–Epstein CIMA reaction model. Initially, we establish the global existence, uniqueness, and boundedness of solutions. We then examine the system's main analytical properties. Subsequently, we derive the conditions on the reactor size and diffusion coefficient that preclude the existence of non-constant positive steady-state solutions. We further analyze the stability of the constant steady-state solution for both the ODE and PDE models. Finally, we establish sufficient conditions for the global asymptotic stability of the positive steady-state solution using the direct Lyapunov method.
Key words: *fractional reaction–diffusion systems, asymptotic stability, Lyapunov functional, global and local stability of solutions.*

Résumé

L'objectif de cette thèse est d'étudier une variante spatiale fractionnaire du modèle de réaction CIMA de Lengyel–Epstein conventionnel. Dans un premier temps, nous établissons l'existence globale, l'unicité et la bornitude des solutions. Nous examinons ensuite les principales propriétés analytiques du système. Nous établissons par la suite les conditions portant sur la taille du réacteur et le coefficient de diffusion qui excluent l'existence de solutions stationnaires positives non constantes. Nous analysons également la stabilité de la solution stationnaire constante pour les modèles d'équations différentielles ordinaires (EDO) et d'équations aux dérivées partielles (EDP). Enfin, nous établissons des conditions suffisantes assurant la stabilité asymptotique globale de la solution stationnaire positive au moyen de la méthode directe de Lyapunov.
Mots clés : *Les systèmes de réaction-diffusion fractionnaires, la stabilité asymptotique, les fonctions de Lyapunov, la stabilité globale et locale des solutions.*

ملخص

تهدف هذه الأطروحة إلى دراسة نسخة كسرية مكانية من نموذج تفاعل سيما التقليدي للنجل–إبستين. في البداية، نثبت الوجود العالمي، ووحدانية الحل، ومحدوديته. ثم ندرس الخصائص التحليلية الأساسية للنظام. بعد ذلك، نستخلص الشروط المتعلقة بحجم المفاعل ومعامل الانتشار التي تمنع وجود حلول اتزان موجبة غير ثابتة مكانيًا. كما نحلل استقرار حالة الاتزان الثابتة لكلٍ من نموذج المعادلات التفاضلية العادية ونموذج المعادلات التفاضلية الجزئية. وأخيرًا، نحدد الشروط الكافية لتحقيق الاستقرار التقاربي العالمي لحالة الاتزان الموجبة باستخدام الطريقة المباشرة لليابونوف.

كلمات مفتاحية: : أنظمة التفاعل-الانتشار الكسرية، الاستقرار العالمي والمحلي للحلول, النظام الكسري لنجل-ابستين, الاستقرار التقاربي.

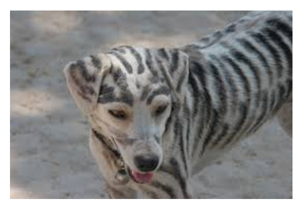
Contents

General introduction

For decades, scientists have endeavored to ascertain the origins of the patterns and structures observed in nature. These patterns and forms encompass a wide range of features, ranging from the markings on a leopard to the elegant spirals on a snail shell to the elaborate designs on a butterfly's wing. Morphogenesis refers to the biological phenomenon through which organisms generate their physical structures and hues [48]. In the 1950s, Alan Turing, a British mathematician, introduced a revolutionary model that elucidates and encompasses the process of pattern formation in morphogenesis. This model represented a significant advancement in the study of morphogenesis.

(a) The Dermis of a Panthera Tigris

(b) The leopard's markings



(c) The dog's stripes

(d) On a cat's stripes

(e) Spiral's-Nautilus

(f) Angelfish's-stripes

Turing's work established the theoretical possibility that disturbances might cause instabilities, which may lead to the formation of concentrated patterns. In other words, if a system is initially stable according to ordinary differential equations (ODEs) with specific parameters and initial conditions but becomes unstable when distortion is included, the solution's spatial dimensions will display a range of distinct patterns. Turing's work exclusively addressed the linear regime, and this aspect should be taken into account. A presentation in [39] showcased the physical understanding and possibilities of spatial oscillations in non-equilibrium systems. The authors of this work have shown that the transition to a spatial order does not violate the second law of thermodynamics, even when applied to systems that are not in equilibrium. The reader can refer to [46],[45] to gain a deeper understanding of dissipative structures from both a theoretical and experimental perspective. This concept about the formation of patterns was not subjected to experimentation for many decades. The authors in [23] investigated a chemical reaction using chlorite-iodide malonic acid (CIMA) that took place across a diffusive membrane. This breakthrough is considered to be one of the earliest tangible manifestations of this concept. The Lengyel-Epstein reaction-diffusion model has been extensively studied to explore the presence and characteristics of solutions, criteria for asymptotic stability, bifurcation analysis, and pattern creation. These experiments were undertaken subsequent to the initial development of a mathematical model, which was presented in [25, 26]. Several studies stand out, namely [30, 35, 52, 53]. A morphological analysis is performed to compare the Turing patterns generated in the chlorite-iodide-malonic acid (CIMA) reaction, a chemical reaction-diffusion system in a non-equilibrium state, with those produced in the Lengyel-Epstein model. The Lengyel-Epstein model is founded on a nonlinear partial differential equation and is regarded as a description of the CIMA response. The original idea has undergone significant improvement throughout the years. These novel models were derived from either theoretical conceptualization or adjustments made to the reaction itself [10, 17, 33, 40]. For instance, the alteration in the light intensity that was applied to the reactants serves as an example of a modification. Several generalizations have been suggested for the Lengyel-Epstein system. Lengyel and Epstein successfully simplified the examination of the reaction mechanism and its relation to the CDMIA reaction by identifying two primary variables: iodide and chlorite. In contrast to the substantial fluctuations in the levels of the other two kinds, they based their assumption on the relative consistency of the concentrations of chlorine dioxide, iodine, and malonic acid.

Three unique chemical reaction schemes can be used to characterize the CIMA reaction, as shown below

$$\left\{\begin{array}{c} MA + I_2 \rightarrow IMA + I^- + H^+ \\ CIO_2 + I^- \rightarrow \frac{1}{2} I_2 + CIO_2^- \\ CIO_2^- + 4I^- + 4H^+ \rightarrow 2I_2 + CI^- + 2H_2O \end{array}\right.$$

This thesis is structured into four chapters.

Chapter 1: In the first chapter, we review the essential terminology and initial definitions related to stability, fractional calculus, and formulas that will be beneficial in the subsequent chapters.

Chapter 2: This chapter has four subsections: Subsection one offers a thorough introduction to reaction-diffusion systems, supplemented by illustrated examples. In part two, we examine the stability analysis and local stability in both ODE and PDE contexts, defining all aspects pertinent to the Lyapunov theorem for global asymptotic stability and the relationship between Lyapunov theory and LaSalle's theorem. In subsection three, introduce Turing instability. Subsequently, we delineate the activator-inhibitor dynamics of a system. We are also applying all prior research on ordinary differential equations (ODE) and partial differential equations (PDE) to the Lengyel-Epstein reaction-diffusion model. In the last subsection, we propose the fractional Lengyel-Epstein fractional Laplacian model.

Chapter 3: A unique and bounded solution is established in this chapter. Additionally, we have examined the basic properties of positive non-constant solutions. When the reactor attains a suitable size or when the diffusion coefficient falls below a threshold dictated by the reactor dimensions, it will be demonstrated that non-constant positive solutions do not exist.

Chapter 4: Determining the conditions for Turing instability and proving the system's global asymptotic stability are the main goals of this chapter.

Nomenclature

- $\mathbb{R}$: Numbers that are real.
- $\mathbb{R}^{+}$: all non-negative real numbers.
- $\mathbb{R}^{n}$: The collection of all N-tuples $x = (x_1, x_2, \ldots, x_n)$.
- $C(\Omega)$: The function region of specified continuous functions.
- $C^{k}(\Omega), k = 1, 2$: Within the domain Ω, a collection of functions that are continuously differentiable k times.
- $C^{s,2}(\Omega) = \left\{ u \in C^{2}(\Omega) : \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^{2}}{|y-x|^{n+2S}} dxdy < \infty \right\}, s \in]0, 1[$.
- $W^{s,2}(\Omega) = \left\{ u \in L^{2}(\Omega) : \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^{2}}{|y-x|^{n+2S}} dxdy < \infty \right\}, s \in]0, 1[$.
- $L^{2}(\Omega)$: A collection of functions that are square-integrable on Ω.
- $L^{\infty}(\Omega)$: space of all measurable functions $u : \mathbb{R}^{n} \to \mathbb{R}$ with a constant value $M \geq 0$ such that $|u(x)| \leq M$ for each $x \in \mathbb{R}^{n}$.
- The Sobolev space $H^{1}(\Omega) = W^{1,2}(\Omega)$ is defined by

$$H^{1}(\Omega) = \left\{ u \in L^{2}(\Omega) : D^{\alpha}u \in L^{2}(\Omega) \text{ for all } |\alpha| \leq 1 \right\}.$$

- H^{2}: Hilbert space.
- $\Gamma(.)$: The gamma function.
- The left Riemann–Liouville fractional derivative is defined by

$${}_{RL}D^{\gamma}_{-\infty,x}u(x,y,t) = \frac{1}{\Gamma(2-\gamma)} \frac{\partial^{2}}{\partial x^{2}} \int_{-\infty}^{x} (x-s)^{1-\gamma} u(s,y,t)\, ds, \qquad 1 < \gamma < 2.$$

- The right Riemann–Liouville fractional derivative is defined by

$${}_{RL}D^{\gamma}_{x,+\infty}u(x,y,t) = \frac{1}{\Gamma(2-\gamma)} \frac{\partial^{2}}{\partial x^{2}} \int_{x}^{+\infty} (s-x)^{1-\gamma} u(s,y,t)\, ds, \qquad 1 < \gamma < 2.$$

- Ω: restricted region of $\mathbb{R}^{n}$.
- $\partial\Omega$: Domain bounds Ω.

- $\bar{\Omega}$: domain closure Ω.
- $|\Omega|$: Volume of the domain Ω.
- $Re(\lambda)$: Represents the real component of a complex number λ.
- $E^{'}(u)$: The derivative of E, denoted as $E'(u) = \frac{d}{dt}E(u)$.
- $\frac{\partial u}{\partial t}, \partial_t u$: The partial derivative in relation to t.
- $\frac{\partial u}{\partial v}$: Outside of $\partial\Omega$, the normal derivative of u.
- v : The exterior unit normal vector of the boundary is denoted as $\partial\Omega$.
- $(-\Delta)^s_\Omega u(x)$: This is the fractional Laplacian operator for a region.
- $F\left(-(-\Delta)^{\frac{\gamma}{2}}u\right)(\xi)$: Fourier transform of $(-\Delta)^{\frac{\gamma}{2}}u$.
- $PV \int_\Omega u(x)dy$: The principal value associated with Cauchy
- $\max u(x), \min u(x)$: Maximum of $u(x)$, minimum of $u(x)$.
- $\sup u(x)$: Superior $u(x)$.
- $\cdot$: Innerproduct.
- $\|u\|$: Norm of u.

Chapter 1

Preliminary

In this chapter, we will provide the foundational theories and definitions used in this thesis, along with the pertinent notation associated with fractional calculus.

1.1 Basic Theorems and Definitions

Let $\Omega \subset R^n$, where $n \geq 1$, be a bounded domain with a suitably smooth boundary $\partial\Omega$. We examine the subsequent system of reaction-diffusion equations for $(x, t) \in \Omega \times \mathbb{R}^+$:

$$U_t - D\Delta U = H(U) \tag{1.1}$$

where D is a matrix of positive constant values and $U = (u_1, u_2, \ldots, u_n)$, with $n \geq 1$. We presume that U fulfills the first condition

$$U(x, 0) = U_0(x), \quad x \in \Omega. \tag{1.2}$$

In combination with (1.1), a Neumann boundary condition is defined as follows:

$$\tfrac{\partial U}{\partial v} = 0, \quad \text{on} \quad \partial\Omega \tag{1.3}$$

on the boundary $\partial\Omega$, where v denotes the outside unit normal vector.

Definition 1. *[38] An equilibrium point of system (1.1) is a vector $U^* \in \mathbb{R}^n$ such that*

$$H(U^*) = 0.$$

The system (1.1) can be transformed to

$$\begin{cases} u_t - d_1\Delta u = f(u, v), \\ v_t - d_2\Delta v = g(u, v), \end{cases} \tag{1.4}$$

if $U = (u, v)$, where u and v represent the chemical species and d_1 and d_2 denote the specific diffusion coefficients.

Definition 2 (Invariant Set [51])**.** *A set $S \subset \mathbb{R}^m$ is called an* invariant set *for a parabolic system if, whenever the initial and boundary data belong to S, the corresponding solution remains in S for all $t \geq 0$.*

Definition 3. *[53] If the vector field (f, g) on the border $\partial\Re$ directs inward, then the rectangle $\Re = (0, r_1) \times (0, r_2)$ is a rectangle that is invariant. To offer more clarification,*

$$\begin{cases} f(0, v) \geq 0 \text{ and } f(r_1, v) \leq 0 \text{ for } 0 < v < r_2 \\ g(u, 0) \geq 0 \text{ and } g(u, r_2) \leq 0 \text{ for } 0 < u < r_1. \end{cases}$$

Lemma 1. *[27]*

Let $g \in C(\Omega \times \mathbb{R})$, and let w be a subset of the domain of $(-\Delta)^s_\Omega$.

1. *If*

$$(-\Delta)^s_\Omega w(x) + g(x, w(x)) \leq 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad w(x_0) = \max_{\overline{\Omega}} w,$$

then

$$g(x_0, w(x_0)) \leq 0$$

is fulfilled by $w \in C(\Omega)$.

2. *If*

$$(-\Delta)^s_\Omega w(x) + g(x, w(x)) \geq 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad w(x_0) = \min_{\overline{\Omega}} w,$$

then

$$g(x_0, w(x_0)) \geq 0$$

is fulfilled by $w \in C(\Omega)$.

Theorem 1. *(Global existence) If an invariant rectangle exists for the system (1.1), then the system (1.1), given specific initial and boundary conditions in $\Re$, possesses a unique global solution.*

Theorem 2. *[28] (Hölder's inequality) Given that $1 < p < \infty$, and let q be the conjugate exponent of p, that is $\frac{1}{p} + \frac{1}{q} = 1$, then for $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, Hölder's inequality holds:*

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \left(\int_\Omega |u(x)|^p dx \right)^{1/p} \left(\int_\Omega |v(x)|^q dx \right)^{1/q}.$$

For $p = q = 2$, it is referred to as the Cauchy-Schwarz inequality:

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \left(\int_\Omega |u(x)|^2 dx \right)^{1/2} \left(\int_\Omega |v(x)|^2 dx \right)^{1/2}.$$

1.1.1 Stability and asymptotic stability

Definition 4. *[43] There exists a $\delta > 0$ such that if condition*

$$\|U\,(0) - U^*\| < \delta \Rightarrow \|U\,(t) - U^*\| < \varepsilon, \forall t \geq 0,$$

is fulfilled, then U^* is deemed stable for every $\varepsilon > 0$.

- *If it is not stable, U^* is unstable.*

Definition 5. *[43]*
If U^* is stable and there exists a $\delta > 0$ such that

$$\|U\,(0) - U^*\| < \delta \Rightarrow \|U\,(t) - U^*\| \to 0, \forall t \geq 0,$$

it is deemed asymptotically stable.

Definition 6 (Local Asymptotic Stability [43]). *Let U^* be an equilibrium point of the system. The equilibrium point U^* is said to be* locally asymptotically stable *if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|U(0) - U^*\| < \delta$$

implies

$$\|U(t) - U^*\| < \varepsilon, \qquad \forall t \geq 0,$$

and

$$\lim_{t\to\infty} \|U(t) - U^*\| = 0.$$

Definition 7 (Global Asymptotic Stability [43]). *Let U^* be an equilibrium point of the system. The equilibrium point U^* is said to be* globally asymptotically stable *if, for every initial condition $U(0) \in \Omega$,*

$$\lim_{t\to\infty} \|U(t) - U^*\| = 0.$$

1.2 Fractional calculus

One method used to ascertain the fractional Laplacian on a confined domain is to use the real space formula [29]

$$(-\Delta_{Riesz})^s_\Omega\, u(x) = C_{n,s} PV \int_{\mathbb{R}^n} \frac{u\,(x) - u\,(y)}{|y - x|^{n+2S}} dy, \tag{1.5}$$

for functions on Ω. This results in the Riesz fractional Laplacian in . Initially, let us examine Dirichlet border conditions. Following the formula (1.5) . Where *P.V.* represents the primary value of the integral:

$$PV\int_{\mathbb{R}^n}\frac{u\left(x\right)-u\left(y\right)}{\left|y-x\right|^{n+2S}}dy=\lim_{\varepsilon\to0}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u\left(x\right)-u\left(y\right)}{\left|y-x\right|^{n+2S}}dy$$

an outside boundary condition $u=g$ in $\mathbb{R}^n\setminus\Omega$ necessitates values of u throughout every domain of $\mathbb{R}^n$.

- For functions u that agree with $u=g$ in $\mathbb{R}^n\setminus\Omega$, the Riesz fractional Laplacian can be defined for $x\in\Omega$ by (see [29],[50])

$$\begin{aligned}
(-\Delta_{Riesz})^s_\Omega u(x) &= C_{n,s}PV\int_{\mathbb{R}^n}\frac{u\left(x\right)-u\left(y\right)}{\left|y-x\right|^{n+2S}}dy,\\
&= C_{n,s}\left(PV\int_\Omega\frac{u\left(x\right)-u\left(y\right)}{\left|y-x\right|^{n+2S}}dy+\int_{\mathbb{R}^n\setminus\Omega}\frac{u\left(x\right)-u\left(y\right)}{\left|y-x\right|^{n+2S}}dy\right),\\
&= C_{n,s}PV\int_\Omega\frac{u\left(x\right)-u\left(y\right)}{\left|y-x\right|^{n+2S}}dy-u\left(x\right)\int_{\mathbb{R}^n\setminus\Omega}\frac{C_{n,s}}{\left|y-x\right|^{n+2S}}dy.
\end{aligned}$$

The Riesz fractional Laplacian is contingent upon Ω and the exterior boundary conditions g.

1.2.1 Basic Formulas

The Taylor series for the function $f:\mathbb{R}^2\to\mathbb{R}$ is defined by Taylor's formula

$$f(u)=f\left(u^*\right)+\frac{\partial f}{\partial u}\left(u-u^*\right)+\frac{1}{2!}\frac{\partial^2 f}{\partial u^2}\left(u-u^*\right)^2+\cdots+\frac{1}{n!}\frac{\partial^n f}{\partial u^n}\left(u-u^*\right)^n+O\left(u,u^*\right).$$

Consequently, we can derive

$$\left\{\begin{array}{c} f\left(u,v\right)=f\left(u^*,v^*\right)+\frac{\partial f}{\partial u}\left(u-u^*,v^*\right)+\frac{\partial f}{\partial v}\left(u^*,v-v^*\right)+\frac{1}{2!}\frac{\partial^2 f}{\partial u^2}\left(u-u^*\right)^2+\frac{1}{2!}\frac{\partial^2 f}{\partial v^2}\left(v-v^*\right)^2\\ +\frac{\partial^2 f}{\partial u\partial v}\left(u-u^*,v^*\right)\left(u^*,v-v^*\right)+O\left[\left(u-u^*\right)^2,\left(v-v^*\right)^2\right].\end{array}\right. \tag{1.6}$$

Green's Formula

For functions u,v belonging to the Sobolev space $H^1(\Omega)$ and with $\partial\Omega$ being smooth, we obtain

$$\int_\Omega\frac{\partial u}{\partial x_i}vdx=-\int_\Omega\frac{\partial v}{\partial x_i}udx+\int_{\partial\Omega}uv\eta_i d\sigma,\quad 1\le i\le n.$$

Here, $\eta=(\eta_1,\dots,\eta_n)$ denotes the outward unit normal vector on $\partial\Omega$, $\eta_i=\eta\cdot e_i$ is its ith component, and $d\sigma$ is the surface measure on $\partial\Omega$. The subsequent inequalities are available in [50].

- $$\int_{\Omega} v\left(x\right)\left(-\Delta\right)_{\Omega}^{s} u(x)dx = \frac{C_{n,s}}{2}\int_{\Omega}\int_{\Omega}\frac{\left(u\left(x\right)-u\left(y\right)\right)\left(v\left(x\right)-v\left(y\right)\right)}{\left|y-x\right|^{n+2s}}dxdy, \tag{1.7}$$

- $$\lambda_{1,s}^{N}\int_{\Omega}\left|u\right|^{2} \leq \frac{C_{n,s}}{2}\int_{\Omega}\int_{\Omega}\frac{\left(u\left(x\right)-u\left(y\right)\right)^{2}}{\left|y-x\right|^{n+2s}}dxdy. \tag{1.8}$$

Chapter 2

Reaction diffusion systems and Lengyel-Epstein model

This chapter focuses on reaction-diffusion systems and stability analysis of partial differential equation (PDE) systems. We provide a comprehensive overview of reaction-diffusion systems, including several examples. We analyze the stability of a $2D$ system by examining both local and global stability, as well as investigating Turing instabilities.We are utilizing prior research on a $2-D$ Lengyel-Epstein reaction-diffusion system with a homogeneous Neumann boundary condition.

2.1 Summary of Reaction-Diffusion Systems

Reaction-diffusion systems have attracted significant interest in recent years due to their prevalent occurrence in models of biological and chemical processes, as well as the complexity of their solution sets. Given the wide range of applications for these systems, the methods used to describe specific chemical challenges, such as reactions involving oscillating chemicals, are important to consider (Brussellateur). People vary in their approach to different problems:

In the field of chemistry, chemical compounds constitute a prominent example. In biochemistry, the term "they" may refer to molecules. The study of atoms in relation to metals is known as metallurgy. Humans serve as the topic of study in the field of population dynamics. In population genetics, characters denote traits or features. In the field of biophysics, electrical charges or potential differences are studied. Within the natural habitat, they might symbolize the fauna or flora found in a woodland, a body of water, or a vast expanse of the sea. For the majority of these issues, we demonstrate outcomes in reaction-diffusion systems.

The origin and characteristics of the problem under investigation will determine the conditions at the boundaries. In the absence of individual migration over the domain border, the conditions at the regular boundaries will be Neumann conditions. For the margins, we use the homogeneous Dirichlet conditions if no one is at the boundary. The unknown, or the answer being sought, can be represented as a vector consisting of positive functions. In the realm of chemistry, this vector denotes the concentrations of several compounds. In biology and metallurgy, a concentration vector denotes the quantities of molecules or atoms. During discussion on population dynamics and the environment, a vector refers to an assemblage of densities pertaining to flora, fauna, or people.

The initial conditions are often favorable since they encompass concentrations, densities, electrical charges, and similar factors. The aforementioned issues can be represented as a partial differential equation:

$$\frac{\partial U}{\partial t} = D\Delta U + H(U). \tag{2.1}$$

In the field of chemistry, the variable U denotes a vector that reflects the concentrations of chemicals, whereas H indicates the chemical reactions that occur based on these concentrations. The word $D\Delta U$ denotes the molecular changes that occur across the reaction barrier.

2.1.1 Examples for reaction-diffusion systems

Brusselator model

Ilya Prigogine, the 1977 Nobel Prize winner in Chemistry, and his associates made the suggestion in Brussels in 1971. The objective was to elucidate the widespread occurrence of autocatalytic oscillations in natural systems, particularly within living organisms. The model includes the reaction equations

$$\begin{aligned}
&A \xrightarrow{K_1} X\\
&2X + Y \xrightarrow{K_2} 3X\\
&B + X \xrightarrow{K_3} D + Y\\
&X \xrightarrow{K_4} E
\end{aligned}$$

In the initial example, the primary products are X and Y, the final outputs are D and E, and the reactants are A and B. The symbols for the reaction rate constants are K_1, K_2, K_3, and K_4. The equations that describe the development of chemicals X and Y are as follows:

$$\begin{cases} \frac{dX}{dt} = k_1 A + k_2 X^2 Y - k_3 BX - k_4 X, \\ \frac{dY}{dt} = -k_2 X^2 Y + k_3 BX. \end{cases}$$

After the above transformations, the reaction-diffusion system exhibits Brusselator kinematics characterized by the terms

$$\begin{cases} \frac{du}{dt} = D_u \nabla^2 u + a - (b+1)\, u + u^2 v, \\ \frac{dv}{dt} = D_v \nabla^2 v + bu - u^2 v, \end{cases}$$

where $a = \frac{Ak_1}{k_3}, b = \frac{Bk_2}{k_1}, k_4 = k_3$, X is substituted with u, Y is substituted with v, $\frac{D_X}{k_3}$ is substituted by D_u, and $\frac{D_Y}{k_3}$ is substituted with D_v.

Degn-Harrison model

$$A \longrightarrow Y$$
$$B \rightleftharpoons X$$
$$X + Y \longrightarrow P$$

The intermediate components in this model are denoted as X and Y, corresponding to oxygen and nutrition, respectively. A and B represent external parameters that act as "sources", and their concentrations are assumed to remain constant throughout the reactor vessel. P denotes the ultimate product, with its concentration presumed to remain constant. The final phase of the reaction process is hindered by an excessive amount of oxygen present in the reactor. The first and final stages are presumed to be irreversible, although the intermediate stage is reversible. Degn and Harrison initially proposed that a non-linear rate equation of the type $\frac{XY}{1+qX^2}$ regulates the end period, where the variable q signifies the extent of the inhibitory impact. The Degn-Harrison reaction scheme, along with homogeneous Neumann boundary conditions, is represented by the variables

$$\begin{cases} X_t - D_X \Delta X = k_2 B - k_3 X - \frac{k_1 XY}{1+qX^2}, \quad x \in \Omega, t > 0, \\ Y_t - D_Y \Delta Y = k_1 A - \frac{k_4 XY}{1+qX^2}, \quad x \in \Omega, t > 0, \\ \frac{\partial X}{\partial \nu} = \frac{\partial Y}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0. \end{cases}$$

In this method, the reactants' concentrations are represented by dimensionless variables A, B, X, and Y. The reaction rates are denoted by the constants as follows k_I (where $i = 1, 2, 3, 4$). D_X and D_Y are the Fickian molecular diffusion coefficients of X and Y, respectively. There is an assumption that these coefficients are positive constants throughout the reactor vessel. The rate and diffusion constants are intrinsic properties of a specific system, whereas the concentrations of A and B are adjustable factors that may be manipulated during the reaction process.

The Gray-Scott Model

The autocatalytic model of glycolysis, which was initially postulated by [42], is a variant of the Gray-Scott (GS) model [16]. The cubic autocatalytic chemical reaction of type

$$\begin{aligned} U + 2V &\rightarrow 3V, \\ V &\rightarrow P, \end{aligned}$$

is taken into account in an open flow reactor, where the product P is removed and U is consistently supplied. After rescaling and incorporating diffusion, the kinetic equations of this reaction are represented as [37]

$$\begin{aligned} \frac{du}{dt} &= D_u \nabla^2 u + F(1-u) - u^2 v, \\ \frac{dv}{dt} &= D_v \nabla^2 v - (F+k)v + u^2 v, \\ \frac{\partial X}{\partial \nu} &= \frac{\partial Y}{\partial \nu} = 0, \quad x \in \partial\Omega, \end{aligned}$$

where F, k, D_u, and D_v are constants. The amounts of both species are represented by U and V, respectively; k denotes the effective rate constant for the conversion of V into P, and F represents the inflow rate; $(F+k)$ indicates the rate at which V is eliminated from the reaction. In contrast to the Lengyel-Epstein model, which features activation and inhibition, the Gray-Scott model is characterized by activation and substrate depletion. As a result, the two concentrations u and v converge towards each other. In this situation, V is identified as the activator, whereas U is acknowledged as the substrate depleter.

2.2 The study of stability and local stability in both ODE and PDE

2.2.1 Stability analysis of ODE systems

This part presents definitions and theorems pertaining to dynamic systems. We analyze the system

$$\begin{cases} \frac{\partial U}{\partial t} = f(U), \\ U(0) = U_0. \end{cases} \tag{2.2}$$

Definition 8. *[6] In the case of an equation defined by $U'(t) = f(U(t))$ has an equilibrium point at $U^* = 0$. Its linearization at $U^* = 0$. The variable can be represented as the linear system $U'(t) = AU(t)$, where $A = \nabla f(0)$. By expanding f using Taylor's expansion, we obtain the result: $f(U) = AU + O(|U|^2)$.*

The linearized system can be defined as the equation $U' = AU(t)$. The asymptotic stability of $U = 0$ for $U' = AU(t)$ requires that all eigenvalues of A possess negative real parts. If any eigenvalue is higher than zero or possesses a positive real part, then $U = 0$ is considered unstable. The following theorem broadens the scope of this conclusion to encompass nonlinear contexts.

Definition 9 (Linear Stability)**.** *An equilibrium point x^* of the nonlinear system*

$$\dot{x} = f(x)$$

is said to be linearly stable *if the equilibrium of its linearized system*

$$\dot{z} = J(x^*)z,$$

is stable, where

$$J(x^*) = \left.\frac{\partial f}{\partial x}\right|_{x=x^*}$$

is the Jacobian matrix evaluated at the equilibrium point.

Theorem 3. *[6] Assume that all eigenvalues of $\nabla f(0)$ possess negative real parts. The equilibrium $U^* = 0$ is asymptotically stable concerning the model $U'(t) = \nabla f\,(0)\,U + O\,(|U|)$. If the gradient of $\nabla f\,(0)$ contains at least one eigenvalue with a positive real portion, the equilibrium $U^* = 0$ is considered unstable.*

Theorem 4. *[6] Let us suppose that A is a constant $n \times n$ matrix that is not unique.*

- *In the event that every eigenvalue in matrix A has a negative real part, the system that matrix A represents is asymptotically stable when $U^* = 0$. Specifically, for any $p \in R^n$ one has that the solution $U\,(t, p) \to 0$ as $t \to \infty$.*
- *If one eigenvalue of matrix A has a positive real part, then the equilibrium point $U^* = 0$ is considered unstable.*

2.2.2 Stability theory

One may examine the stability of fixed points in a system of first-order differential equations that are linear with constant coefficients by evaluating the values of eigenvalues of the corresponding matrix. This investigation primarily examines the stability of the (2×2) variant of the model (2.2). The system will be expressed as

$$\begin{cases} \frac{\partial u(t)}{\partial t} = a_{11}u + a_{12}v = f\,(u, v)\,, \\ \frac{\partial v(t)}{\partial t} = a_{21}u + a_{22}v = g\,(u, v)\,, \end{cases} \tag{2.3}$$

with the coefficients a_{ij} being real values. Given the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

with the equations

$$\begin{cases} a_{11}u + a_{12}v = f(u,v), \\ a_{21}u + a_{22}v = g(u,v). \end{cases} \tag{2.4}$$

possesses a distinct equilibrium point $(u^*, v^*) = (0,0)$ that satisfies condition:

$$\begin{cases} f(u^*, v^*) = 0, \\ g(u^*, v^*) = 0. \end{cases}$$

The system (2.3) is defined as the following Jacobian matrix:

$$J = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The characteristic polynomial is defined by equation

$$p(\lambda) = \det(J - \lambda I) = \lambda^2 - \operatorname{tr}(J)\lambda + \det(J). \tag{2.5}$$

We are utilizing equation

$$\Delta = \operatorname{tr}(J)^2 - 4\det(J) \tag{2.6}$$

to determine Δ for the resolution of the characteristic polynomial. The determinant and trace of the Jacobian matrix are denoted as $\det(J)$ and $\operatorname{tr}(J)$, respectively. In

Cas I : we get the condition $\Delta \geq 0$ followed by the expression

$$\lambda_{1,2} = \frac{\operatorname{tr}(J) \pm \sqrt{\Delta}}{2}, (\lambda_{1,2} \text{ are reals})$$

The stability necessities and the attributes of the points that are fixed may be derived using this information.

- The equilibrium (u^*, v^*) is asymptotically stable given that the eigenvalues are real and negative. Conversely, (u^*, v^*) is deemed unstable if any eigenvalue is positive.
- The main point of origin is a stable node that is asymptotically stable if $\lambda_1 < \lambda_2 < 0$ or $\lambda_2 < \lambda_1 < 0$. Conversely, if $\lambda_1, \lambda_2 > 0$, we possess an unstable node.
- If $\lambda_1.\lambda_2 < 0$, the origin is unstable and is referred to as a saddle point.

Cas II : $\Delta < 0$ then

$$\lambda_{1,2} = \frac{\operatorname{tr}(J) \pm i\sqrt{-\Delta}}{2}, (\lambda_{1,2} \text{ are complex conjugates })$$

- The origin is asymptotically stable if $\lambda_{1,2} = \alpha \pm \beta i$ with $\alpha < 0$; conversely, the origin is unstable if $\alpha > 0$.
- If $\lambda_{1,2} = \pm\beta i$, where $\beta \neq 0$, and $\alpha = 0$, then the origin is stable but not asymptotically stable. The equilibrium point that is called a center.

When A is nonsingular, the equilibrium (u^*, v^*) can be classified according to the following table.

Eigenvalues	Equilibrium
$\lambda_{1,2} \in R, \lambda_1, \lambda_2 < 0$	Asymptotically stable node
$\lambda_{1,2} \in R, \lambda_1, \lambda_2 > 0$	Unstable node
$\lambda_{1,2} \in R, \lambda_1.\lambda_2 < 0$	Unstable saddle
$\lambda_{1,2} = \alpha \pm \beta i, \alpha < 0$	Asymptotically stable focus
$\lambda_{1,2} = \alpha \pm \beta i, \alpha > 0$	Unstable focus
$\lambda_{1,2} = \pm\beta i$	Stable center

In addition to the classification of solutions for planar linear systems, which are determined by the eigenvalues of the system's matrix (2.3), a plane characterized by (tr, det) can also incorporate this classification. The solutions corresponding to the characteristic equation (2.5) are the eigenvalues of the matrix J.

2.2.3 Theories of local stability in case PDEs

A prevalent technique for examining the local asymptotic stability of systems of partial differential equations is the eigenfunction expansion approach [8]. Examining the theoretical underpinnings of the eigenvalue associated with the Laplace operator is crucial.

2.2.3.1 Characteristics of the Eigenvalues of the Laplace Operator

We represent these eigenvalues as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ and the associated normalized eigenfunctions in Ω as $\Phi_0, \cdots, \Phi_k, \cdots$. We implement Neumann conditions for boundaries. The eigenvalue issue is fulfilled by the value eigenvalue and eigenfunctions

$$-\Delta\Phi_k = \lambda_k\Phi_k,$$

in Ω, with $\frac{\partial \Phi_k}{\partial v} = 0$ on the boundary of Ω, and

$$\|\Phi_k\|_2 = \int_\Omega \Phi_k^2(x)dx = 1.$$

2.2.3.2 Local Stability

In general, the system (1.1) consists of two components, each of which is represented by a linearized reaction in the form

$$\frac{\partial}{\partial t}U - D\Delta U = J_0 U.$$

In the simplest scenario, D is presumed to be diagonal, and cross-diffusion is disregarded, i.e.,

$$D = \begin{pmatrix} d_u & 0 \\ 0 & d_v \end{pmatrix}.$$

Components u and v are represented by the diffusion rate constants d_u and d_v, respectively. The corresponding system of ordinary differential equations assessed at the equilibrium point possesses a Jacobian matrix referred to as J_0. Let the linearizing operator be denoted as $L = J_0 U + D\Delta U$. Consider the eigenfunction $(\phi(x), \psi(x))$ of L, which is associated with the eigenvalue ξ. We possess

$$L(\phi(x), \psi(x))^t = \xi(\phi(x), \psi(x))^t,$$

resulting in

$$(L - \xi I)\binom{\phi}{\psi} = \binom{0}{0}.$$

For the sake of simplicity, let us designate

$$\begin{cases} \phi = \sum_{0\le i\le\infty, 1\le j\le m_i} a_{ij}\Phi_{ij}, \\ \psi = \sum_{0\le i\le\infty, 1\le j\le m_i} b_{ij}\Phi_{ij}. \end{cases}$$

We are now able to write $(\phi(x), \psi(x))^T = \sum_{0\le i\le\infty, 1\le j\le m_i} (a_{ij}, b_{ij})^T \Phi_{ij}$.
This may be restructured into the form

$$\sum_{0\le i\le\infty, 1\le j\le m_i} (J_0 - \lambda_i D - \xi I)\binom{a_{ij}}{b_{ij}}\Phi_{ij} = \binom{0}{0}.$$

If every eigenvalue of L has negative real components, the equilibrium point is locally asymptotically stable.

Theorem 5. *[8]*

- *If the eigenvalue of $J_0 - \lambda_i D$ possesses negative real parts associated with every positive integer i, then the equilibrium value of (1.1) is asymptotically stable globally. Furthermore, there exist supplementary positive constants K and ω such that for any $t > 0$ Moreover, there are positive constants K and ω such that*

$$U(t,x) \leq Ke^{-\omega t} \|U_0\|$$

holds for all $t > 0$.

- *For any nonnegative value i, the equilibrium is stable if the eigenvalues of $J_0 - \lambda_i D$ have nonpositive real parts, with those exhibiting zero real parts indicating basic fundamental divisors.*

- *An equilibrium is considered unstable if an eigenvalue of $J_0 - \lambda_i D$ occurs with a nonsimple elementary divisor and a positive real component or a zero real component.*

2.2.4 Stability in the sense of Lyapunov

Methods that use Lyapunov functions to evaluate system stability, such as the LaSalle invariance notion, have been used since the 1960s. Nonetheless, these concepts cannot be directly implemented in time-varying systems because of the non-invariant nature of such systems. A straightforward stability criterion for time-varying systems is proposed to overcome this issue. Limit systems are developed in lieu of the ω-limit set, and two detectability conditions are articulated in relation to limit systems. The bounded solutions of the dynamic system are demonstrated to converge towards a predetermined equilibrium set. The relationship between the suggested system and the concepts of integral invariance and the LaSalle invariance principle is investigated. This method is relevant for assessing the stability of time-varying systems.

2.2.4.1 The Direct Lyapunov Method

The Lyapunov direct method, as defined below, was developed by Russian mathematician Aleksandr Lyapunov in the early 1900s and is an essential and powerful instrument for assessing global asymptotic stability. For further information on the approach, consult, for example, [6].

Definition 10. *[6] A real-value function $V \in C^1(\Omega, \mathbb{R})$ is classified as a Lyapunov function if:*

$$U \in \Omega, U \neq U^*, V(U) > V(U^*),$$

$$\frac{dV(U(t))}{dt} \leq 0, \ \textit{for every } U \in \Omega.$$

Here, $U^* \in \mathbb{R}^n$ represents the state of equilibrium point of the reaction-diffusion system (2.2).

Definition 11 (Derivative of a Lyapunov Function Along Solutions [51])**.** *Let*

$$V : D \to \mathbb{R}$$

be a continuously differentiable function, and consider the nonlinear system

$$\dot{x} = f(x).$$

The derivative of V along the trajectories of the system is denoted by $\dot{V}(x)$ and is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = \nabla V(x)^T f(x).$$

2.2.4.2 Lyapunov stability

Stability, by definition, signifies that a system in equilibrium will maintain that state despite variations in external variables. Analyzing a system's pathways when its initial state is close to an equilibrium state is known as Lyapunov stability analysis. The objective of stability is to derive conclusions regarding the system's behavior without explicitly computing its trajectories in order to examine classical results pertaining to the concept of stability in the Lyapunov sense.

Theorem 6. *(Theorem of Lyapunov Stability)[6]*

- *In the reaction-diffusion system (2.2), a Lyapunov function exists, indicating that u^* is stable.*
- *For each $u \neq u^*$, u^* is asymptotically stable if $\frac{dV(u(t))}{dt} < 0$.*

2.2.4.3 Positive definite (semi-definite) function

Definition 12. *A function that is continuously differentiable [36] In the region $\Omega \subset R^n$ that includes the origin, the transformation $V : R^n \to R$ is considered positive definite if:*

a) $V(0) = 0$

b) $V(u) > 0,$ *and* $u \neq 0.$

A function is deemed positive semi-definite if the condition "b" is substituted by $V(u) \geq 0$.

2.2.4.4 Lyapunov theorem

Theorem 7. *Regarding autonomous systems [36]. Let $\Omega \subset R^n$ be a domain that encompasses the equilibrium point at the origin. A continuously differentiable positive determined function $V : \Omega \to \mathbb{R}$ satisfies the following equation:*

$$V_t = \frac{\partial V}{\partial u}\frac{du}{dt} = \frac{\partial V}{\partial u} f(u) = -W(u).$$

If Ω is not positive semi-definite, the point of equilibrium at 0 is stable. Moreover, the point of equilibrium is asymptotically stable when $W(u)$ is positively specific. Furthermore, if $D = R^n$ and V is radially unbounded, defined as

$$\|u\| \to \infty \Rightarrow V(u) \to \infty$$

It is a globally asymptotically stable origin. The following reduced form of LaSalle's theorem can be used to ensure asymptotic stability in autonomous systems even if $W(u)$ in the previously discussed theory is only positive semi-definite.

2.2.4.5 Lyapunov theorem for global asymptotic stability

The Lyapunov function $V(u)$ delineates the region in the state space where our prior conclusions hold. The collection of beginning circumstances resulting in paths converging at this juncture, known as the "area of attraction" of an asymptotically stable equilibrium point, holds considerable significance. A point of equilibrium is asymptotically global if its region of attraction includes the whole state space. The equilibrium point at the origin is globally asymptotically stable if a function $V(u)$ is positively determined through every point in the state space, meets the criterion that $|V(u)| \nearrow \infty$ as $\|u\| \nearrow \infty$, and possesses a derivative V_t that is negative certain in the all state space.

2.2.4.6 Global stability definitions

Lemma 2. *[24] If the positive limit set ω-limit of a bounded solution $u(t)$ of (2.2) is compact, invariant, and belongs to Ω for $t \geq 0$, then it is a nonempty set. Additionally,*

$$\lim_{t\to\infty} u(t) \to \omega\text{-limit}.$$

where

$$\omega\text{-limit} = \left\{ p \in \Omega \subset R^n : \exists t_n \to \infty, \lim_{n\to\infty} x(t_n) = p. \right\}$$

2.2.4.7 LaSalle's invariance principle theorem

Theorem 8. *When it comes to autonomous systems [36]. Let $D \subset R^n$ be a domain that includes the equilibrium point at the origin. Let there exist a positive determined function $V : D \to \mathbb{R}$ that is continuously differentiable and fulfills the the necessities*

$$V_t = \frac{\partial V}{\partial x} f(x) = -W(x) \leq 0$$

in D. Let $S = \{x \in D \mid V_t(x) = 0\}$, with the condition that no solution can stay invariant in S, unless the origin. The origin is asymptotically stable. Additionally, provided that $D = R^n$ and V are regionally boundless, the point of origin is asymptotically stable globally.

2.2.4.8 Relation betwen Lyapunov theory and LaSalle's theorem

Lyapunov's second theorem leads to the global asymptotic stability of the origin when $\frac{\partial}{\partial t} V(u)$ is negative definite. When $\frac{\partial}{\partial t} V(u)$ is only negative semidefinite, the invariance principle provides a criterion for asymptotic stability.

Theorem 9. *[18] A function $V \in C^0(\mathbb{R}^n, \mathbb{R}^+)$ that meets the subsequent criteria:*

- *For every $u \in \mathbb{R}^n$, $V(u) \geq 0$, and $V(0) = 0$.*
- *For every $u \in \mathbb{R}^n$, $\frac{\partial}{\partial t} V(u) \leq 0$.*
- *The value 0 at S^* is globally asymptotically stable, with S^* being the largest positively invariant set in $S = \left\{u \in R^n : \frac{\partial}{\partial t} V(u) = 0\right\}$.*
- *The system (2.2) has bounded solutions.*

As a result, the origin is asymptotically stable globally.

2.3 The CIMA Reaction-Diffusion model

A. Gierer and H. Meinhardt successfully expanded the theoretical modeling of hydra regeneration [15]. Subsequent mathematical verification demonstrated its capacity to generate peaks and spikes. (see, for instance, [34]). Turing's proposal has been effectively developed based on theoretical foundations and has attracted considerable interest from many scholars.Epstein [14] has examined it within the domains of semiconductor physics, economics, and stellar formation, as well as biology and chemistry. However, it was challenging to identify Turing patterns in honest chemical or biological systems. In the early 1990s, De

Kepper et al. investigated the chlorite-iodide-malonate acid (CIMA) reaction.DeKepper [21] delineated the formation of stationary three-dimensional formations that are fundamentally two-dimensional. Nearly 40 years subsequent to the publication of [48], this was the inaugural experimental validation of Turing patterns. The CIMA reaction involves five reactants, challenging the mathematical comprehension of the system. Epstein and Lengyel The original system's simplification successfully produced the Lengyel-Epstein model, a two-dimensional framework. References: [25, 26]. Pojman and Epstein [14] offer a comprehensive historical account of the development of the CIMA reaction model and its associated experiments. Three chemical reaction pathways clarify the CIMA reaction:

$$\begin{cases} MA + I_2 \to IMA + I^- + H^+, \\ CIO_2 + I^- \to \frac{1}{2} I_2 + CIO_2^-, \\ CIO_2^- + 4I + 4H^+ \to CI^- + 2I_2 + 2H_2O. \end{cases}$$

The actual rate laws related to these processes were analyzed, and constant components were disregarded to simplify the response model to the standard Lengyel-Epstein model.This model makes use of two independent variables, U and V, which stand for the concentrations of iodide (I^-) and chlorite (CIO_2). This model is derived from two differential equations. Converting the reactions into differential equations is challenging, and studies must balance the rates of change in different concentrations. Let U represent the concentration of iodide and V represent the concentration of chlorite. Consequently, the suggested rate equations are labeled as

$$\begin{cases} \frac{dU}{d\tau} = \frac{k_1 [MA][I_2]}{k_2 [I_2]} - k_4 [CIO_2] U - 4 \left(k_5 [H^+] UV + k_6 [I_2] \dfrac{UV}{\ell + U^2} \right), \\ \frac{dV}{d\tau} = k_4 [CIO_2] U - \left(k_5 [H^+] UV + k_6 [I_2] \dfrac{UV}{\ell + U^2} \right). \end{cases}$$

The rate constants k_1, k_2, k_3, k_4, k_5, and k_6 are specified, with ℓ also being a constant. All particles, including those associated with the method [9], are associated with the rate conditions. The iodide and chlorite concentrations, however, might be hypothesized to change more quickly than the other components during the course of the study. In this sense, it makes sense to believe that, provided the reaction is maintained, efficacy is coherent with the aggregation of diverse molecules. This assumption does not need to be used any further. But we have to choose to use real values in any mathematical models. After choosing the best course for sustainability, the k_5 term will be ignored, as stated in Lengyel and Epstein (1991)) . This may be ascribed to the comparatively low concentration of hydrogen ions H^+, the lightest nucleus. The subsequent system of differential equations arises from the aforementioned hypotheses:

$$\begin{cases} \frac{dU}{d\tau} = m - nU - 4p\left(\dfrac{UV}{\ell + U^2}\right), \\ \frac{dV}{d\tau} = pU - p\left(\dfrac{UV}{\ell + U^2}\right), \end{cases}$$

where $m, n, p > 0$. Through the implementation of a simple normalization and consideration of the spatial diffusion of substances, we derive the reaction-diffusion system.

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + a - u - \frac{4uv}{1+u^2}, \\ \frac{\partial v}{\partial t} = (\sigma b)\Delta v + (\sigma c)\left(u - \frac{uv}{1+u^2}\right). \end{cases} \tag{2.7}$$

The Lengyel-Epstein model is defined as follows with the completion of all operations, as per Jang2004 [19]:

$$\begin{cases} \frac{du}{dt} = \Delta u + a - u - \dfrac{4uv}{1+u^2}, \\ \frac{dv}{dt} = \sigma\left[c\Delta v + b\left(u - \dfrac{uv}{1+u^2}\right)\right], \\ u(x,0) = u_0 > 0, \quad v(x,0) = v_0 > 0, \\ \partial_\nu u(x) = \partial_\nu v(x) = 0, \end{cases} \tag{2.8}$$

where $\Omega \subset \mathbb{R}^n$ denotes a limited domain with a regular border $\partial\Omega$, often for $n = 2$ or 3. The activator iodide (I^-) is denoted by the corresponding chemical quantities $u = u(x,t)$ and $v = v(x,t)$ within that context. For $t > 0$ and point $x \in \Omega$, the inhibitor chlorite $\left(ClO_2^-\right)$ is extant. a, b, and c are all strictly positive constants. The authors in [3] provide a thorough derivation of the two-component model, beginning with reaction laws and elucidating the associated processes, while using CIO_2, I_2, and MA as constants. Ultimately, more flexible requirements for asymptotic stability were established in [5].

2.3.1 Turing Instability

One of the initial applications of reaction-diffusion systems in research is the elucidation of pattern development in natural organisms, such as the spots on a leopard's skin. Comprehending the emergence and organization of these patterns, referred to as morphogenesis, holds substantial importance for both scientists and physicists alike. Alan Turing (1912-1954), the British mathematician, is recognized as a key pioneer of the concept of pattern generation. In 1952, he examined the concept of morphogenesis and linked it to reaction-diffusion systems. The overarching concept he proposed can be encapsulated in the subsequent points:

- Active properties within the natural cell are responsible for stimulating the production and commencement of synthetic agents known as morphogenesis.

- Chemical processes are inadequate for pattern generation due to their high symmetry.
- Instabilities permitted by the diffusion of chemical agents are the primary impetus for primer pattern creation. The fundamental patterns thereafter undergo specific developments due to the reaction process.

The second issue that becomes of significance is whether diffusion may possibly destabilize a system. Once again, that response is indeed, and this is the element that Turing favored as the primary catalyst for pattern formation. DeKepper recognized the Chlorite-Iodide Malonic-Acid (CIMA) reaction in 1990, [22] a period during which Turing's proposal was considered somewhat extreme and remained an unverified theory. An interesting generic term for the diffusion-driven instability known as "Turing Instability" is provided here.

Definition 13. *A diffusion-driven instability, also known as Turing instability, arises when a steady state, stable without diffusion, becomes unstable in the presence of diffusion.*

2.3.2 Activator-Inhibitor Nature

This subsection may be found at [11]. Typically, a reaction-diffusion system reliant on diffusion-driven instability may be categorized into one of two groups of systems that demonstrate distinct behaviours: activator-inhibitor and positive input. Section 7.8 of [13] contains a concise and informative review of the properties of these two classes. Essentially, when we analyze the signs of the Jacobian components evaluated in a specific steady state, we get two types of matrices of the form

$$\begin{pmatrix} + & - \\ + & - \end{pmatrix} \text{ or } \begin{pmatrix} + & + \\ - & - \end{pmatrix}$$

The activator-inhibitor feature is crucial due to the scientific rationale underlying pattern development. Science is especially focused on pattern formation, notably in morphogenesis, which we previously identified as the impetus for Turing's groundbreaking work. Science generates patterns from spatially homogeneous states. For instance, little zebras begin with uniform skin pigmentation and subsequently generate a variety of examples. This has attracted the attention of researchers and, as a result of its expansion, applied mathematicians. The primary motivation behind pattern formation is the activator-inhibitor property, given the current state of affairs. Turing's instability denotes a situation when the inhibitor diffuses at a rate exceeding that of the activator by a defined margin, generally larger than 10. The experimental execution of Turing-type chemical reactions was frustrating due to the significant difference in diffusivities. This is the primary an argument for including the starch indicator into the solution structure for the CIMA reaction. It is crucial to acknowledge that Turing's instability is insufficient for the formation of patterns. In order to undermine the firm positive input, specific nonlinearities in the reaction terms are necessary. Additional information regarding pattern formation is available in [11].

2.3.3 Local stability in the ODE sense

Multiple writers who have examined the Lengyel-Epstien model have expressed interest in its local stability [30],[52],[53]. By excluding the diffusion terms in system Lengyel-Epstien, we obtain the ODE system

$$\begin{cases} u_t = a - u - \frac{4uv}{1+u^2} = f(u,v), \\ v_t = \sigma b\left(u - \frac{uv}{1+u^2}\right) = \sigma b g(u,v). \end{cases} \tag{2.9}$$

The following lemma demonstrates that the positive constant (u^*, v^*) of (2.9) is locally asymptotically stable under specific conditions.

Lemma 3. *If condition $3\alpha^2 - 5 < \sigma b\alpha$ is satisfied, then the system (2.9) has a locally asymptotically stable equilibrium point (u^*, v^*).*

Proof. System (2.9) possesses a unique steady-state solution

$$(u^*, v^*) = \left(\alpha, 1+\alpha^2\right),$$

where

$$\alpha = \frac{a}{5}.$$

Currently, we give some results derived from [35]. The Jacobian matrix assessed at (u^*, v^*) for (2.9) is

$$J(u^*, v^*) = \begin{pmatrix} f_0 & f_1 \\ \sigma b g_0 & \sigma b g_1 \end{pmatrix},$$

where

$$f_0 = \frac{3\alpha^2-5}{1+\alpha^2}, \quad f_1 = -\frac{4\alpha}{1+\alpha^2}, \quad g_0 = \frac{2\alpha^2}{1+\alpha^2}, \quad g_1 = -\frac{\alpha}{1+\alpha^2}. \tag{2.10}$$

Note that $f_v(u^*, v^*) < 0$, $g_u(u^*, v^*) > 0$ and $g_v(u^*, v^*) < 0$. If

$$f_u(u^*, v^*) = \frac{3\alpha^2 - 5}{1+\alpha^2} > 0 \tag{2.11}$$

We identify the system (2.8) as an activator-inhibitor system, u as an activator, and v as an inhibitor. If

$$3\alpha^2 - 5 < \sigma b\alpha, \tag{2.12}$$

then the equilibrium of the ODE system (2.9) is asymptotically stable (see [35]). Combining (2.11) and (2.12), we conclude the condition holds

$$0 < 3\alpha^2 - 5 < \sigma b\alpha. \tag{2.13}$$

□

2.3.4 Local stability in the PDE sense

The linearization of the Lengyel-Epetein reaction-diffusion system (2.8) at the equilibrium point $(u^*, v^*) = (\alpha, 1 + \alpha^2)$ may be expressed as

$$LU = \frac{DU}{Dt} = D\Delta U + J_0 U. \tag{2.14}$$

The Jacobian matrix of the ODE system at equilibrium is represented as J_0, whilst the diffusion coefficient matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma c \end{pmatrix} \tag{2.15}$$

is designated as D. In accordance with conventional linear operator theory [8], the model (u^*, v^*) is asymptotically stable if all eigenvalues of the operator L have non-positive real components. The model (u^*, v^*) is considered unstable if all eigenvalues have positive real components. Let $(\phi(x), \psi(x))$ be an eigenfunction of L with the eigenvalue λ. Considering the equation

$$\begin{pmatrix} \Delta + \frac{3\alpha^2-5}{1+\alpha^2} - \lambda(\zeta_i) & -\frac{4\alpha}{1+\alpha^2} \\ (\sigma b)\frac{2\sigma\alpha^2 b}{1+\alpha^2} & \sigma c\Delta + (\sigma b)\frac{-\sigma\alpha b}{1+\alpha^2} - \lambda(\zeta_i) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.16}$$

with

$$\phi = \textstyle\sum_{0\le i\le\infty, 1\le j\le m_i} a_{ij}\Phi_{ij} \quad \text{and} \quad \psi = \textstyle\sum_{0\le i\le\infty, 1\le j\le m_i} b_{ij}\Phi_{ij},$$

we make the assumption that

$$\Gamma_k = J_0 - Dk^2, (k = 1, 2, 3, \ldots).$$

For $k = 0, 1, 2, \ldots$, the eigenvalues of the matrix Γ_k correspond to the eigenvalues of the matrix L. The essential equation for Γk is articulated as $\lambda^2 - P_k\lambda - Q_k = 0$, where

$$\begin{aligned} P_k &= \operatorname{tr}(J)\,\Gamma_k = \operatorname{tr}(J)_0 - (1 + \sigma c)\,k^2 && (2.17) \\ &= \frac{3\alpha^2 - 5 - \sigma\alpha b}{1 + \alpha^2} - (1 + \sigma c)\,k^2 && (2.18) \end{aligned}$$

and

$$Q_k = \det \Gamma_k = \det J_0 - k^2 \left(\frac{(\sigma b)\, 3\alpha^2 - 5}{1+\alpha^2} - \frac{\sigma \alpha b}{1+\alpha^2} \right) + (\sigma b)\, k^4 \tag{2.19}$$

$$= (\sigma b)\, k^4 - k^2 \sigma \left(\frac{b\,(3\alpha^2 - 5)}{1+\alpha^2} - \frac{\alpha b}{1+\alpha^2} \right) + \frac{5\sigma \alpha b}{1+\alpha^2}. \tag{2.20}$$

Consider the system described by equation

$$\begin{cases} u_t = \frac{\partial^2 u}{\partial^2 x} a - u - \frac{4uv}{1+u^2}, \\ v_t = \sigma \left[c \frac{\partial^2 u}{\partial^2 x} + b \left(u - \frac{u}{1+u^2} \right) \right], \end{cases} \tag{2.21}$$

with the no-flux boundary condition in a one-dimensional space across the interval $]0, \pi[$. Additionally, the no-flux boundary condition

$$u_x(0,t) = v_x(\pi,t) = v_x(0,t) = u_x(\pi,t)$$

is satisfied by this system.

Theorem 10. *Assume that* $b > b_0 = \frac{(3\alpha^2 - 5)}{\sigma \alpha}$, *which signifies that* (u^*, v^*) *is a locally asymptotically stable equilibrium for* (2.9) *. If*

$$\alpha^2 > 3 \text{ and } c > \frac{3\alpha b}{\alpha^2 - 3},$$

then (u^*, v^*) *is an unstable equilibrium solution of* (2.21) *. Conversely, if*

$$\begin{cases} \frac{5}{3} < \alpha^2 \leq 3, \\ \qquad or \\ \alpha^2 < 3 \text{ and } 0 < c \leq \frac{3\alpha b}{\alpha^2 - 3}, \end{cases}$$

then (u^*, v^*) *constitutes a locally asymptotically stable equilibrium solution of* (2.21) *.*

2.3.5 Global asymptotic stability

The authors investigate the global stability of (u^*, v^*) assuming that the iodide feeding rate is less than $\sqrt{27}$, without imposing any restrictions on the other parameters. They provide enough requirements for the global asymptotic stability of the singular constant steady state see [30], [35], [53].

Theorem 11. *[30] The equilibrium point* (u^*, v^*) *is globally asymptotically stable for the ODE system* (2.9)

$$\sigma b > \left(\frac{32}{25} + a^2 \right) - 4.$$

Theorem 12. *Assume that $0 < a^2 < 27$. The solution $(u(x), v(x))$ of the system (2.8) for each nonnegative starting function $u_0(x), v_0(x) \in C^2(\Omega) \cap C\left(\bar{\Omega}\right)$, converges uniformly to (u^*, v^*) in x. The comprehensive demonstration of Theorem 11 mentioned before may be located in [53]. It is important to mention that the proof relies on the Lyapunov approach. The authors prove that the function is equal to*

$$E(t) = \int_{\Omega} \left[\sigma b \int \left(u^2 - (u^*)^2\right) du + 4 \int (v - v^*)\, dv\right] dx. \tag{2.22}$$

Lemma 4. *If $0 < a$, then the equilibrium solution (u^*, v^*), which is globally asymptotically stable, is the only possible solution of equation (2.8).*

Theorem 13. *Suppose there is $a^2 \leq \frac{125}{4}$. Subsequently, by considering any solution (u, v) to equation (2.8), we obtain equation*

$$\lim_{t \to \infty} |u(\cdot, t) - u^*|_{L^2(\Omega)} = \lim_{t \to \infty} |v(\cdot, t) - v^*|_{L^2(\Omega)} = 0.$$

In order to demonstrate this theorem, we employ a novel Lyapunov function denoted as

$$E(t) = \int_{\Omega} \left[(\sigma b)(u - u^*) \frac{(u + 2u^*)}{3} + 2 (v - v^*)^2\right] dx.$$

Chapter 3

The fundamental characteristics and non-existence results

This chapter[1] discusses the fractional Lengyel-Epstein fractional Laplacian model, in particular its essential properties and non-existence results.

3.1 Lengyel-Epstein fractional Laplacian model

In 1744, A. Trembley [47] discovered the regeneration process of hydra, a prominent and early example of morphogenesis. To replicate the emergence of biological patterns and notable forms and events. In 1952, A. Turing [48] established the notion of "diffusion-driven instability." Turing contended that heterogeneous distributions of reactants may arise from differing diffusion rates in a system comprising two interacting substances. Ni and Tang [35] executed a notable investigation on the Lengyel-Epstein system. The study's authors have shown through experimental and statistical approaches that the system cannot sustain non-constant stable states when the effective diffusion rate, reactor size, or initial reactant concentrations are inadequately big. In 2008, Yi et al. Conducted research that explicitly outlined the system properties necessary to determine if the spatially homogeneous equilibrium and the spatially homogeneous periodic solutions exhibit Turing instability or diffusive instability. This work exhibits scientific inquiry. In 2009, Yi et al. [53] demonstrated that the constant equilibrium solution is globally asymptotically stable by the application of a certain Lyapunov functional. Du and Wang [12] presented a range of spatially nonhomogeneous periodic solutions, whereas Jin et al. [20] identified different bifurcation parameters for the same model, in accordance with [12]. In 2013, Wang and Zhao [49] employed the Hopf bifurcation theorem to determine the essential conditions for the stability of equilibrium points. They illustrated their conclusions using numerical examples. In 2014, Lisena conducted a study [30] that examined the dynamics of the Lengyel-Epstein system and relaxed the criteria for the global asymptotic stability of the steady-state solution. Shoji and Ohta [44] introduced computer simulations of three-dimensional Turing patterns derived from the Lengyel-Epstein model in 2015. The preliminary effort to improve the model in [2] was successful, showcasing a comprehensive examination of several approximations of the response term. The authors of [4] recently conducted a reanalysis of the same model, in which they proved sufficient conditions for the nonexistence of Turing patterns. Mansouri [32] analyzed the time-fractional Lengyel-Epstein system, establishing the necessary conditions for the local and global asymptotic stability of the unique equilibrium. The authors in [31] utilize the following model:

$$\begin{cases} u_t = \nabla^\gamma u + a - u - \dfrac{4uv}{1+u^2}, \\ v_t = d\nabla^\gamma v + b\left(u - \dfrac{uv}{1+u^2}\right), \end{cases}$$

where ∇^γ is the fractional Riesz operator for $1 < \gamma < 2$, as defined as

$$\nabla^\gamma u = \frac{\partial^\gamma u}{\partial |x|^\gamma} + \frac{\partial^\gamma u}{\partial |y|^\gamma} = -\frac{1}{2\cos\left(\frac{\pi\gamma}{2}\right)}({}_{RL}D^{\gamma}_{-\infty,x}u + {}_{RL}D^{\gamma}_{x,+\infty}u) - \frac{1}{2\cos\left(\frac{\pi\gamma}{2}\right)}({}_{RL}D^{\gamma}_{-\infty,y}u + {}_{RL}D^{\gamma}_{y,+\infty}u).$$

According to

$$F\left(-(-\Delta)^{\frac{\gamma}{2}}u\right)(\xi) = -|\xi|^\gamma Fu(\xi), \quad \xi \in R^n, \quad \gamma > 0.$$

The anomalously diffusive operator $(-\Delta)^{\frac{\gamma}{2}}$ will be defined. If $0 < \gamma < 1$, it is classified as subdiffusion; if $1 < \gamma < 2$, it is characterized by the Fourier transform, as indicated in [41], and is termed superdiffusion. In the fundamental scenario when $\gamma = 2$, the operator $(-\Delta)^{\frac{\gamma}{2}}$ is equivalent to the conventional diffusion operator (Laplacian operator). Therefore, we can articulate

$$F\left(\nabla^{\frac{\gamma}{2}}u\right)(\xi) = -|\xi|^\gamma Fu(\xi), \quad \xi \in R^n, \quad \gamma > 0.$$

The following is the fractional Lengyel-Epstein regional fractional laplacian model after all of these operations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\left(-\Delta\right)^s_\Omega u + a - u - \dfrac{4uv}{1+u^2}, \\ \dfrac{\partial v}{\partial t} = \sigma\left[-c\left(-\Delta\right)^s_\Omega v + b\left(u - \dfrac{uv}{1+u^2}\right)\right], \\ u(x,0) = u_0 > 0, \quad v(x,0) = v_0 > 0, \\ \partial_\nu u(x) = \partial_\nu v(x) = 0, \qquad x \in \partial\Omega \end{cases} \tag{3.1}$$

[1]*Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

where Ω is an open bounded domain in $\mathbb{R}^n (n \geq 1)$ with a Lipschitz continuous border $\partial\Omega$, specifically for $n = 2$ or 3 in practical applications. The eigenvalue problem for the fractional Laplacian operator $(-\Delta)^s_\Omega$ is expressed as (see [27])

$$\begin{cases} (-\Delta)^s_\Omega \varphi_i = \lambda^N_{i,s}\varphi_i & x \in \partial\Omega, \\ \partial_\nu \varphi_i = 0 & x \in \partial\Omega, \end{cases}$$

with the definition domain of $(-\Delta)^s_\Omega$

$$D\left[(-\Delta)^s_\Omega\right] = \left\{ u \in L^2(\Omega), \|(-\Delta)^s_\Omega\|_{L^2(\Omega)} = \sum_{i=1}^{\infty} \left|\lambda^N_{i,s} \langle u, \varphi_i\rangle\right|^2 < \infty, \partial_\nu u = 0 \right\},$$

where $\lambda^N_{i,s}$ denotes the corresponding eigenvalue of φ_i; φ_i represents the corresponding eigenfunction of $(-\Delta)^s_\Omega$,and $\{\varphi_i\}_{i=1}^{i=\infty}$ forms an orthonormal basis of $L^2(\Omega)$; $0 = \lambda^N_{0,s} < \lambda^N_{1,s} < \lambda^N_{2,s} < ...\lambda^N_{n,s} < ...$that converges to $+\infty$. The expression of $(-\Delta)^s_\Omega u$ for $u \in W^{s,2}_0(\Omega)$ [29, 50] also comes by

$$\begin{aligned} (-\Delta)^s_\Omega u(x) &= C_{n,s} PV \int_\Omega \frac{u(x) - v(y)}{|y-x|^{n+2S}} dy, \\ &= C_{n,s} \lim_{\delta\to 0} \int_{\{y\in\Omega, |y-x|>\delta\}} \frac{u(x) - v(y)}{|y-x|^{n+2S}} dy, \quad \delta > 0, \end{aligned}$$

where PV denotes the Cauchy principal value, and the constant $C_{n,s}$ serves as a normalizing constant

$$C_{n,s} = \frac{2^{2s}\Gamma\left(\frac{n+2s}{2}\right) s}{\pi^{\frac{n}{2}}\Gamma(1-s)}, \quad \text{where} \quad s \in \left]\tfrac{1}{2}, 1\right[.$$

3.2 Invariant region

In this section, we will establish that the system (3.1) possesses an invariant region.

$$\Re =]0, a[\times \left]0, 1 + a^2\right[\tag{3.2}$$

Proposition 1. *For every $x \in \Omega$ and $t > 0$, the initial-boundary value problem (3.1) The equation has a distinct solution (u, v). Furthermore, there exist two positive constants C_1 and C_2 that are solely contingent upon a, u_0, and v_0 to obtain*

$$C_1 < u(x,t), v(x,t) < C_2 \qquad x \in \Omega, t > 0. \tag{3.3}$$

[1] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

Proof. The uniqueness and local presence of solutions for the system (3.1) are examined in reference [27], [7]. We utilize theorem [54] to establish the conditions for both global existence and boundedness. We will build this rectangle

$$\Re =]U_1, U_2[\times]V_1, V_2[\, , \tag{3.4}$$

where

$$\begin{aligned} U_2 &= \max\left\{a, \max_{x\in\overline{\Omega}} u_0(x)\right\}, & V_2 = \max\left\{2+U_2^2, \max_{x\in\overline{\Omega}} v_0(x)\right\}, \\ U_1 &= \min\left\{\frac{a}{1+4V_2}, \min_{x\in\overline{\Omega}} u_0(x)\right\}, & V_1 = \min\left\{\frac{1}{2}, \min_{x\in\overline{\Omega}} v_0(x)\right\}. \end{aligned}$$

The rectangle clearly contains the starting functions $u_0(x)$ and $v_0(x)$. Employing a computation similar to that of [35]. This will demonstrate the vector field.The value is 2.5.

$$\left(\ a - u - \tfrac{4uv}{1+u^2}, \quad \sigma b\left(u - \tfrac{uv}{1+u^2}\right) \ \right). \tag{3.5}$$

When considering the left side, we have the following conditions $u = U_1$ and $V_1 \leq v \leq V_2$. This is expressed as equation

$$\begin{aligned} a - u - \frac{4uv}{1+u^2} &= a - U_1 + \frac{-4U_1 v}{1+U_1^2} = f(U_1, v) && (3.6) \\ &> a - U_1 - 4U_1V_2 = \left(\underbrace{\frac{a}{(1+4V_2)} - U_1}_{\geq 0}\right)(1+4V_2) && (3.7) \\ &\geq 0 && (3.8) \end{aligned}$$

At the right side, where $u = U_2$ and $V_1 \leq v \leq V_2,$ we obtain the value

$$\begin{aligned} a - u - \frac{4uv}{1+u^2} &= a - U_2 + \frac{-4U_2 v}{1+U_2^2} = f(U_2, v) && (3.9) \\ &\leq a - U_2 && (3.10) \\ &\leq 0 && (3.11) \end{aligned}$$

At the lower boundary, when $U_1 \leq u \leq U_2$ and $v = V_1,$ we have equation

$$\begin{aligned} u - \frac{uv}{1+u^2} &= u\left(\frac{u^2 + \overbrace{1 - V_1}^{\geq 0}}{1+u^2}\right) = g(u, V_1) && (3.12) \\ &\geq u\left(\frac{u^2}{1+u^2}\right) && (3.13) \\ &\geq 0 && (3.14) \end{aligned}$$

[1] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

At the upper boundary, when $U_1 \leq u \leq U_2$ and $v = V_2$, we have equation

$$u - \frac{uv}{1+u^2} = u\left(\frac{1+u^2-V_2}{1+u^2}\right) = g(u, V_2) \tag{3.15}$$

$$\leq u\left(\frac{\overbrace{1+U_2^2-V_2}^{\leq 0}}{1+u^2}\right) \tag{3.16}$$

$$\leq 0 \tag{3.17}$$

Therefore, for the system (3.1), the area $\Re$ remains positively invariant. The solutions exhibit their comprehensiveness and distinctiveness. At last, we have the ability to select two constants, labeled as C_1 and C_2, respectively:

$$C_1 = \min\{U_1, V_1\}, \quad C_2 = \max\{U_2, V_2\}.$$

□

Proposition 2. *Let $u(x,t)$ and $v(x,t)$ denote the distinct solutions of the systems (3.1). Therefore, for any $x \in \overline{\Omega}$:*

$$\limsup_{t\to\infty} u < a \quad \textit{and} \quad \limsup_{t\to\infty} v < 1 + a^2$$

Proof. For all $x \in \overline{\Omega}$ and $t > 0$, Proposition 1 asserts the existence of a positive value ε such that

$$\varepsilon < \frac{4uv}{1+u^2}.$$

Analyze the Cauchy problem that has a unique solution. $\tilde{u} = \tilde{u}(t)$:

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = \tilde{a} - \tilde{u}, \\ \tilde{u}(0) = 2\max_{x\in\overline{\Omega}} u_0(x), \end{cases} \tag{3.18}$$

with:

$$\tilde{a} = a - \frac{\varepsilon}{2}.$$

Let $\hat{u} = u - \tilde{u}$. From equations (3.1) and (3.18), it follows that by $u = \hat{u} + \tilde{u}$ and $a = \tilde{a} + \frac{\varepsilon}{2}$ we have

$$\begin{aligned} \frac{d\hat{u}}{dt} + \tilde{a} - \tilde{u} &= -(-\Delta^s)\,\hat{u} + a - u - \frac{4uv}{1+u^2} \\ \frac{4uv}{1+u^2} + \tilde{a} - a + \underbrace{u - \tilde{u}}_{=\hat{u}} &= -\hat{u}_t - (-\Delta^s)\,\hat{u} \end{aligned}$$

$$\begin{cases} -\hat{u}_t - (-\Delta)^s_\Omega \hat{u} - \hat{u} = \dfrac{4uv}{1+u^2} + \tilde{a} - a > 0, \\ \quad \hat{u}(x,0) < 0. \end{cases} \tag{3.19}$$

This unambiguously results in $\hat{u}(x,t) < 0$ for any $t > 0$ and $x \in \overline{\Omega}$. Undoubtedly, the subsequent argument relies on a contradiction. Assume the statement to be untrue. Therefore, a value of $T > 0$ exists such that $\hat{u}(x,t) < 0, \quad \forall (x,t) \in \overline{\Omega} \times]0,T[$ and $\hat{u}(x,T) = 0$ for a certain $x \in \overline{\Omega}$. Through the application of continuity, we have attained

$$\max_{x\in\overline{\Omega}} \hat{u}(x,T) = 0.$$

If there is an $x_1 \in \Omega$ such that $\hat{u}(x_1,T) = 0$, then it follows that

$$\hat{u}_t(x_1,T) \geq 0,$$

and

$$(-\Delta)^s_\Omega \hat{u}(x_1,T) \geq 0.$$

This implies that

$$-\hat{u}_t(x_1,T) - (-\Delta)^s_\Omega \hat{u}(x_1,T) - \hat{u}(x_1,T) \leq 0, \tag{3.20}$$

as stated in (3.19) and (3.20), leads to a contradiction. If the function $\hat{u}(x,T) < 0$, then all $x \in \Omega$. Subsequently, there would exist a certain $x_1 \in \partial\Omega$ such that $\hat{u}(x_1,T) = 0$. Next, we have

$$-\hat{u}_t(x_1,T) - (-\Delta)^s_\Omega \hat{u}(x_1,T) - \hat{u}(x_1,T) > 0. \tag{3.21}$$

By the principle of continuity, there is a compact subset $\Omega' = B(x_1) \backslash \{x_1\}$ in Ω with $\partial\Omega' \cap \partial\Omega = \{x_1\}$. Furthermore, $\hat{u}(x,T) < 0$, for all $x \in \Omega'$. The equation (3.19) may be expressed as

$$-(-\Delta)^s_\Omega \hat{u} = \hat{u} + \frac{4uv}{1+u^2} + \tilde{a} - a > 0, \quad \text{in } \Omega' \times \{T\}.$$

Therefore, we get $(-\Delta)^s_\Omega(-\hat{u}) > 0$ and $-\hat{u} > 0, \quad \text{in } \Omega' \times \{T\}$. Next, using the Hopf lemma [1], we find that $\partial_v \hat{u}(x_1,T) < 0$, which contradicts (3.1). Therefore,

$$u(x,t) < \tilde{u}(t) \quad \text{for all } t > 0 \text{ and } x \in \overline{\Omega}. \tag{3.22}$$

The comparison result between $v = v(x,t)$ and $\tilde{v} = \tilde{v}(t)$ is displayed below. Let C_1 denote the constant referenced in Proposition 1, and let $\varepsilon_0 > 0$ represent a positive value. The function $\widetilde{v}(t)$ is uniquely defined as the solution to the initial value problem

$$\begin{cases} \dfrac{d\tilde{v}}{dt} = \sigma b \tilde{g}(\tilde{u},\tilde{v}), \\ \tilde{v}(0) = 2\max_{x\in\overline{\Omega}} v_0(x), \end{cases} \tag{3.23}$$

[1] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

where

$$\tilde{g}\left(\tilde{u},\tilde{v}\right)=\sup_{C_1<\varepsilon<\widetilde{u}}\left(\varepsilon-\frac{\varepsilon\left(\tilde{v}-\varepsilon_0\right)}{1+\varepsilon^2}\right),$$

and

$$\tilde{a}^2+\varepsilon_0<a^2.$$

Examine the function $\hat{v}=v-\tilde{v}$. Thus, for any $x\in\overline{\Omega}$, $\hat{v}(x,t)<0$. This unequivocally indicates that $\hat{v}\left(x,t\right)<0$ for every $t>0$ and $x\in\overline{\Omega}$. We will, indeed, employ a contradiction argument. Presume that this is fabricated. Subsequently, there exists a $T>0$ such that $\hat{v}\left(x,t\right)<0,\quad\forall\left(x,t\right)\in\overline{\Omega}\times\left]0,T\right[,$ and $\hat{v}\left(x,t\right)=0$ for some $x\in\overline{\Omega}$. It is possible to establish $\max_{x\in\overline{\Omega}}\hat{v}\left(x,T\right)=0.$ through the application of continuity. Consequently, if $x_1\in\Omega$ is such that $\hat{v}\left(x_1,T\right)=0$, then

$$\hat{v}_t\left(x_1,T\right)\geq 0,$$

and

$$\left(-\Delta\right)_{\Omega}^{s}\hat{v}\left(x_1,T\right)\geq 0.$$

Consequently, we have

$$-\hat{v}_t\left(x_1,T\right)-\left(-\Delta\right)_{\Omega}^{s}\hat{v}\left(x_1,T\right)\leq 0.$$

Conversely, we have

$$-\hat{v}_t-\sigma c\left(-\Delta\right)_{\Omega}^{s}\hat{v}=\sigma b\left[\tilde{g}\left(\tilde{u},\tilde{v}\right)-\left(u-\frac{uv}{1+u^2}\right)\right], \tag{3.24}$$

in (3.1) and (3.23). Setting $\tilde{v}=v$ and $\tilde{u}>u$. We can observe that

$$\begin{aligned}
-\hat{v}_t-\sigma c\left(-\Delta^s\right)\hat{v} &= \sigma b\left[\tilde{g}\left(\tilde{u},\tilde{v}\right)-\left(u-\frac{uv}{1+u^2}\right)\right]\\
&= \sigma b\left[\sup_{C_1<\varepsilon<\widetilde{u}}\left(\varepsilon-\frac{\varepsilon\left(v-\varepsilon_0\right)}{1+\varepsilon^2}\right)-\left(u-\frac{uv}{1+u^2}\right)\right]\\
&> \sigma b\left[\sup_{C_1<\varepsilon<\widetilde{u}}\left(\varepsilon-\frac{\varepsilon v}{1+\varepsilon^2}\right)-\left(u-\frac{uv}{1+u^2}\right)\right]\\
&\geq \sigma b\left[\sup_{C_1<\varepsilon<u}\left(\varepsilon-\frac{\varepsilon v}{1+\varepsilon^2}\right)-\left(u-\frac{uv}{1+u^2}\right)\right]\\
&> 0
\end{aligned}$$

as demonstrated in the proof of Proposition 2.2 in [35]. This results in a contradiction. If the condition $\hat{v}\left(x,T\right)<0$ holds, then it applies to all $x\in\Omega$. Subsequently, there exists an $x_1\in\partial\Omega$ such that $\hat{v}\left(x_1,T\right)=0$. Consequently, we possess

$$-\hat{v}_t\left(x_1,T\right)-\sigma c\left(-\Delta\right)_{\Omega}^{s}\hat{v}\left(x_1,T\right)\geq 0.$$

[1] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

Utilizing the principle of continuity, we may identify a small ball $\Omega'' = B(x_1)$. $\{x_1\}$ In Ω, the intersection of the boundary of Ω'' and Ω comprises a single point, designated as x_1. Moreover, it is guaranteed that $\hat{v}\,(x,T) < 0$, for every $x \in \Omega''$. The expression (3.19) may be equivalently expressed as

$$-\hat{v}_t - \sigma c\,(-\Delta)^s_\Omega\,\hat{v} > 0, \quad \text{in } \Omega'' \times \{T\}\,.$$

When examining equation (3.24) in the domain $\Omega'' \times \{T\}$, we observe that $(-\Delta)^s_\Omega\,(-\hat{v}) \geq 0$. According to the Hopf lemma, [1] states that $\partial_\nu \hat{v}\,(x_1,T) > 0$. This contradicts equation (3.1) and, consequently, equation

$$v\,(x,t) < \tilde{v}\,(t)\,, \quad \text{for all } t > 0, \quad x \in \overline{\Omega}. \tag{3.25}$$

Lastly, we examine the subsequent system:

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = \tilde{a} - \tilde{u}, \\ \dfrac{d\tilde{v}}{dt} = \sigma b \tilde{g}\,(\tilde{u},\tilde{v})\,, \end{cases}$$

in $\Re$. By applying the logic stated in the [35] research and using the definition of $\tilde{u}$ and equation (3.18), the resulting consequence is obtained:

$$\tilde{u}\,(t) = (\tilde{u}\,(0) - \tilde{a})\exp(-t) + \tilde{a}$$

This suggests that

$$\begin{aligned} \lim_{t\to\infty} \tilde{u}\,(t) &= (\tilde{u}\,(0) - \tilde{a}) \lim_{t\to\infty} \exp(-t) + \tilde{a} \\ &= \tilde{a} \end{aligned}$$

Based on the definitions of v and g, it can be deduced that

- if $\tilde{g} < 0$, then $\sup_{C_1<\varepsilon<\tilde{u}} \left[\frac{\varepsilon}{1+\varepsilon^2}\left(1+\varepsilon^2 - (\tilde{v}-\varepsilon_0)\right)\right] < 0$ and subsequently $1+\tilde{u}^2 - (\tilde{v}-\varepsilon_0) < 0$, leading to the conclusion that $\tilde{v} > 1+\tilde{u}^2+\varepsilon_0$.

- If $\tilde{g} > 0$, then $\sup_{C_1<\varepsilon<\tilde{u}} \left[\frac{\varepsilon}{1+\varepsilon^2}\left(1+\varepsilon^2 - (\tilde{v}-\varepsilon_0)\right)\right] > 0$, which in turn implies that $1+\tilde{u}^2 - (\tilde{v}-\varepsilon_0) > 0$. In order to get $\tilde{v} < 1+\tilde{u}^2+\varepsilon_0$.

- If $\tilde{g} = 0$, then $\sup_{C_1<\varepsilon<\tilde{u}} \left[\frac{\varepsilon}{1+\varepsilon^2}\left(1+\varepsilon^2 - (\tilde{v}-\varepsilon_0)\right)\right] = 0$, and hence, $1+\tilde{u}^2-(\tilde{v}-\varepsilon_0) = 0$. Consequently,

  $$\begin{cases} \lim \tilde{v} \geq 1 + \lim \tilde{u}^2 + \varepsilon_0 \\ \lim \tilde{v} \leq 1 + \lim \tilde{u}^2 + \varepsilon_0 \end{cases}$$

  subsequently leads to $\lim_{t\to\infty} \tilde{v}\,(t) = 1+\tilde{u}^2+\varepsilon_0$.

From the result of (3.25) and (3.22), we obtain $\tilde{a} < a, \quad 1+\tilde{u}^2+\varepsilon_0 < 1+a^2,$ thereby establishing the conclusion. □

3.3 Properties of nonconstant

This subsection will analyze the fundamental characteristics of non-homogeneous stable states in the Lengyel-Epstein system with regional fractional Laplacian. More specifically, we will focus on positive solutions that are not constant. The expression $d = \frac{c}{b}$ is evaluated in equation

$$\begin{cases} -\left(-\Delta\right)_{\Omega}^{s} u + a - u - \dfrac{4uv}{1+u^2} = 0, \\ -d\left(-\Delta\right)_{\Omega}^{s} v + u - \dfrac{uv}{1+u^2} = 0, \\ \partial_v u\left(x\right) = \partial_v v\left(x\right) = 0 \cdot \end{cases} \tag{3.26}$$

Applying Lemma 1, one employs the same methodology as in demonstrating proposition 3.1 (see [35]) to get the following outcome:

Proposition 3. *If the function $(u, v) = (u(x), v(x))$ satisfies the boundary value problem (3.26) and is positive, then it also satisfies equation*

$$\frac{a}{5+4a^2} < u < a \text{ and } 1 + \left(\frac{a}{5+4a^2}\right)^2 < v < l + a^2, \quad x \in \Omega. \tag{3.27}$$

Proof. The upper bounds may be obtained directly from Proposition 2. To determine the lower limits in equation (3.27), we examine the case where u reaches its minimal value, first

$$a - u - \frac{4uv}{1+u^2} \leq 0,$$

and then

$$\begin{aligned} a &\leq u\left(1 + \frac{4v}{1+u^2}\right), \\ &< u\left(1+4v\right). \end{aligned}$$

If we consider proposition 2, we may obtain the following equation:

$$a < u\left(5 + 4a^2\right),$$

which leads to the lower bound for u stated in equation

$$u > \frac{a}{4a^2+5}.$$

Supposing that v has a minimum at a certain location within $\overline{\Omega}$ results in

$$u - \frac{uv}{1+u^2} \leq 0,$$

[1] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

which thus implies

$$v \geq 1 + u^2.$$

Based on the constraint provided in 2, we may determine the minimum value for v, which is represented by equation

$$v > 1 + \left(\frac{a}{4a^2+5}\right)^2.$$

The mean outcomes of the positive solutions $u = u(x)$ and $v = v(x)$ The values of equation (3.26) over the set $\{y \in \Omega, |x-y| > \delta\}$ are computed using formula

$$\overline{u} = \frac{1}{|\Omega|} \int_\Omega u(x)\, dx, \quad \overline{v} = \frac{1}{|\Omega|} \int_\Omega v(x)\, dx,$$

where $|\Omega|$ denotes the magnitude of the volume of Ω. Let us examine the variables $\phi = u - \overline{u}$ and $\psi = v - \overline{v}$. Following that, we present equation

$$\int_\Omega \phi dx = \int_\Omega \psi dx = 0. \tag{3.28}$$

□

Lemma 5. *Let u denote a solution to equation (3.26). Therefore, $\overline{u} = \frac{a}{5}$.*

Proof. Write

$$w(x) = 4dv(x) - u(x). \tag{3.29}$$

By combining equations (3.26), we get equation

$$-4d(-\Delta)^s_\Omega v(x) + (-\Delta)^s_\Omega u(x) - a + 5u = 0,$$

$$(-\Delta)^s_\Omega w = 5u - a. \tag{3.30}$$

By performing the integration of equation (3.30) across the domain $\{y \in \Omega, |x-y| > \delta\}$, we get equation

$$\begin{aligned} PV \int_\Omega (5u - a)\, dx &= PV \int_\Omega (-\Delta)^s_\Omega w(x)\, dx \\ &= 5C_{n,s} PV \int_\Omega PV \int_\Omega \frac{u(x)}{|y-x|^{n+2s}} dxdy \\ &\quad -5C_{n,s} PV \int_\Omega PV \int_\Omega \frac{u(y)}{|y-x|^{n+2s}} dydx. \end{aligned}$$

Owing to the symmetry of the roles of x and y in the preceding integrals, we may deduce that

$$PV \int_\Omega (5u - a)\, dx = 0,$$

[1]*Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

implies $\overline{u}=\dfrac{a}{5}$. Let the variables be defined as $\phi=u-\overline{u}$ and $\psi=v-\overline{v}$. Thereafter,

$$\int_{\Omega}\phi dx=\int_{\Omega}\psi dx=0. \tag{3.31}$$

$\square$

Lemma 6. *Assume that* (u,v) *is a nonconstant solution to* (3.26). *Subsequently,*

$$\textstyle\int_{\Omega}\phi\psi dx>0, \quad \textit{and}\ \Upsilon>0,$$

where

$$\Upsilon=\int_{\Omega}\int_{\Omega}\frac{\left(\phi\left(x\right)-\phi\left(y\right)\right)\left(\psi\left(x\right)-\psi\left(y\right)\right)}{\left|y-x\right|^{n+2s}}dxdy$$

Proof. Rewrite the expression (3.30) as

$$\begin{aligned}\left(-\Delta\right)_{\Omega}^{s}w &= -a+5u,\\ &= 5\left(u-\frac{a}{5}\right),\\ &= 5\left(u-\overline{u}\right),\\ &= 5\phi,\end{aligned}$$

by multiplying it by $w=4dv-u$ and integrate over Ω, resulting in

$$\int_{\Omega}w\left(-\Delta\right)_{\Omega}^{s}wdx=5\int_{\Omega}\phi wdx.$$

This is done considering the variables $\phi=u-\overline{u}$, $\psi=v-\overline{v}$, $\int_{\Omega}\phi=\int_{\Omega}\psi=0$ and (1.7). Therefore, we acquire

$$\begin{aligned}\frac{C_{n,s}}{2}\int_{\Omega}\int_{\Omega}\frac{\left|\left(w\left(x\right)-w\left(y\right)\right)\right|^{2}}{\left|y-x\right|^{n+2s}}dxdy &= 5\int_{\Omega}\phi wdx,\\ &= 5\int_{\Omega}\phi\left(4dv-u\right)dx,\\ &= 20d\int_{\Omega}\phi vdx-5\int_{\Omega}\phi udx,\\ &= 20d\int_{\Omega}\phi vdx-5\int_{\Omega}\phi udx,\\ &= 20d\int_{\Omega}\phi vdx-5\int_{\Omega}\phi\underbrace{\left(u-\overline{u}\right)}_{=\phi^{2}}dx-5\overline{u}\underbrace{\int_{\Omega}\phi dx}_{=0},\end{aligned}$$

$$\int_{\Omega}\phi\psi dx=\frac{1}{20d}\left(\frac{C_{n,s}}{2}\int_{\Omega}\int_{\Omega}\frac{\left|\left(w\left(x\right)-w\left(y\right)\right)\right|^{2}}{\left|y-x\right|^{n+2s}}dxdy+5\int_{\Omega}\phi^{2}dx\right). \tag{3.32}$$

[1]*Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

Multiplying the equation (3.30) by ϕ and integrating over Ω yields the value of

$$\int_{\Omega}\phi\left(-\Delta\right)^{s}wdx=-a\int_{\Omega}\phi dx+5\int_{\Omega}u\phi dx$$

$$\frac{C_{N,s}}{2}\int_{\Omega}\int_{\Omega}\frac{\left(\phi\left(x\right)-\phi\left(y\right)\right)\left(w\left(x\right)-w\left(y\right)\right)}{\left|y-x\right|^{N+2s}}dxdy=-a\underbrace{\int_{\Omega}\phi dx}_{=0}+5\int_{\Omega}\phi^{2}dx$$

$$\begin{aligned}5\int_{\Omega}\phi^{2}dx &= \frac{C_{N,s}}{2}\int_{\Omega}\int_{\Omega}\frac{\left(\phi\left(x\right)-\phi\left(y\right)\right)\left(\left(4dv-u\right)\left(x\right)-\left(4dv-u\right)\left(y\right)\right)}{\left|y-x\right|^{N+2s}}dxdy\\ &= \frac{C_{N,s}}{2}\int_{\Omega}\int_{\Omega}\frac{\left(\phi\left(x\right)-\phi\left(y\right)\right)\left(4dv\left(x\right)-4dv\left(y\right)\right)}{\left|y-x\right|^{N+2s}}dxdy\\ &\quad+\frac{C_{N,s}}{2}\int_{\Omega}\int_{\Omega}\frac{\left(\phi\left(x\right)-\phi\left(y\right)\right)\left(u\left(y\right)-u\left(x\right)\right)}{\left|y-x\right|^{N+2s}}dxdy\\ &= \frac{4dC_{N,s}}{2}\int_{\Omega}\int_{\Omega}\frac{\phi\left(x\right)\psi\left(x\right)-\phi\left(x\right)\psi\left(y\right)-\phi\left(y\right)\psi\left(x\right)+\phi\left(y\right)\psi\left(y\right)}{\left|y-x\right|^{N+2s}}dxdy\\ &\quad+\frac{C_{N,s}}{2}\int_{\Omega}\int_{\Omega}\frac{\phi\left(x\right)\phi\left(y\right)-\phi\left(x\right)^{2}-\phi\left(y\right)^{2}+\phi\left(y\right)\phi\left(x\right)}{\left|y-x\right|^{N+2s}}dxdy\end{aligned}$$

$$5\int_{\Omega}\phi^{2}dx=2dC_{N,s}\underbrace{\int_{\Omega}\int_{\Omega}\frac{\left(\phi\left(x\right)-\phi\left(y\right)\right)\left(\psi\left(x\right)-\psi\left(y\right)\right)}{\left|y-x\right|^{N+2s}}dxdy}_{=\Upsilon}-\frac{C_{N,s}}{2}\underbrace{\int_{\Omega}\int_{\Omega}\frac{\left(\phi\left(x\right)-\phi\left(y\right)\right)^{2}}{\left|y-x\right|^{N+2s}}dxdy}_{=\Phi}$$

$$5\int_{\Omega}\phi^{2}dx=2dC_{n,s}\Upsilon-\frac{C_{n,s}}{2}\Phi$$

hence implying the value of

$$\Upsilon=\frac{1}{2dC_{n,s}}\left(\frac{C_{n,s}}{2}\Phi+5\int_{\Omega}\phi^{2}dx\right). \tag{3.33}$$

The conclusion of this lemma immediately follows equations (3.32) and (3.33). The estimate (3.27) guarantees the presence of a fixed c_g that depends on a, such that

$$\begin{aligned}\left|g\left(u,v\right)\right| &= \left|u-\frac{uv}{1+u^{2}}\right|=\left|u\right|\left|1-\frac{v}{1+u^{2}}\right|\\ &< \left|u\right|\left|1+\frac{v}{1+u^{2}}\right|\\ &< \underbrace{a\left(1+\frac{1}{1+\left(\frac{a}{5+4a^{2}}\right)^{2}}\right)}_{=c_g}\\ &< c_g\end{aligned}$$

This allows us to infer. ∎

[1] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

Lemma 7. *There is a constant* $C_g \equiv C_g\left(a, C_{ns}, \lambda_{1,s}^N, |\Omega|\right)$ *such that equation*

$$\int_\Omega \psi^2 dx + \Psi \leq C_g d^{-2}, \tag{3.34}$$

is satisfied, where

$$\Psi = \int_\Omega \int_\Omega \frac{(\psi(x) - \psi(y))^2}{|y-x|^{n+2s}} dxdy.$$

Proof. By utilizing the second equation of systems (3.26) in conjunction with the Schwarz inequality

$$\begin{aligned} d\int_\Omega \psi(-\Delta)_\Omega^s v dx &= \int_\Omega \psi g(u,v)\,dx \leq \int_\Omega |\psi|\,|g(u,v)|\,dx \leq c_g \int_\Omega |\psi|\,dx \\ &\leq c_g\sqrt{|\Omega|}\left(\int_\Omega |\psi|^2 dx\right)^{\frac{1}{2}}, \end{aligned}$$

we may get the following. Using equation (1.8), we derive equation

$$\begin{aligned} d\frac{C_{n,s}}{2}\Psi &= d\frac{C_{n,s}}{2}\int_\Omega\int_\Omega \frac{(\psi(x)-\psi(y))^2}{|y-x|^{N+2s}}dxdy = d\int_\Omega \psi(-\Delta)_\Omega^s v dx, \\ &\leq c_g\sqrt{|\Omega|}\left(\int_\Omega |\psi|^2 dx\right)^{\frac{1}{2}}. \end{aligned}$$

Furthermore,

$$\int_\Omega |\psi|^2 dx \leq \frac{C_{n,s}}{2\lambda_{1,s}^N}\Psi.$$

may be deduced from the Poincare inequality. This yields a result of

$$\begin{aligned} d\frac{C_{n,s}}{2}\Psi &\leq c_g\sqrt{|\Omega|}\left(\frac{C_{n,s}}{2\lambda_{1,s}^N}\Psi\right)^{\frac{1}{2}} \\ d^2\frac{C_{n,s}}{2}\Psi &\leq \frac{c_g^2|\Omega|}{\lambda_{1,s}^N} \\ \Psi &\leq \frac{2}{C_{n,s}}\frac{c_g^2|\Omega|}{\lambda_{1,s}^N}d^{-2}. \end{aligned}$$

[1]*Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

Applying equation (1.8) once more, we derive equation

$$\begin{aligned}
\int \psi^2 dx + \Psi &\leq \frac{C_{n,s}}{2\lambda_{1,s}^N}\Psi + \Psi \\
&\leq \frac{C_{n,s}}{2\lambda_{1,s}^N}\frac{2}{C_{n,s}}\frac{c_g^2\,|\Omega|}{\lambda_{1,s}^N}d^{-2} + \frac{2}{C_{n,s}}\frac{c_g^2\,|\Omega|}{\lambda_{1,s}^N}d^{-2} \\
&\leq \frac{c_g^2\,|\Omega|}{\left(\lambda_{1,s}^N\right)^2}d^{-2} + \frac{2}{C_{N,s}}\frac{c_g^2\,|\Omega|}{\lambda_{1,s}^N}d^{-2} \\
&\leq \left(\frac{1}{\left(\lambda_{1,s}^N\right)^2} + \frac{2}{C_{n,s}}\frac{1}{\lambda_{1,s}^N}\right)c_g^2\,|\Omega|\,d^{-2} \\
&\leq \frac{C_{n,s} + 2\lambda_{1,s}^N}{\left(\lambda_{1,s}^N\right)^2}c_g^2\,|\Omega|\,d^{-2}
\end{aligned}$$

$$\int \psi^2 dx + \Psi \leq C_g d^{-2}$$

using value

$$C_g = c_g^2\,|\Omega|\left(C_{n,s} + 2\lambda_{1,s}^N\right)/\left(\lambda_{1,s}^N\right)^2.$$

□

Lemma 8. *Let (u, v) be a solution to the problem (3.26) that is not constant. Subsequently*

$$\frac{3\left(\lambda_{1,s}^N\right)^2}{3\left(\lambda_{1,s}^N\right)^2 + 15\lambda_{1,s}^N + 20} < \frac{\Phi}{d^2\Psi} < 16. \tag{3.35}$$

Proof. Using equations (3.29) and (3.33), we derive the following result:

$$\begin{aligned}
\int_\Omega\int_\Omega \frac{(w(x) - w(y))^2}{|y-x|^{n+2s}}dxdy &= \int_\Omega\int_\Omega \frac{(4dv(x) - u(x) - 4dv(y) + u(y))^2}{|y-x|^{n+2s}}dxdy \\
&= \int_\Omega\int_\Omega \frac{[4d(v(x) - v(y)) + (u(y) - u(x))]^2}{|y-x|^{n+2s}}dxdy \\
&= \int_\Omega\int_\Omega \frac{[4d(\psi(x) - \psi(y)) - (\phi(x) - \phi(y))]^2}{|y-x|^{n+2s}}dxdy \\
&= 16d^2\Psi - 8d\Upsilon + \Phi \\
&= 16d^2\Psi - 8d\left[\frac{1}{2dC_{n,s}}\left(\frac{C_{n,s}}{2}\Phi + 5\int_\Omega \phi^2 dx\right)\right] + \Phi \\
&= 16d^2\Psi - \Phi - \frac{20}{C_{n,s}}\int_\Omega \phi^2 dx \\
&\geq 0
\end{aligned}$$

[1] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

as a result,

$$\Phi + \frac{20}{C_{n,s}} \int_\Omega \phi^2 dx \leq 16d^2\Psi. \tag{3.36}$$

Therefore, the second inequality (3.35) is derived. We commence the proof by asserting

$$16d^2\Psi = \Phi + \frac{10}{C_{n,s}} \int_\Omega \phi^2 dx + \frac{40d}{C_{n,s}} \int_\Omega \phi\psi dx. \tag{3.37}$$

Continuing the aforementioned calculation, we deduce that

$$16d^2\Psi = \int_\Omega \int_\Omega \frac{(w(x) - w(y))^2}{|y-x|^{n+2s}} dxdy + \Phi + \frac{20}{C_{n,s}} \int_\Omega \phi^2 dx.$$

Using equation (3.32), we obtain

$$\begin{aligned} 16d^2\Psi &= \frac{2}{C_{n,s}} \left(20d \int \phi\psi dx - 5 \int \phi^2 dx \right) + \Phi + \frac{20}{C_{n,s}} \int_\Omega \phi^2 dx \\ 16d^2\Psi &= \frac{40d}{C_{n,s}} \int_\Omega \phi\psi dx + \frac{10}{C_{n,s}} \int_\Omega \phi^2 dx + \Phi. \end{aligned}$$

Next, we will review the elementary Cauchy inequality, which states that for all given real numbers ξ, ζ and $\kappa = \frac{3}{4} d\lambda_{1,s}^N$ equation

$$\xi\zeta \leq \frac{1}{4\kappa}\xi^2 + \kappa\zeta^2, \tag{3.38}$$

holds. Based on equations (3.37) and (3.38), it can be deduced that

$$\begin{aligned} 16d^2\Psi &\leq \Phi + \frac{10}{C_{n,s}} \int_\Omega \phi^2 dx + \frac{40}{3\lambda_{1,s}^N C_{n,s}} \int_\Omega \phi^2 dx + \frac{30d^2\lambda_{1,s}^N}{C_{n,s}} \int_\Omega \psi^2 dx \\ &\leq \Phi + \frac{10}{C_{n,s}} \frac{C_{n,s}}{2\lambda_{1,s}^N} \Phi + \frac{40}{3\lambda_{1,s}^N C_{n,s}} \frac{C_{n,s}}{2\lambda_{1,s}^N} \Phi + \frac{30d^2\lambda_{1,s}^N}{C_{n,s}} \frac{C_{n,s}}{2\lambda_{1,s}^N} \Psi \\ &\leq \left[1 + \frac{20}{3\left(\lambda_{1,s}^N\right)^2} + \frac{5}{\lambda_{1,s}^N} \right] \Phi + 15d^2\Psi. \end{aligned}$$

As a result, equation

$$d^2\Psi \leq \left[1 + \frac{20}{3\left(\lambda_{1,s}^N\right)^2} + \frac{5}{\lambda_{1,s}^N} \right] \Phi, \tag{3.39}$$

proves the first inequality of equation (3.35). □

Lemma 9. *Let (u, v) be a solution to the system (3.26) that is not constant. Subsequently,*

$$\frac{6\left(\lambda_{1,s}^N\right)^3}{\left(2\lambda_{1,s}^N + C_{n,s}\right)\left(3\left(\lambda_{1,s}^N\right)^2 + 15\lambda_{1,s}^N + 20\right)} < \frac{\Phi + \frac{1}{C_{n,s}} \int_\Omega \phi^2}{d^2\left(\Psi + \int_\Omega \psi^2\right)} < 16. \tag{3.40}$$

Proof. Applying equation (1.8), we obtain result

$$\begin{aligned}\Psi + \int_\Omega \psi^2 &\leq \Psi + \frac{C_{n,s}}{2\lambda_{1,s}^N}\Psi \\ &\leq \frac{2\lambda_{1,s}^N + C_{n,s}}{2\lambda_{1,s}^N}\Psi.\end{aligned}$$

This, along with the initial inequality of (3.35), demonstrates the first inequality of (3.40)

$$\begin{aligned}\frac{3\left(\lambda_{1,s}^N\right)^2}{3\left(\lambda_{1,s}^N\right)^2 + 15\lambda_{1,s}^N + 20} &< \frac{\Phi}{d^2\Psi} = \frac{\frac{2\lambda_{1,s}^N + C_{n,s}}{2\lambda_{1,s}^N}\Phi}{d^2\frac{2\lambda_{1,s}^N + C_{n,s}}{2\lambda_{1,s}^N}\Psi} \\ \frac{3\left(\lambda_{1,s}^N\right)^2}{3\left(\lambda_{1,s}^N\right)^2 + 15\lambda_{1,s}^N + 20} &< \frac{\frac{2\lambda_{1,s}^N + C_{n,s}}{2\lambda_{1,s}^N}\Phi}{d^2\left(\Psi + \int_\Omega \psi^2\right)} \\ \frac{6\left(\lambda_{1,s}^N\right)^3}{\left(2\lambda_{1,s}^N + C_{n,s}\right)\left(3\left(\lambda_{1,s}^N\right)^2 + 15\lambda_{1,s}^N + 20\right)} &< \frac{\Phi + \frac{1}{C_{n,s}}\int_\Omega \phi^2}{d^2\left(\Psi + \int_\Omega \psi^2\right)}.\end{aligned}$$

The second inequality in equation (3.40) is established by equation (3.36)

$$\begin{aligned}\Phi + \frac{1}{C_{n,s}}\int_\Omega \phi^2 dx &\leq \Phi + \frac{20}{C_{N,s}}\int_\Omega \phi^2 dx \\ &\leq 16d^2\Psi \\ &< 16d^2\left(\Psi + \int_\Omega \psi^2 dx\right).\end{aligned}$$

□

3.4 Non-existence of nonconstant positive solutions

This section will focus on nonconstant positive solutions to equation (3.26) . Our ideas indicate that the reactor's dimensions and the diffusion coefficients are pivotal in precluding the existence of nonconstant positive solutions.

Theorem 14. *There is a constant* $d_0 = d_0\left(a, C_{n,s}, \lambda_{1,s}^N\right) > 0$ *for which the problem* (3.26) *does not have any nonconstant solution for* $0 < d < d_0$.

[1]*Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

Proof. Multiplying the equation (3.26) by ψ and integrating over Ω produces the formula

$$\begin{aligned}
d\frac{C_{n,s}}{2}\Psi &= \int_\Omega \phi\psi dx - \int_\Omega \frac{uv\psi}{1+u^2}dx,\\
&= \int_\Omega \phi\psi dx - \int_\Omega \left[\frac{uv\psi}{1+u^2} - \frac{u\overline{v}\psi}{1+u^2} + \frac{u\overline{v}\psi}{1+u^2}\right]dx + \int_\Omega \frac{\overline{uv}\psi}{1+\overline{u}^2}dx\\
&= \int_\Omega \phi\psi dx - \underbrace{\int_\Omega \left(\frac{uv}{1+u^2} - \frac{u\overline{v}}{1+u^2}\right)\psi dx}_{=I_1} - \underbrace{\int_\Omega \left(\frac{u\overline{v}}{1+u^2} - \frac{\overline{uv}}{1+\overline{u}^2}\right)\psi dx}_{=I_2}
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_\Omega \left(\frac{uv}{1+u^2} - \frac{u\overline{v}}{1+u^2}\right)\psi dx = \int_\Omega \frac{u}{1+u^2}\underbrace{(v-\overline{v})}_{=\psi}\psi dx = \int \frac{u\psi^2}{1+u^2}dx\\
I_2 &= \int_\Omega \left(\frac{u\overline{v}}{1+u^2} - \frac{\overline{uv}}{1+\overline{u}^2}\right)\psi dx = \int_\Omega \left(\frac{u}{1+u^2} - \frac{\overline{u}}{1+\overline{u}^2}\right)\overline{v}\psi dx\\
&= \int_\Omega \left(\frac{u\left(1+\overline{u}^2\right) - \overline{u}\left(1+u^2\right)}{\left(1+u^2\right)\left(1+\overline{u}^2\right)}\right)\overline{v}\psi dx = \int_\Omega \left(\frac{u+u\overline{u}^2-\overline{u}-\overline{u}u^2}{\left(1+u^2\right)\left(1+\overline{u}^2\right)}\right)\overline{v}\psi dx\\
&= \int_\Omega \left(\frac{u-\overline{u}+u\overline{u}\left(\overline{u}-u\right)}{\left(1+u^2\right)\left(1+\overline{u}^2\right)}\right)\overline{v}\psi dx = \int_\Omega \left(\frac{1-u\overline{u}}{\left(1+u^2\right)\left(1+\overline{u}^2\right)}\right)\overline{v}\phi\psi dx
\end{aligned}$$

$$d\frac{C_{n,s}}{2}\Psi = \int_\Omega \phi\psi dx - \int_\Omega \frac{u\psi^2}{1+u^2}dx - \int_\Omega \left(\frac{1-u\overline{u}}{\left(1+u^2\right)\left(1+\overline{u}^2\right)}\right)\overline{v}\phi\psi dx.$$

Proposition 3 provides a priori estimations, which lead to the conclusion that

$$\begin{aligned}
d\frac{C_{n,s}}{2}\Psi &= \int_\Omega \phi\psi dx - \int_\Omega \frac{u\psi^2}{1+u^2}dx + \frac{\overline{vu}}{\left(1+\overline{u}^2\right)}\int_\Omega \frac{u\phi\psi}{\left(1+u^2\right)}dx + \frac{-\overline{v}}{\left(1+\overline{u}^2\right)}\int_\Omega \frac{\phi\psi}{\left(1+u^2\right)}dx\\
&< \int_\Omega \phi\psi dx - \int_\Omega \frac{u\psi^2}{1+u^2}dx + \frac{\overline{vu}a}{\left(1+\overline{u}^2\right)}\int_\Omega \phi\psi dx + \frac{-\overline{v}}{\left(1+\overline{u}^2\right)\left(1+a^2\right)}\int_\Omega \phi\psi dx\\
&= \underbrace{\left(1+\frac{\overline{vu}a}{\left(1+\overline{u}^2\right)} + \frac{-\overline{v}}{\left(1+\overline{u}^2\right)\left(1+a^2\right)}\right)}_{=C_1(a)}\int_\Omega \phi\psi dx - \frac{1}{1+a^2}\int_\Omega u\psi^2 dx\\
&= C_1(a)\int_\Omega \phi\psi dx - \underbrace{\frac{a}{\left(1+a^2\right)\left(5+4a^2\right)}}_{=C_2(a)}\int_\Omega \psi^2 dx \Big/ \frac{a}{5+4a^2} < u\\
&\le C_1(a)\int_\Omega \phi\psi dx - C_2(a)\int_\Omega \psi^2 dx.
\end{aligned}$$

[1]*Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

By employing the Cauchy inequality and the inequality (1.8), we derive

$$\begin{aligned}
d\frac{C_{n,s}}{2}\Psi &\leq C_1(a)\left(\int_\Omega |\phi|^2\psi dx\right)^{\frac{1}{2}}\left(\int_\Omega |\psi|^2 dx\right)^{\frac{1}{2}} - \left(C_2(a)\int_\Omega \psi^2 dx + \frac{(C_1(a))^2}{4C_2(a)}\int_\Omega \phi^2 dx\right) \\
&\quad + \frac{(C_1(a))^2}{4C_2(a)}\int \phi^2 dx \\
&\leq -\left(C_2(a)\int_\Omega \psi^2 dx + \frac{(C_1(a))^2}{4C_2(a)}\int_\Omega \phi^2 dx - C_1(a)\left(\int_\Omega |\phi|^2\psi dx\right)^{\frac{1}{2}}\left(\int_\Omega |\psi|^2 dx\right)^{\frac{1}{2}}\right) \\
&\quad + \frac{(C_1(a))^2}{4C_2(a)}\int_\Omega \phi^2 dx \\
&\leq -\left(\frac{C_1(a)}{2\sqrt{C_2(a)}}\left(\int_\Omega |\phi|^2 dx\right)^{\frac{1}{2}} - \sqrt{C_2(a)}\left(\int_\Omega |\psi|^2 dx\right)^{\frac{1}{2}}\right)^2 + \frac{(C_1(a))^2}{4C_2(a)}\int_\Omega \phi^2 dx \\
&\leq \frac{(C_1(a))^2}{4C_2(a)}\int_\Omega \phi^2 dx \leq \frac{(C_1(a))^2}{4C_2(a)}\frac{C_{n,s}}{\lambda_{1,s}^N}\Phi,
\end{aligned}$$

$$d\Psi \leq \frac{(C_1(a))^2 C_{n,s}}{8C_2(a)\lambda_{1,s}^N}\Phi, \tag{3.41}$$

where

$$C_1(a) = 1 + \tfrac{\overline{v}\overline{u}a}{(1+\overline{u}^2)} + \tfrac{-\overline{v}}{(1+\overline{u}^2)(1+a^2)}, \quad C_2(a) = \tfrac{a}{(1+a^2)(5+4a^2)}.$$

Thus, it can be demonstrated that equation

$$\begin{aligned}
d\Psi &\leq \frac{(C_1(a))^2 C_{n,s}}{8C_2(a)\lambda_{1,s}^N}\left(16d^2\Psi\right), \\
&\leq \frac{2d(C_1(a))^2 C_{n,s}}{C_2(a)\lambda_{1,s}^N}\Psi,
\end{aligned}$$

$$\Psi \leq \frac{d}{d_0}\Psi,$$

by (3.35), where

$$d_0 = d_0\left(a, C_{n,s}, \lambda_{1,s}^N\right) = \frac{C_2(a)\lambda_{1,s}^N}{2(C_1(a))^2 C_{n,s}}. \tag{3.42}$$

Clearly, if $d < d_0$ holds, then $\Psi = 0$ and consequently $\Phi = 0$ may be derived from (3.35) once more. Consequently, φ and ψ are constants. □

Theorem 15. *The boundary value problem (3.26) does not have a nonconstant solution if $a^2 \leq 75$ and*

$$\frac{1}{d} > \frac{8a}{5} - \frac{25}{a}, \tag{3.43}$$

is satisfied. In particular, if $a^2 \leq \frac{125}{8}$, there is no nonconstant solution for all $d > 0$.

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[1]*Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

Proof. Multiplying (3.26) by $(1+u^2(x))\phi(x)$ yields

$$\left[1+u^2(x)\right]\phi(x)(-\Delta)^s_\Omega u(x)=\left[(a-\overline{u}-\phi(x))+(a-u(x))u^2(x)\right]\phi(x)-4\int_\Omega u(x)v(x)\phi(x)\,dx.$$

Utilization of 1.7. Consequently, this leads to the result of

$$\begin{aligned}
\int_\Omega \phi(x)(-\Delta)^s_\Omega u(x)\,dx &= -\int_\Omega u^2(x)\phi(x)(-\Delta)^s_\Omega u(x)\,dx-\int_\Omega \phi^2(x)\,dx\\
&\quad +(a-\overline{u})\underbrace{\int_\Omega \phi(x)\,dx}_{=0}+\int_\Omega (a-u(x))u^2(x)\phi(x)\,dx\\
&\quad -4\int_\Omega u(x)v(x)\phi(x)\,dx,\\
&= -\frac{C_{n,s}}{2}\underbrace{\int_\Omega\int_\Omega \frac{(u^2(x)\phi(x)-u^2(y)\phi(y))(u(x)-u(y))}{|y-x|^{n+2s}}dxdy}_{=A_1}\\
&\quad -\int_\Omega \phi^2(x)\,dx+\int_\Omega\left[\underbrace{(a-u(x))u^2(x)-\frac{4}{125}a^3}_{=A_2}\right]\phi(x)\,dx\\
&\quad \underbrace{-4\int_\Omega (u(x)v(x)-\overline{uv})\phi(x)\,dx}_{=A_3},
\end{aligned}$$

$$\begin{aligned}
A_1 &= \int_\Omega\int_\Omega \frac{\begin{bmatrix} u^2(x)\phi(x)-u^2(x)\phi(y)\\ +u^2(x)\phi(y)-u^2(y)\phi(y)\end{bmatrix}(\phi(x)-\phi(y))}{|y-x|^{n+2s}}dxdy\\
&= \int_\Omega\int_\Omega \frac{\begin{bmatrix} u^2(x)(\phi(x)-\phi(y))\\ +\phi(y)(\phi(x)-\phi(x))(u(x)+u(y))\end{bmatrix}(\phi(x)-\phi(y))}{|y-x|^{n+2s}}dxdy\\
&= \int_\Omega\int_\Omega \frac{\left[u^2(x)+\underbrace{\phi(y)}_{=u(y)-\overline{u}}(u(x)+u(y))\right](\phi(x)-\phi(y))^2}{|y-x|^{n+2s}}dxdy\\
&= \int_\Omega\int_\Omega \frac{\begin{bmatrix} u^2(x)+u(y)^2+[-\overline{u}(u(x)+u(y))]\\ +u(y)u(x)\end{bmatrix}(\phi(x)-\phi(y))^2}{|y-x|^{n+2s}}dxdy\\
&\le -\frac{1}{5}\int_\Omega\int_\Omega \frac{(2a-15u(x))u(x)(\phi(x)-\phi(y))^2}{|y-x|^{n+2s}}dxdy\Big/\overline{u}=\frac{a}{5}
\end{aligned}$$

[1]*Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

$$\begin{aligned} A_2 &= -u^3 + au^2 - \frac{4a^3}{125} \\ &= -u^3 + \left(\frac{4a}{5}u^2 + \overline{u}u^2\right) - \underbrace{\frac{4a^3}{125}}_{=\frac{4a^2}{25}\overline{u}} + \frac{4a}{5}\left(\frac{a}{5} - \overline{u}\right)u \Big/ \overline{u} = \frac{a}{5} \\ &= -u^3 + \frac{4a}{5}u^2 + \frac{4a^2}{25}u + \overline{u}u^2 - \frac{4a}{5}\overline{u}u - \frac{4a^2}{25}\overline{u} \\ &= u\left(-u^2 + \frac{4a}{5}u + \frac{4a^2}{25}\right) - \left(-u^2 + \frac{4a}{5}u + \frac{4a^2}{25}\right)\overline{u} \\ &= \phi\left(-u^2 + \frac{4a}{5}u + \frac{4a^2}{25}\right) \end{aligned}$$

$$\begin{aligned} A_3 &= -4\int_\Omega \left(\left(u - \overline{u} + \overline{u}\right)v - \overline{uv}\right)\phi dx \\ &= -4\int_\Omega \left(\phi v + \overline{u}v - \overline{uv}\right)\phi dx \\ &= -4\int_\Omega \phi^2 v dx - 4\int_\Omega \left(v - \overline{v}\right)\overline{u}\phi dx \\ &= -4\int_\Omega \phi^2 v dx - \frac{4a}{5}\int_\Omega \phi\psi dx \Big/ \overline{u} = \frac{a}{5} \end{aligned}$$

$$\begin{aligned} \int_\Omega \phi(x)\left(-\Delta\right)_\Omega^s u(x)\,dx &\le \frac{C_{n,s}}{10}\int_\Omega\int_\Omega \frac{\left(2a - 15u(x)\right)u(x)\left(\phi(x) - \phi(y)\right)^2}{|y-x|^{n+2s}}dxdy \\ &\quad + \int_\Omega \left(-u^2(x) + \frac{4a}{5}u(x) + \frac{4a^2}{25} - 1\right)\phi(x)^2\,dx \\ &\quad -4\int_\Omega v(x)\,\phi^2(x)\,dx - \frac{4a}{5}\int_\Omega \phi\psi dx. \end{aligned}$$

Given that

$$(2a - 15u)\,u \le \tfrac{a^2}{15}, \quad -u^2 + \tfrac{4a}{5}u + \tfrac{4a^2}{25} \le \tfrac{8a^2}{25},$$

and $v \ge 1$, this results in

$$\begin{aligned} \frac{C_{n,s}}{2}\Phi &\le \frac{a^2 C_{n,s}}{150}\Phi + \int_\Omega \left(\frac{8a^2}{25} - 5\right)\phi(x)^2\,dx \\ &\quad -\frac{4a}{5}\int_\Omega \phi(x)\,\psi(x)\,dx. \end{aligned}$$

Applying (3.32) to $\int_\Omega \phi(x)\,\psi(x)\,dx$ we obtain

$$\frac{C_{n,s}}{2}\Phi \le \frac{a^2 C_{N,s}}{150}\Phi + \int_\Omega \left(\frac{8a^2}{25} - \frac{a}{5d} - 5\right)\phi(x)^2\,dx \underbrace{-\frac{aC_{n,s}}{50d}\int_\Omega\int_\Omega \frac{\left|\left(w(x) - w(y)\right)\right|^2}{|y-x|^{n+2s}}dx}_{\le 0},$$

[1] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

$$\frac{C_{n,s}}{2}\Phi \leq \frac{a^2 C_{n,s}}{150}\Phi + \left(\frac{8a^2}{25} - \frac{a}{5d} - 5\right)\int_\Omega \phi(x)^2 dx. \tag{3.44}$$

Now, if

$$\begin{aligned} \frac{2}{C_{n,s}}\left(a^2 C_{n,s}\right) &\leq 150 \\ a^2 &\leq 75 \end{aligned}$$

and

$$\begin{aligned} \frac{2}{C_{n,s}}\left(\frac{8a^2}{25} - \frac{a}{5d} - 5\right) &< 0 \\ \frac{8a^2}{25} - 5 &< \frac{a}{5d} \\ \frac{8a}{5} - \frac{25}{a} &< \frac{1}{d} \end{aligned}$$

holds, then for a nonconstant solution (u, v), we deduce that $\Phi < \Phi$, which is a contradiction. Therefore, $\phi \equiv 0$. This concludes the proof. □

Theorem 16. *There is a positive constant $\Lambda = \Lambda(a) > 0$ such that if $\lambda_{1,s}^N > \Lambda$, is the problem (3.26) , it does not have any solutions that are not constant.*

Proof. The product of multiplying (3.26) by ϕ and integrating over Ω results in:

$$\begin{aligned} \frac{C_{n,s}}{2}\Phi &= \int_\Omega \left(a - u - \frac{4uv}{1+u^2}\right)\phi dx \\ &= -\int_\Omega \phi^2 - 4\int_\Omega \frac{uv\phi}{1+u^2} dx. \end{aligned}$$

Given this,

$$\begin{aligned} \int_\Omega \frac{uv\phi}{1+u^2} dx &= \int_\Omega \left[\begin{array}{c} \underbrace{\frac{uv}{1+u^2} - \frac{\overline{u}v}{1+u^2}} + \underbrace{\frac{\overline{u}v}{1+u^2} - \frac{\overline{uv}}{1+u^2}} \\ + \underbrace{\frac{\overline{uv}}{1+u^2} - \frac{\overline{uv}}{1+\overline{u}^2}} \end{array} \right] \phi dx \\ &= \int_\Omega \frac{v\phi^2}{1+u^2} dx + \int_\Omega \frac{\overline{u}\phi\psi}{1+u^2} dx - \int_\Omega \frac{\overline{uv}\left(\overline{u}+u\right)\phi^2}{\left(1+u^2\right)\left(1+\overline{u}^2\right)} dx, \end{aligned}$$

suggests that

$$\frac{C_{n,s}}{2}\Phi = -\int_\Omega \phi^2 - 4\int_\Omega \frac{v\phi^2}{1+u^2} dx - 4\int_\Omega \frac{\overline{u}\phi\psi}{1+u^2} dx + 4\int_\Omega \frac{\overline{uv}\left(\overline{u}+u\right)\phi^2}{\left(1+u^2\right)\left(1+\overline{u}^2\right)} dx \tag{3.45}$$

$$\frac{C_{n,s}}{2}\Phi \leq 4\int_\Omega \frac{\overline{uv}\left(\overline{u}+a\right)}{\left(1+\overline{u}^2\right)}\phi^2 dx - 4\int_\Omega \frac{\overline{u}}{1+u^2}\phi\psi dx. \tag{3.46}$$

[1] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

By employing the previously established estimates in proposition 3, we derive the estimate $(u < a)$ and $\left(\frac{1}{(1+u^2)} \leq 1\right)$, leading to the estimate

$$\begin{aligned}
\frac{C_{n,s}}{2}\Phi &\leq 4\int_\Omega \frac{\overline{uv}\,(\overline{u}+a)}{(1+\overline{u}^2)}\phi^2 dx + 4\int_\Omega \frac{\overline{u}}{1+u^2}\phi\psi dx \\
&\leq 4\frac{\overline{uv}\,(\overline{u}+a)}{(1+\overline{u}^2)}\int_\Omega \phi^2 dx + \frac{4\left(5+4a^2\right)^2\overline{u}}{\left(5+4a^2\right)^2+a^2}\int_\Omega \phi\psi dx \Big/ \frac{1}{1+u^2} \leq \frac{\left(5+4a^2\right)^2}{\left(5+4a^2\right)^2+a^2},
\end{aligned}$$

$$\frac{C_{n,s}}{2}\Phi \leq C_3\int_\Omega \phi^2 dx + C_3\int_\Omega \phi\psi dx, \tag{3.47}$$

where

$$C_3 = \max\left(4\frac{\overline{uv}\,(\overline{u}+a)}{(1+\overline{u}^2)}, \frac{4\left(5+4a^2\right)^2\overline{u}}{\left(5+4a^2\right)^2+a^2}\right).$$

By combining the Cauchy inequality with (1.8) and (3.41), we obtain

$$\begin{aligned}
\int_\Omega \phi\psi dx &\leq \left(\int_\Omega |\phi|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\psi|^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{C_{n,s}}{2\lambda_{1,s}^N}\Phi^{\frac{1}{2}}\Psi^{\frac{1}{2}}.
\end{aligned}$$

Consequently,

$$\int \phi\psi dx \leq \frac{C_{n,s}}{2\lambda_{1,s}^N}\Phi^{\frac{1}{2}}\left(\frac{\left(C_1(a)\right)^2 C_{n,s}}{8dC_2(a)\,\lambda_{1,s}^N}\Phi\right)^{\frac{1}{2}},$$

$$\textstyle\int_\Omega \phi\psi dx \leq C_4\,\frac{C_{n,s}}{2}d^{-\frac{1}{2}}\left(\lambda_{1,s}^N\right)^{-\frac{3}{2}}\Phi, \quad \text{where} \quad C_4 = \left(\frac{\left(C_1(a)\right)^2 C_{n,s}}{8C_2(a)}\right)^{\frac{1}{2}}.$$

Based on (3.47) we may deduce that

$$\begin{aligned}
\frac{C_{n,s}}{2}\Phi &\leq C_3\int_\Omega \phi^2 dx + C_3\left(C_4\,\frac{C_{n,s}}{2}d^{-\frac{1}{2}}\left(\lambda_{1,s}^N\right)^{-\frac{3}{2}}\Phi\right) \\
&\leq C_3\left(\frac{C_{n,s}}{2\lambda_{1,s}^N}\Phi\right) + C_3\left(C_4\,\frac{C_{n,s}}{2}d^{-\frac{1}{2}}\left(\lambda_{1,s}^N\right)^{-\frac{3}{2}}\Phi\right) \\
&\leq \frac{C_{n,s}}{2\lambda_{1,s}^N}\left(C_3 + C_3C_4\,\left(d\lambda_{1,s}^N\right)^{-\frac{1}{2}}\right)\Phi,
\end{aligned}$$

$$\Phi \leq \frac{C}{\lambda_{1,s}^N}\left(1+\ \left(d\lambda_{1,s}^N\right)^{-\frac{1}{2}}\right)\Phi, \tag{3.48}$$

with $C = \max\left(C_3, C_3 \times C_4\right)$ as its components. If $d \geq 1$, then there exists a $\Lambda > 0$ such that, upon selecting $\lambda_{1,s}^N > \Lambda$, we have

$$\frac{C}{\lambda_{1,s}^N}\left(1+\ \left(d\lambda_{1,s}^N\right)^{-\frac{1}{2}}\right) \leq \frac{C}{\lambda_{1,s}^N}\left(1+\ \left(\lambda_{1,s}^N\right)^{-\frac{1}{2}}\right) < 1.$$

Consequently, the inequality in equation (3.48) indicates that the value of $\Phi = 0$. The estimates presented in equation (3.35) also suggest the presence of $\Psi = 0$ and the absence of nonconstant positive solutions. Furthermore, if $d < 1$, we can choose $\lambda_{1,s}^{N}$, which, as indicated in Theorem 14, results in nonexistence. $\square$

[1] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

Chapter 4

On Local and Global Asymptotic Stability

This chapter examines the local and global stability conditions of the fractional Lengyel-Epstein model.

4.1 Turing instability

This section[2] focuses mostly on the stability of steady-state constant solutions. We will define the sufficient conditions for Turing instability. A constant solution is considered Turing unstable if it remains stable without diffusion but becomes unstable in the presence of diffusion. This needs satisfying of the following two conditions:

(i) It represents a stable equilibrium within the system of ordinary differential equations (2.9).

(ii) The equilibrium of the reaction-diffusion equations (3.1) with homogeneous Neumann boundary conditions is unstable.

Utilizing the theory of ordinary differential equations, we easily established a necessary and sufficient condition for (i) (see Lemma3). We will now establish a sufficient condition for (ii) above. The model (3.1) creates a steady activator-inhibitor system without diffusion. The spectral issue for the fractional Laplacian operator $(-\Delta)^s_\Omega$ is

$$\begin{cases} (-\Delta)^s_\Omega \phi_i = \lambda^N_{i,s}\phi_i, & x \in \Omega, \\ \frac{\partial \phi_i}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

where each $\lambda_{i,s}^N$ has an algebraic multiplicity of $m_i \geq 1$. The normalized eigenfunctions $\{\phi_{i,j}\}, i \geq 0, 1 \leq j \leq m_i$, form an orthonormal basis in $L^2(\Omega)$. All of them are given the condition

$$\lambda_{1,s}^N < f_0 = \frac{3\alpha^2 - 5}{1 + \alpha^2}, \tag{4.1}$$

We define $i_\alpha = i_\alpha(\alpha, \Omega)$ as the largest positive integer such that

$$\lambda_{i,s}^N < f_0 \quad \text{for} \quad i \leq i_\alpha.$$

If equation (4.1) is met, then $1 < i < \infty$. In this specific instance, we permit

$$\tilde{d} = \tilde{d}(\alpha, \Omega) = \min_{1 \leq i \leq i_\alpha} d_i, \quad d_i = \tfrac{\alpha}{1+\alpha^2} \tfrac{\lambda_{i,s}^N + 5}{\lambda_{i,s}^N \left(f_0 - \lambda_{i,s}^N\right)}. \tag{4.2}$$

The following theorem demonstrates that the positive constant (u^*, v^*) of (3.1) is locally asymptotically stable under specific conditions.

Lemma 10. *Assume that condition (2.13) is satisfied. The constant steady state (u^*, v^*) is asymptotically stable if $\lambda_{1,s}^N \geq f_0$ or $\lambda_{1,s}^N < f_0$ and $0 < d = \frac{c}{b} < \tilde{d}$. If $\lambda_{1,s}^N < f_0$ and $d > \tilde{d}$, then (u^*, v^*) is unstable.*

Proof. We investigate a perturbation that is expressed in the form of

$$\begin{pmatrix} \tilde{u}(t) \\ \tilde{v}(t) \end{pmatrix} = \begin{pmatrix} u(t) - u^* \\ v(t) - v^* \end{pmatrix}.$$

For simplicity, we will refer to $\tilde{u}(t)$ and $\tilde{v}(t)$ as $u(t)$ and $v(t)$, respectively. The linear and nonlinear elements of the system (3.1) are reformulated as

$$\frac{\partial U}{\partial t} = DA^s U + F(U),$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & \sigma c \end{pmatrix}, \quad A^s U = \begin{pmatrix} -(-\Delta)_\Omega^s & 0 \\ 0 & -(-\Delta)_\Omega^s \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

and

$$F(U) = \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix}.$$

Now, let us analyze the given eigenvalue problem: The expression

$$\frac{\partial U}{\partial t} = DA^s U + JU,$$

is being referred to. The equation

$$(DA^s + J)\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \mu\begin{pmatrix}\phi\\ \psi\end{pmatrix},$$

delineates an eigenvalue issue, as μ represents an eigenvalue of $DA^s + J\,(u^*, v^*)$. The matrix $-\lambda_{i,s}^N D + J$ possesses μ as an eigenvalue if and only if for every $i \geq 0$. The assessment of local stability of (u^*, v^*) necessitates the application of the characteristic polynomial

$$\det\left(\mu I - \lambda_{i,s}^N D - J\,(u^*, v^*)\right) = \mu^2 + P_i\mu + \sigma b Q_i.$$

Alternatively, it can be represented as

$$\sum_{0\leq i\leq\infty, 1\leq j\leq m_i}\begin{pmatrix} f_0 - \lambda_{i,s}^N - \mu & f_1 \\ \sigma b g_0 & \sigma b g_1 - \sigma c\lambda_{i,s}^N - \mu\end{pmatrix}\begin{pmatrix} a_{i,j}\\ b_{i,j}\end{pmatrix}\phi_{i,j} = \begin{pmatrix}0\\0\end{pmatrix}$$

based on

$$\phi = \sum_{0\leq i\leq\infty, 1\leq j\leq m_i} a_{i,j}\phi_{i,j} \quad\text{and}\quad \psi = \sum_{0\leq i\leq\infty, 1\leq j\leq m_i} b_{i,j}\phi_{i,j}$$

after (2.10), and condition (2.13), and

$$\begin{aligned} P_i &= -\left(f_0 - \lambda_{i,s}^N + \sigma b g_1 - \sigma c\lambda_{i,s}^N\right)\\ &= (1+\sigma c)\,\lambda_{i,s}^N - (f_0 + \sigma b g_1)\\ &= (1+\sigma c)\,\lambda_{i,s}^N - \left(\frac{3\alpha^2 - 5}{1+\alpha^2} - \frac{\sigma b\alpha}{1+\alpha^2}\right) > 0\end{aligned}$$

and

$$\begin{aligned} \sigma b Q_i &= \left(f_0 - \lambda_{i,s}^N\right)\left(\sigma b g_1 - \sigma c\lambda_{i,s}^N\right) - \sigma b g_0 f_1\\ Q_i &= \underbrace{\frac{c}{b}}_{=d}\lambda_{i,s}^N\left(\lambda_{i,s}^N - f_0\right) + \left(f_0 - \lambda_{i,s}^N\right) g_1 - g_0 f_1\\ &= d\lambda_{i,s}^N\left(\lambda_{i,s}^N - f_0\right) + \left(\left(\frac{3\alpha^2-5}{1+\alpha^2} - \lambda_{i,s}^N\right)\left(-\frac{\alpha}{1+\alpha^2}\right) - \frac{2\alpha^2}{1+\alpha^2}\left(-\frac{4\alpha}{1+\alpha^2}\right)\right)\\ &= d\lambda_{i,s}^N\left(\lambda_{i,s}^N - f_0\right) + \left(\frac{3\alpha^2-5}{1+\alpha^2} - \lambda_{i,s}^N - \frac{8\alpha^2}{1+\alpha^2}\right)\left(-\frac{\alpha}{1+\alpha^2}\right)\\ &= d\lambda_{i,s}^N\left(\lambda_{i,s}^N - f_0\right) + \left(5 + \lambda_{i,s}^N\right)\left(\frac{\alpha}{1+\alpha^2}\right)\end{aligned}$$

$$Q_i = d\lambda_{i,s}^N\left(\lambda_{i,s}^N - f_0\right) + \frac{\alpha\left(5+\lambda_{i,s}^N\right)}{1+\alpha^2} \tag{4.3}$$

results from $c = bd$ and $f_0 g_1 - g_0 f_1 = 5\alpha/\left(1+\alpha^2\right)$. It is evident that $Q_0 > 0$ when $\lambda_0 = 0$. Currently, we are confronted with two possible results.

[2]*Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

(i) If $\lambda_{1,s}^N \geq f_0$, then according to equation (4.3), $Q_i > 0$ for all $i \geq 1$. Therefore, the equilibrium (u^*, v^*) is asymptotically stable.

(ii) If $\lambda_{1,s}^N < f_0$. In this instance, we have two separate instances.

- If $0 < d < \tilde{d}$, then

$$\lambda_{i,s}^N < f_0 \quad \text{and} \quad d < d_i, \quad i \in [1, i_\alpha].$$

As a result,

$$\begin{aligned}
Q_i &= d\lambda_{i,s}^N \left(\lambda_{i,s}^N - f_0\right) + \frac{\alpha\left(5 + \lambda_{i,s}^N\right)}{1+\alpha^2} \\
&> \tilde{d}\lambda_{i,s}^N \left(\lambda_{i,s}^N - f_0\right) + \frac{\alpha\left(5 + \lambda_{i,s}^N\right)}{1+\alpha^2} \\
&= \lambda_i \left(\min_{1\leq i\leq i_\alpha} d_i\right) \left(\lambda_{i,s}^N - f_0\right) + \frac{\alpha\left(5 + \lambda_{i,s}^N\right)}{1+\alpha^2} \\
&> \lambda_{i,s}^N d_i \left(\underbrace{\lambda_{i,s}^N - f_0}_{\leq 0}\right) + \frac{\alpha\left(5 + \lambda_{i,s}^N\right)}{1+\alpha^2} \\
&= \lambda_{i,s}^N \left(\frac{\alpha}{1+\alpha^2} \frac{\lambda_{i,s}^N + 5}{\lambda_{i,s}^N\left(f_0 - \lambda_{i,s}^N\right)}\right) \left(\lambda_{i,s}^N - f_0\right) + \frac{\alpha\left(5 + \lambda_{i,s}^N\right)}{1+\alpha^2} \\
&> 0
\end{aligned}$$

for any $i \in [1, i_\alpha]$. Furthermore, if $i > i_\alpha$, then $\lambda_{i,s}^N \geq f_0$ and $Q_i > 0$ according to equation (4.3). The argument once again demonstrates the asymptotic stability of (u^*, v^*).

- If $d > \tilde{d}$, we may conclude that $k \in [1, i_\alpha]$ and obtain its minimum value at expression (4.2). Therefore,

$$d > d_k \quad \text{and} \quad \lambda_{k,s}^N < f_0,$$

which indicates that

$$\begin{aligned}
Q_k &= d\lambda_{k,s}^N \left(\lambda_{k,s}^N - f_0\right) + \frac{\alpha\left(5 + \lambda_{k,s}^N\right)}{1+\alpha^2} \\
&< d_k\lambda_{k,s}^N \left(\underbrace{\lambda_{k,s}^N - f_0}_{\leq 0}\right) + \frac{\alpha\left(5 + \lambda_{k,s}^N\right)}{1+\alpha^2} \\
&= \lambda_{k,s}^N \left(\frac{\alpha}{1+\alpha^2} \frac{\lambda_{k,s}^N + 5}{\lambda_{k,s}^N\left(f_0 - \lambda_{k,s}^N\right)}\right) \left(\lambda_{k,s}^N - f_0\right) + \frac{\alpha\left(5 + \lambda_{k,s}^N\right)}{1+\alpha^2} \\
&< 0,
\end{aligned}$$

[2] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

and consequently, (u^*, v^*) is unstable.

$\square$

[2] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

4.2 Global Asymptotic Stability

This subsection presents the established, necessary conditions for global asymptotic stability.

Theorem 17. *Assume that $0 < a^2 < 27$. The solution is $(u(t,x), v(t,x))$ for System (3.1). It converges uniformly in x to (u_*, v_*) for any nonnegative u_0, v_0.*

Proof. In order to guarantee global asymptotic stability. The Lyapunov function given in [53] is defined as

$$E(t) = \int_\Omega \left[E_1(u(t,x)) + E_2(v(t,x))\right] dx.$$

where

$$E_1(u) = \sigma b \int \left(u^2 - u_*^2\right) du = \sigma b \left(\tfrac{u^3}{3} - u_*^2 u\right), \quad E_2(v) = 4\int (v - v_*)\, dv = 2v^2 - 4v_* v.$$

Therefore,

$$\begin{aligned}
E'(t) &= \int_\Omega \left[\frac{\partial}{\partial u} E_1(u(t,x))\right] u_t dx + \int_\Omega \left[\frac{\partial}{\partial v} E_2(v(t,x))\right] v_t dx \\
&= \sigma b \int_\Omega \left[\left(u^2 - u_*^2\right)\left(a - u - \frac{4uv}{1+u^2}\right) + 4(v - v_*)\left(\left(u - \frac{uv}{1+u^2}\right)\right)\right] dx \\
&\quad -\sigma b \int_\Omega \left(u^2 - u_*^2\right)(-\Delta)^s u dx - 4\sigma c \int_\Omega (v - v_*)(-\Delta)^s v dx \\
&= \sigma b \int_\Omega \underbrace{\frac{u}{1+u^2}}_{=\varphi(u)} \left[\left(u^2 - u_*^2\right)\left(\underbrace{\frac{(a-u)(1+u^2)}{u}}_{=h(u)} - 4v\right) + 4(v - v_*)\left(1 + u^2 - v\right)\right] dx \\
&\quad -\sigma b \left[\int_\Omega \left(u^2 - u_*^2\right)(-\Delta)^s u - \frac{4c}{b}\int_\Omega (v - v_*)(-\Delta)^s v \right] dx.
\end{aligned}$$

Using equation (1.7), we deduce that

$$\begin{aligned}
E'(t) &= \sigma b \int_\Omega \varphi(u) \left[\left(u^2 - u_*^2\right)(h(u) - 4v) + 4(v - v_*)\left(1 + u^2 - v\right)\right] dx - \sigma b I(t), \\
&= \sigma b \int_\Omega \varphi(u) \left[\begin{array}{c} (u^2 - u_*^2)(h(u) - 4v) + 4(v - v_*)(u^2 - u_*^2) \\ +4(v - v_*)(1 + u_*^2 - v) \end{array}\right] dx - \sigma b I(t), \\
&= \sigma b \int_\Omega \varphi(u) \left[\left(u^2 - u_*^2\right)\left(h(u) - \underbrace{4v_*}_{=h(u_*)}\right) + 4(v - v_*)\left(\underbrace{1 + u_*^2}_{=v_*} - v\right)\right] dx - \sigma b I(t), \\
&= \sigma b \int_\Omega \varphi(u) \left[\left(u^2 - u_*^2\right)(h(u) - h(u_*)) - 4(v - v_*)^2\right] dx - \sigma b I(t),
\end{aligned}$$

where

$$\begin{aligned}
I(t) &= \frac{1}{2}\int_\Omega \int_\Omega \frac{(u(t,x) - u(t,y))^2 (u(t,x) + u(t,y))}{|x-y|^{n+2s}} dxdy \\
&\quad + \frac{2}{\sigma b}\int_\Omega \int_\Omega \frac{(v(t,x) - v(t,y))^2}{|x-y|^{n+2s}} dxdy,
\end{aligned}$$

[2] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

and

$$\varphi(u) = \frac{u}{1+u^2}, \quad h(u) = \frac{(a-u)\left(1+u^2\right)}{u}.$$

If condition

$$0 < a^2 < 27$$

is satisfied, with $u = u_*$ then equation

$$h(u_*) = \frac{(a-u_*)\left(1+u_*^2\right)}{u_*}.$$

Consequently, equation

$$h'(u) < 0.$$

As a result, the function $h(u)$ is a strictly decreasing function, and we have

$$\left(u^2 - u_*^2\right)(h(u) - h(u_*)) \leq 0.$$

As a result,

$$E^{'}(t) \leq 0.$$

Put simply, E decreases monotonically if

$$E^{'}(t) = 0 \Leftrightarrow (u,v) = (u_*, v_*).$$

Using the argument presented in [53], we deduce that (u_*, v_*) is the only possible ω-limit set. $\square$

Theorem 18. *The equilibrium* (u_*, v_*) *is globally asymptotically stable, according to condition* $0 < a^2 \leq 27$.

Proof. The functional $V(t)$ has a non-positive derivative and is positive-definite.

Furthermore, if $(u(t,x), v(t,x)) \in \Re$ constitutes a solution to system (3.1) Given the condition $\frac{\partial V(t)}{\partial t} = 0$, it is established that $\int_\Omega \int_\Omega \frac{(u(t,x)-u(t,y))^2}{|x-y|^{n+\alpha}} dxdy$ remains constant. This means that u and v have spatial *homogeneity. Hence, the ODE system is fulfilled by the values* (u,v). The LaSalle invariance theorem

$$\lim_{\to\infty} |u(t,x) - u_*| = \lim_{t\to\infty} |v(t,x) - v_*| = 0$$

is established by observing that (u_*, v_*) constitutes the biggest subset

$$\left\{(u;v) \in \Re \middle| \frac{\partial V(t)}{\partial t} = 0\right\},$$

of the differential ODE system that remains invariant uniformly in x. The point is further elucidated by [30],[52]. *As a result,*

$$\lim_{t\to\infty} \int_\Omega (u(t,x) - u_*)^2 dx = \lim_{t\to\infty} \int_\Omega (v(t,x) - v_*)^2 dx = 0.$$

$\square$

[2] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

4.3 Numerical Examples

Example 1 Presents the simulation settings for the Lengyel-Epstein fractional Laplacian model of the CIMA reaction as delineated in (3.1).

		a	σ	b	c	$u(x,0)$	$v(x,0)$	s
Stable	No diffusion	3.5	1	3	–	1	2	$\frac{2}{3}$
Stable	Diffusion	3.5	1	3	5	$1+\sin(x)$	$2+\sin(x)$	$\frac{2}{3}$

The equilibrium solution

$$(u^*, v^*) = (\alpha, 1+\alpha^2) = (0.7, 1.49)\,, \quad \text{where} \quad \alpha = \tfrac{a}{5},$$

is stable in the context of ODE, given condition

$$3\alpha^2 - 5 = 3 \times \left(\frac{3.5}{5}\right)^2 - 5 = -3.53 < 2.1 = 1 \times \frac{3.5}{5} \times 3 = \sigma\alpha b.$$

These parameters evidently satisfy the first criterion

$$a^2 = (3.5)^2 = 12.25 \leq 27.$$

Consequently, prior findings ensure the global asymptotic stability of the solutions.

Example 2 Specifies the simulation parameters for the Lengyel-Epstein fractional Laplacian model of the CIMA reaction as specified in (3.1).

		a	σ	b	c	$u(x,0)$	$u(x,0)$	s
Stable	No diffusion	5.1	0.5	1	–	1	2	$\frac{2}{3}$
Stable	Diffusion	5.1	0.5	1	1	4	3	$\frac{2}{3}$

The equilibrium solution

$$(u^*, v^*) = (\alpha, 1+\alpha^2) = (1.02, 2.0404)\,, \quad \text{where} \quad \alpha = \tfrac{a}{5} = 1.02,$$

is stable in the context of ordinary differential equations, given condition

$$3\alpha^2 - 5 = 3 \times \left(\frac{5.1}{5}\right)^2 - 5 = -1.8788 < 0.51 = 0.5 \times \frac{5.1}{5} \times 1 = \sigma\alpha b.$$

These parameters evidently satisfy this initial criterion

$$a^2 = (5.1)^2 = 26.01 \leq 27.$$

Therefore, previous results can ensure the global asymptotic stability of the solutions.

[2] *Article published in Boundary Value Problems, 2024, pages 1-20. See [54]*

General conclusion

This study investigates the chloriteiodidemalonic acid (CIMA) chemical reaction using a space-fractional version of the LengyelEpstein system. The main objective of this study was to establish the existence of an invariant region. In addition, the global existence, uniqueness, and positivity of solutions were established. Furthermore, the boundary value problem associated with the regional fractional Laplacian was investigated. Fundamental estimates for nonconstant steady-state solutions were derived, leading to improved sufficient conditions for the nonexistence of positive nonconstant steady-state solutions, depending on both the size of the spatial domain and the diffusion coefficient. Finally, sufficient conditions ensuring the global asymptotic stability of the unique positive steady-state solution were established using the direct Lyapunov method.

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