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*Stabilization of Chaotic Fractional
Discrete Neural Networks Based on the
Caputo -h- Difference Operator*

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Dedication

To my dear father,

the man who has always been my support in every step, who taught me that strength is not in words, but in patience, endurance, and rising again no matter the circumstances.

You have always been the steady ground I return to, and the unwavering support I rely on. I carry my deepest gratitude and love for you.

This achievement carries a great part of you.

To my beloved mother,

the heart that never grew tired of praying for me, the hand that wiped away my exhaustion before I even spoke of it, and the soul that has accompanied me since my first day at school until this very moment.

You were with me in every small success before the big ones, and in every moment of weakness, fear, and fatigue.

No words can ever describe your right over me, for some forms of giving are beyond expression.

To my dear grandfather,

who was like a second father to me, a fundamental part of my memories and life, and whose hands I grew up under while learning countless things without even realizing it.

Your presence in my life has been a great blessing that I can never repay.

To my beloved grandmother,

whose care and fear for me were pure love, and whose attention to every detail of my life made me feel constantly surrounded by a family that would never let life break me easily.

May God always protect you for me.

To my dear siblings,

thank you for being part of my days, my laughter, my stress, and my academic journey, and for making everything lighter simply by being present.

To all my friends,

who shared with me a long journey filled with effort, pressure, and

*beautiful moments, and who made the path lighter despite its difficulties.
University does not only teach knowledge; it teaches one how to survive
within chaos and maintain oneself as much as possible...
quite fitting for someone studying chaos itself.
A beautiful academic paradox indeed.*

*To everyone who supported me, even with a kind word or a sincere prayer,
I dedicate this humble work.*

*“I am not what happened to me, but I am what I choose to become.”
— Carl Gustav Jung*

*In conclusion, I find myself close to this idea; everything I have gone
through was never my definition, but only a path I walked through.
What I am today is shaped by what I chose despite the hardships,
and by what I never gave up on until the very end.*

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And to my friend Mohamed, who is not just a friend to me, but a true brother with whom I shared many aspects of life.

Abstract

The objective of this thesis is to present a general study of discrete fractional-order dynamical systems, with a particular focus on chaotic fractional discrete neural networks, stability theory, and chaotic behavior.

More specifically, we are interested in fractional difference systems, including Caputo and h -fractional operators, where memory effects play a fundamental role in the system dynamics.

The study investigates both linear and nonlinear discrete fractional models and discusses several stability concepts, including Mittag–Leffler stability, which is widely used in the analysis of fractional systems. In addition, chaotic dynamics are examined through mathematical tools such as Lyapunov exponents, bifurcation analysis, and fractal dimensions, providing insight into the complex behavior of nonlinear discrete systems.

Furthermore, fractional-order discrete neural networks are considered as an important application of these systems. Neural networks have demonstrated remarkable capabilities in various tasks, including pattern recognition, image classification, natural language processing, and regression analysis. They have found applications in diverse fields such as finance, healthcare, marketing, robotics, and autonomous vehicles, among others.

Keywords: Chaotic fractional discrete neural networks; Caputo h -difference operator; Fractional difference equations; Stabilization; Mittag-Leffler stability; Lyapunov exponents; Bifurcation analysis; Chaos.

Resumé

L'objectif de cette thèse est de présenter une étude générale des systèmes dynamiques discrets d'ordre fractionnaire, avec un intérêt particulier porté aux réseaux de neurones discrets fractionnaires chaotiques, à la théorie de la stabilité et au comportement chaotique.

Plus précisément, nous nous intéressons aux systèmes aux différences fractionnaires, notamment ceux basés sur les opérateurs de Caputo et h -fractionnaires, dans lesquels les effets de mémoire jouent un rôle fondamental dans la dynamique du système.

Cette étude examine à la fois des modèles discrets fractionnaires linéaires et non linéaires et discute plusieurs concepts de stabilité, notamment la stabilité de type Mittag-Leffler, largement utilisée dans l'analyse des systèmes fractionnaires. En outre, les dynamiques chaotiques sont étudiées à l'aide d'outils mathématiques tels que les exposants de Lyapunov, l'analyse des bifurcations et les dimensions fractales, offrant ainsi un aperçu du comportement complexe des systèmes discrets non linéaires.

Par ailleurs, les réseaux de neurones discrets d'ordre fractionnaire sont considérés comme une application importante de ces systèmes. Les réseaux de neurones ont démontré des capacités remarquables dans diverses tâches, notamment la reconnaissance de formes, la classification d'images, le traitement du langage naturel et l'analyse de régression. Ils trouvent des applications dans divers domaines tels que la finance, la santé, le marketing, la robotique et les véhicules autonomes, entre autres.

Mots-clés : Systèmes dynamiques fractionnaires discrets ; Opérateur de différence h de Caputo ; Réseaux de neurones fractionnaires chaotiques ; Stabilité de Mittag-Leffler ; Exposants de Lyapunov ; Chaos.

ملخص

تهدف هذه المذكرة إلى تقديم دراسة عامة للأنظمة الديناميكية الكسرية المتقطعة، مع التركيز بشكل خاص على الشبكات العصبية المتقطعة الكسرية الفوضوية، ونظرية الاستقرار، والسلوك الفوضوي. وبشكل أكثر تحديداً، نهتم بدراسة أنظمة المعادلات الفرقية الكسرية، بما في ذلك المؤثرات الكسرية من نوع كابوتو ومؤثرات h -الكسرية، حيث تلعب تأثيرات الذاكرة دوراً أساسياً في ديناميكية هذه الأنظمة.

تتناول الدراسة النماذج الكسرية المتقطعة الخطية وغير الخطية، كما تستعرض عدة مفاهيم للاستقرار، من بينها استقرار ميتاغ--ليفلر، الذي يُعد من أكثر المفاهيم استخداماً في تحليل الأنظمة الكسرية. بالإضافة إلى ذلك، تتم دراسة الديناميكيات الفوضوية باستخدام أدوات رياضية مثل أسس ليابونوف، وتحليل التشعبات، والأبعاد الكسرية، مما يتيح فهماً أعمق للسلوك المعقد للأنظمة المتقطعة غير الخطية.

علاوة على ذلك، تُدرس الشبكات العصبية المتقطعة ذات الرتبة الكسرية باعتبارها تطبيقاً مهماً لهذه الأنظمة. وقد أظهرت الشبكات العصبية قدرات متميزة في العديد من المهام، بما في ذلك التعرف على الأنماط، وتصنيف الصور، ومعالجة اللغات الطبيعية، وتحليل الانحدار. كما وجدت تطبيقات واسعة في مجالات متنوعة مثل التمويل، والرعاية الصحية، والتسويق، والروبوتات، والمركبات ذاتية القيادة، وغيرها.

الكلمات المفتاحية: الشبكات العصبية المتقطعة الكسرية الفوضوية؛ مؤثر الفرق h من نوع كابوتو؛ المعادلات الفرقية الكسرية؛ الاستقرار؛ استقرار ميتاغ--ليفلر؛ أسس ليابونوف؛ تحليل التشعبات؛ الفوضى.

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Introduction

The study of discrete fractional-order dynamical systems has become an important field in modern applied mathematics due to its ability to model complex systems with memory and nonlocal behavior. Unlike classical integer-order models, fractional systems incorporate the entire history of the process, which leads to more accurate representations of real-world phenomena.

Discrete fractional calculus extends classical difference equations by introducing fractional sum and difference operators, such as the Caputo fractional difference operator and the h -fractional difference operator. These operators allow the description of systems where past states influence current dynamics.

Stability analysis is a central topic in the study of dynamical systems. In the fractional discrete setting, classical stability concepts are generalized, leading to notions such as Mittag-Leffler stability, which is widely used to describe the asymptotic behavior of fractional systems.

In addition, chaotic behavior in discrete fractional systems has attracted significant attention. This behavior is characterized by sensitivity to initial conditions, bifurcations, and complex geometric structures, which are studied using tools such as Lyapunov exponents and fractal dimensions.

Finally, fractional-order discrete neural networks provide a powerful framework for modeling nonlinear dynamical systems with memory effects, combining neural computation with fractional dynamics.

Stability analysis of neural networks is crucial for assessing their behavior under different conditions and ensuring their robustness and reliability in real-world applications. It involves examining how small perturbations or changes in the network parameters affect their performance and stability over time. Understanding the stability properties of neural networks is essential for designing efficient learning algorithms, optimizing network architectures, and deploying neural networks in safety-critical systems.

This thesis is organized into four chapters. The first chapter introduces discrete fractional calculus and stability theory. The second chapter focuses on chaos in fractional discrete systems. The third chapter presents fractional-order neural networks. The fourth chapter studies stabilization of chaotic fractional discrete neural networks based on the Caputo h -difference operator.

Chapter 1

Discrete Fractional Calculus and Stability Theory

This chapter establishes the theoretical foundations for discrete fractional calculus and the stability analysis of fractional-order discrete systems. It introduces fractional sum and difference operators, Caputo and Riemann–Liouville types, the generalized h -difference operator, and stability notions such as Lyapunov and Mittag–Leffler stability. These preliminaries are essential for the study of chaotic fractional discrete neural networks in later chapters.

1.1 Preliminaries and Notation

This section recalls standard discrete calculus operations and sets the notation used throughout the thesis.

Definition 1.1 (Gamma function). [1] The Gamma function, denoted by $\Gamma(z)$, is defined by

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds,$$

for all $z \in \mathbb{C}$ with $\Re(z) > 0$. It admits a meromorphic extension to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$.

Definition 1.2 (Falling factorial). [1] For $t \in \mathbb{R}$ and $n \in \mathbb{N}_0$, the falling h -factorial of order n is defined by

$$t_h^n = t(t-h)(t-2h) \cdots (t-(n-1)h), \quad t_h^0 = 1.$$

Example (Falling h -factorial). [1] We compute $t_h^{(n)}$ for several concrete values.

For $h = 1$, $t = 5$, $n = 3$:

$$5_1^{(3)} = 5 \cdot 4 \cdot 3 = 60.$$

This coincides with the classical falling factorial $t^n = \frac{t!}{(t-n)!}$, which can be verified: $\frac{5!}{2!} = 60$.

For $h = 0.5$, $t = 2$, $n = 4$:

$$2_{0.5}^{(4)} = 2 \cdot 1.5 \cdot 1.0 \cdot 0.5 = 1.5.$$

For $h = 1$, $t = 3$, $n = 0$, the convention gives $3_1^{(0)} = 1$, regardless of t .

For $h = 2$, $t = 8$, $n = 3$:

$$8_2^{(3)} = 8 \cdot 6 \cdot 4 = 192.$$

These cases illustrate that the falling h -factorial starts at t and descends in steps of size h , stopping after n factors. This finite product structure generalizes to non-integer orders $\alpha \in \mathbb{R}$ via the Gamma function ratio

$$t_h^{(\alpha)} = h^\alpha \frac{\Gamma\left(\frac{t}{h} + 1\right)}{\Gamma\left(\frac{t}{h} + 1 - \alpha\right)},$$

which recovers the product above when $\alpha = n \in \mathbb{N}_0$.

Definition 1.3 (Generalized falling factorial). [1] For $\alpha \in \mathbb{R}$, the generalized falling factorial is defined by

$$t_h^\alpha = h^\alpha \frac{\Gamma\left(\frac{t}{h} + 1\right)}{\Gamma\left(\frac{t}{h} + 1 - \alpha\right)}. \quad (1.1)$$

Definition 1.4 (Binomial coefficient). [2] For $n, k \in \mathbb{N}_0$ with $k \leq n$, the binomial coefficient is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n_h^k}{h^k \Gamma(k+1)}.$$

Definition 1.5 (Generalized binomial coefficient). [2] For $x, y \in \mathbb{R}$, the generalized binomial coefficient is defined by

$$\binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)},$$

whenever the Gamma functions are finite.

Definition 1.6 (Discrete time domains). [1] Let

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}, \quad \mathbb{N} = \{1, 2, 3, \dots\}.$$

For $h > 0$, define

$$h\mathbb{N}_0 = \{0, h, 2h, \dots\}.$$

For $a \in \mathbb{R}$, define

$$(h\mathbb{N})_a = \{a, a+h, a+2h, \dots\}.$$

For $a < b$ with $(b-a)/h \in \mathbb{N}_0$, define

$$[a, b]_h = \{a, a+h, a+2h, \dots, b\}.$$

Definition 1.7 (Forward and backward difference operators). [2] Let $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$. The forward (delta) difference operator is

$$\Delta_h x(t) = x(t+h) - x(t), \quad t, t+h \in (h\mathbb{N})_a,$$

and the backward (nabla) difference operator is

$$\nabla_h x(t) = x(t) - x(t-h), \quad t, t-h \in (h\mathbb{N})_a.$$

Remark. When $h = 1$, we recover $\Delta x(n) = x(n+1) - x(n)$ and $\nabla x(n) = x(n) - x(n-1)$.

Definition 1.8 (Shift operators). [2] Define

$$\sigma_h(t) = t + h, \quad \rho_h(t) = t - h, \quad t \in (h\mathbb{N})_a.$$

Then

$$\Delta_h x(t) = x(\sigma_h(t)) - x(t), \quad \nabla_h x(t) = x(t) - x(\rho_h(t)).$$

Definition 1.9 (Rising factorial). [1] For $t \in \mathbb{R}$ and $n \in \mathbb{N}_0$, define

$$t_h^{\bar{n}} = t(t+h)(t+2h) \cdots (t+(n-1)h), \quad t_h^{\bar{0}} = 1.$$

Definition 1.10 (Generalized rising factorial). [1] For $\alpha \in \mathbb{R}$, define

$$t_h^{\bar{\alpha}} = h^\alpha \frac{\Gamma\left(\frac{t}{h} + \alpha\right)}{\Gamma\left(\frac{t}{h}\right)}.$$

Definition 1.11 (Kernel functions for fractional sums). [2] Define For $\alpha > 0$, define

$$k_\alpha(t, s) = (t - \sigma_h(s))_h^{\alpha-1}, \quad t \in (h\mathbb{N})_a, \quad s \in [a, t-h]_h.$$

Definition 1.12 (Convolution of sequences). [6] Let $f, g : (h\mathbb{N})_a \rightarrow \mathbb{R}$. The discrete convolution is

$$(f * g)(t) = \sum_{s \in [a, t]_h} f(s) g(t-s), \quad t \in (h\mathbb{N})_a.$$

Definition 1.13 (Sequence spaces). [6] The space of bounded sequences:

$$\ell^\infty((h\mathbb{N})_a) = \{x : (h\mathbb{N})_a \rightarrow \mathbb{R} : \|x\|_\infty < \infty\}, \quad \|x\|_\infty = \sup_{t \in (h\mathbb{N})_a} |x(t)|.$$

The space of finitely supported sequences:

$$c_{00}((h\mathbb{N})_a) = \{x \in \ell^\infty : x(t) = 0 \text{ for all but finitely many } t\}.$$

For $1 \leq p < \infty$:

$$\ell^p((h\mathbb{N})_a) = \{x : (h\mathbb{N})_a \rightarrow \mathbb{R} : \|x\|_p < \infty\}, \quad \|x\|_p = \left(\sum_{k=0}^{\infty} |x(a+kh)|^p \right)^{1/p}.$$

Definition 1.14 (Vector-valued sequences). [6] Let $x : (h\mathbb{N})_a \rightarrow \mathbb{R}^n$. Then

$$\|x\|_p = \left(\sum_{k=0}^{\infty} \|x(a+kh)\|^p \right)^{1/p}, \quad \|x\|_\infty = \sup_{k \geq 0} \|x(a+kh)\|,$$

where $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$.

1.2 Fractional Sum and Difference Operators

We define fractional sums and differences as generalizations of integer-order operators, introducing memory into discrete systems.

1.2.1 Fractional Sum Operator

Definition 1.15 (Left fractional h -sum [1]). Let $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$ and $\alpha > 0$. The left fractional h -sum of order α starting at a is defined by

$$({}_a\Delta_h^{-\alpha}x)(t) = \frac{h}{\Gamma(\alpha)} \sum_{k=0}^n (t - \sigma_h(a + kh))_h^{\alpha-1} x(a + kh), \quad t \in [a, a + nh]_h, \quad n \in \mathbb{N}_0.$$

Definition 1.16 (Right fractional h -sum [6]). Let $x : (h\mathbb{N})_b \rightarrow \mathbb{R}$ and $\alpha > 0$. The right fractional h -sum of order α ending at b is

$$(h\Delta_b^{-\alpha}x)(t) = \frac{h}{\Gamma(\alpha)} \sum_{k=0}^m (\rho_h(b - kh) - t)_h^{\alpha-1} x(b - kh), \quad t \in [b - mh, b]_h, \quad m \in \mathbb{N}_0.$$

Definition 1.17 (Kernel functions [5]). The kernel of the left fractional h -sum is

$$k_\alpha(t, s) = (t - \sigma_h(s))_h^{\alpha-1}, \quad s \in [a, t - h]_h.$$

For the right fractional sum, the kernel is

$$\tilde{k}_\alpha(t, s) = (\rho_h(s) - t)_h^{\alpha-1}, \quad s \in [t + h, b]_h.$$

Definition 1.18 (Fractional convolution). [5] For sequences $f, g : (h\mathbb{N})_a \rightarrow \mathbb{R}$, the fractional convolution is

$$(f *_\alpha g)(t) = \sum_{s=a}^{t-h} k_\alpha(t, s) g(s), \quad t \in (h\mathbb{N})_a.$$

Then the left fractional h -sum can be written as

$${}_a\Delta_h^{-\alpha}x = x *_\alpha 1, \quad 1(s) \equiv 1.$$

Example (Left fractional h -sum of a constant function). We apply the left fractional h -sum operator ${}_a\Delta_h^{-\alpha}$ to the constant function $x(t) = 1$ for $a = 0$, $h = 1$, $\alpha = 0.5$, and compute the first few values explicitly.

By Definition:

$$({}_a\Delta_h^{-\alpha}x)(t) = \frac{h}{\Gamma(\alpha)} \sum_{k=0}^n (t - \sigma(kh))_h^{(\alpha-1)} x(kh), \quad t = a + nh.$$

Since $x(kh) = 1$ for all k , $h = 1$, $a = 0$, and $\sigma(kh) = (k + 1)h$, this reduces to:

$$({}_0\Delta_1^{-0.5}1)(n) = \frac{1}{\Gamma(0.5)} \sum_{k=0}^n (n - k)_1^{(-0.5)}.$$

Recall that $\Gamma(0.5) = \sqrt{\pi}$ and the generalized falling factorial is:

$$(n - k)_1^{(-0.5)} = \frac{\Gamma(n - k + 1)}{\Gamma(n - k + 1.5)}.$$

Computing term by term for $n = 0, 1, 2$:

For $n = 0$:

$$({}_0\Delta_1^{-0.5} 1)(0) = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(1)}{\Gamma(1.5)} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\frac{\sqrt{\pi}}{2}} = \frac{2}{\pi} \approx 0.637.$$

For $n = 1$:

$$({}_0\Delta_1^{-0.5} 1)(1) = \frac{1}{\sqrt{\pi}} \left(\frac{\Gamma(2)}{\Gamma(2.5)} + \frac{\Gamma(1)}{\Gamma(1.5)} \right) = \frac{1}{\sqrt{\pi}} \left(\frac{1}{\frac{3\sqrt{\pi}}{4}} + \frac{2}{\sqrt{\pi}} \right) \approx 0.910.$$

For $n = 2$:

$$({}_0\Delta_1^{-0.5} 1)(2) = \frac{1}{\sqrt{\pi}} \left(\frac{\Gamma(3)}{\Gamma(3.5)} + \frac{\Gamma(2)}{\Gamma(2.5)} + \frac{\Gamma(1)}{\Gamma(1.5)} \right) \approx 1.128.$$

The sequence of values (0.637, 0.910, 1.128, ...) is increasing, as expected: the fractional sum of the constant function $x = 1$ accumulates past values with power-law weights $(n - k)_1^{(\alpha-1)}$, producing a growing sequence that generalizes the classical partial sum $\sum_{k=0}^n 1 = n + 1$ corresponding to $\alpha = 1$.

Theorem 1.1 (Linearity of fractional sums). [3] Let $x, y : (h\mathbb{N})_a \rightarrow \mathbb{R}$ and $\alpha > 0$. For any scalars $c_1, c_2 \in \mathbb{R}$,

$$a\Delta_h^{-\alpha}(c_1x + c_2y)(t) = c_1 a\Delta_h^{-\alpha}x(t) + c_2 a\Delta_h^{-\alpha}y(t), \quad t \in [a, a + nh]_h.$$

Proof. By the definition of the fractional sum,

$$\begin{aligned} a\Delta_h^{-\alpha}(c_1x + c_2y)(t) &= \frac{h}{\Gamma(\alpha)} \sum_{k=0}^n (t - \sigma_h(a + kh))_h^{\alpha-1} (c_1x(a + kh) + c_2y(a + kh)) \\ &= c_1 \frac{h}{\Gamma(\alpha)} \sum_{k=0}^n (t - \sigma_h(a + kh))_h^{\alpha-1} x(a + kh) \\ &\quad + c_2 \frac{h}{\Gamma(\alpha)} \sum_{k=0}^n (t - \sigma_h(a + kh))_h^{\alpha-1} y(a + kh) \\ &= c_1 a\Delta_h^{-\alpha}x(t) + c_2 a\Delta_h^{-\alpha}y(t). \end{aligned}$$

This proves the linearity. □

Theorem 1.2 (Semigroup property of fractional sums). [2] Let $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$ and $\alpha, \beta > 0$. Then

$$a\Delta_h^{-(\alpha+\beta)}x(t) = a\Delta_h^{-\alpha}(a\Delta_h^{-\beta}x)(t), \quad t \in [a, a + nh]_h.$$

Proof. Follows from the discrete convolution property:

$$a\Delta_h^{-\alpha}(a\Delta_h^{-\beta}x) = (x *_{\beta} 1) *_{\alpha} 1 = x *_{\alpha+\beta} 1 = a\Delta_h^{-(\alpha+\beta)}x.$$

□

Remark. For $\alpha = 1$, the left fractional sum reduces to the classical forward sum:

$$a\Delta_h^{-1}x(t) = h \sum_{k=0}^n x(a + kh), \quad t = a + nh.$$

Definition 1.19 (Domain of fractional sums). The left fractional sum $a\Delta_h^{-\alpha}x$ is defined for $t \in [a, a + nh]_h$ such that $\sigma_h(a + kh) \leq t$ for all summation indices k . The right fractional sum $h\Delta_b^{-\alpha}x$ is defined for $t \in [b - mh, b]_h$ such that $\rho_h(b - kh) \geq t$ for all summation indices k .

1.2.2 Fractional Difference Operator

Definition 1.20 (Left Riemann–Liouville fractional h -difference [6]). Let $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$ and $\alpha > 0$ with $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$. The left Riemann–Liouville fractional h -difference of order α starting at a is defined by

$$(a\Delta_h^\alpha x)(t) = \Delta_h^m (a\Delta_h^{-(m-\alpha)} x)(t), \quad t \in [a + (m - \alpha)h, \infty)_h,$$

where Δ_h^m is the m -fold forward difference.

Definition 1.21 (Right Riemann–Liouville fractional h -difference [6]). Let $x : (h\mathbb{N})_b \rightarrow \mathbb{R}$ and $\alpha > 0$, $m - 1 < \alpha \leq m$. The right Riemann–Liouville fractional h -difference of order α ending at b is

$$(h\Delta_b^\alpha x)(t) = (-1)^m \nabla_h^m (h\Delta_b^{-(m-\alpha)} x)(t), \quad t \in (-\infty, b - (m - \alpha)h]_h.$$

Definition 1.22 (Caputo fractional h -difference [3]). Let $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$ and $\alpha > 0$, $m - 1 < \alpha \leq m$. The left Caputo fractional h -difference is

$$(a^C \Delta_h^\alpha x)(t) = a\Delta_h^{-(m-\alpha)} (\Delta_h^m x)(t), \quad t \in [a + (m - \alpha)h, \infty)_h.$$

Remark. The Caputo fractional difference allows initial conditions to be expressed in terms of integer-order differences:

$$\Delta_h^k x(a), \quad k = 0, 1, \dots, m - 1.$$

Theorem 1.3 (Relation between Riemann–Liouville and Caputo). *Let $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$ and $\alpha > 0$, $m - 1 < \alpha \leq m$. Then*

$$a^C \Delta_h^\alpha x(t) = a\Delta_h^\alpha x(t) - \sum_{k=0}^{m-1} \frac{(t - \sigma_h(a))_h^{k-\alpha}}{\Gamma(k - \alpha + 1)} \Delta_h^k x(a), \quad t \in [a + (m - \alpha)h, \infty)_h.$$

Proof. From the definitions,

$$a^C \Delta_h^\alpha x = a\Delta_h^{-(m-\alpha)} (\Delta_h^m x) = \Delta_h^m (a\Delta_h^{-(m-\alpha)} x) - \sum_{k=0}^{m-1} \frac{(t - \sigma_h(a))_h^{k-\alpha}}{\Gamma(k - \alpha + 1)} \Delta_h^k x(a),$$

where the second term arises from the discrete fractional integration by parts formula. This proves the relation. $\square$

Definition 1.23 (Domain of fractional differences). [6]

- Left Riemann–Liouville and Caputo differences are defined for $t \in [a + (m - \alpha)h, \infty)_h$.
- Right Riemann–Liouville differences are defined for $t \in (-\infty, b - (m - \alpha)h]_h$.

Remark. For integer $\alpha = m$, the Caputo and Riemann–Liouville differences reduce to the standard m -fold forward or backward differences:

$$a^C \Delta_h^m x(t) = \Delta_h^m x(t), \quad a\Delta_h^m x(t) = \Delta_h^m x(t).$$

1.3 Caputo Fractional Difference Operator

Focuses on the Caputo fractional difference, which allows integer-order initial conditions while preserving fractional memory.

1.3.1 Definition and Basic Properties

Definition 1.24. [8] [Caputo fractional h -difference [4]] Let $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$, $\alpha > 0$, and $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$. The left Caputo fractional h -difference of order α is defined by

$$(a^C \Delta_h^\alpha x)(t) = a \Delta_h^{-(m-\alpha)} (\Delta_h^m x)(t), \quad t \in [a + (m - \alpha)h, \infty)_h.$$

Remark. Initial conditions can be expressed using integer-order differences:

$$\Delta_h^k x(a), \quad k = 0, \dots, m - 1.$$

The operator inherits the memory effect from the fractional sum: each value depends on all previous points of x .

Theorem 1.4 (Linearity). [8] Let $x, y : (h\mathbb{N})_a \rightarrow \mathbb{R}$ and $c_1, c_2 \in \mathbb{R}$. Then

$$a^C \Delta_h^\alpha (c_1 x + c_2 y)(t) = c_1 a^C \Delta_h^\alpha x(t) + c_2 a^C \Delta_h^\alpha y(t).$$

Proof. [8] By definition:

$$a^C \Delta_h^\alpha (c_1 x + c_2 y) = a \Delta_h^{-(m-\alpha)} \Delta_h^m (c_1 x + c_2 y) = a \Delta_h^{-(m-\alpha)} (c_1 \Delta_h^m x + c_2 \Delta_h^m y) = c_1 a^C \Delta_h^\alpha x + c_2 a^C \Delta_h^\alpha y.$$

□

Theorem 1.5 (Composition with fractional sums – semigroup property). [5] Let $\alpha, \beta > 0$, $\alpha > \beta$. Then

$$a \Delta_h^{-\alpha} (a^C \Delta_h^\beta x)(t) = a \Delta_h^{-(\alpha-\beta)} x(t), \quad t \in [a + (m - \beta)h, \infty)_h.$$

Proof.

$$a \Delta_h^{-\alpha} (a^C \Delta_h^\beta x) = a \Delta_h^{-\alpha} (a \Delta_h^{-(m-\beta)} \Delta_h^m x) = a \Delta_h^{-(\alpha+m-\beta)} \Delta_h^m x = a \Delta_h^{-(\alpha-\beta)} x.$$

□

Remark. If $\alpha = m$ is an integer, the Caputo fractional difference reduces to the standard m -th order difference:

$$a^C \Delta_h^m x(t) = \Delta_h^m x(t).$$

It allows specifying initial values in terms of integer-order differences, unlike Riemann–Liouville type operators.

1.3.2 Comparison with Riemann–Liouville Type Operators

Definition 1.25 (Riemann–Liouville fractional h -difference). [7]

$$a\Delta_h^\alpha x(t) = \Delta_h^m(a\Delta_h^{-(m-\alpha)}x)(t), \quad m-1 < \alpha \leq m.$$

Theorem 1.6 (Relation between Caputo and Riemann–Liouville). [7]

$$a^C\Delta_h^\alpha x(t) = a\Delta_h^\alpha x(t) - \sum_{k=0}^{m-1} \frac{(t - \sigma_h(a))_h^{k-\alpha}}{\Gamma(k - \alpha + 1)} \Delta_h^k x(a).$$

Proof. [7] Follows directly from the definitions and the discrete fractional integration by parts formula. The sum accounts for the difference between starting from fractional vs integer-order initial conditions. $\square$

Remark. Caputo type: initial conditions are expressed in integer differences.

Riemann–Liouville type: initial conditions involve fractional sums.

Both coincide when $\alpha = m \in \mathbb{N}$.

1.4 Fractional h -Difference Operator

Extends Caputo operators to uniform-step h -differences for practical discrete modeling.

1.4.1 Definition and Memory Effect

Definition 1.26. [7] [General fractional h -difference operator] Let $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$, $\alpha > 0$, $m-1 < \alpha \leq m$. Then

$$\Delta_h^\alpha x(t) = \Delta_h^m(a\Delta_h^{-(m-\alpha)}x)(t), \quad t \in [a + (m - \alpha)h, \infty)_h.$$

Remark (Memory effect).

$$\Delta_h^\alpha x(t) = \sum_{k=0}^n c_k x(a + kh), \quad t = a + nh, \quad c_k \neq 0 \ \forall k,$$

showing dependence on all past values of x .

Theorem 1.7 (Linearity). For $x, y : (h\mathbb{N})_a \rightarrow \mathbb{R}$ and scalars c_1, c_2 :

$$\Delta_h^\alpha(c_1x + c_2y)(t) = c_1\Delta_h^\alpha x(t) + c_2\Delta_h^\alpha y(t).$$

Proof. Follows from linearity of Δ_h^m and $a\Delta_h^{-(m-\alpha)}$. $\square$

1.4.2 Caputo-Type h -Difference Operator

Definition 1.27. [7] [Caputo-type fractional h -difference]

$$a^C\Delta_h^\alpha x(t) = a\Delta_h^{-(m-\alpha)}(\Delta_h^m x)(t), \quad t \in [a + (m - \alpha)h, \infty)_h.$$

Theorem 1.8 (Semigroup property). Let $\alpha, \beta > 0$, $\alpha > \beta$:

$$a\Delta_h^{-\alpha}(a^C\Delta_h^\beta x)(t) = a\Delta_h^{-(\alpha-\beta)}x(t).$$

Proof.

$$a\Delta_h^{-\alpha}(a^C\Delta_h^\beta x) = a\Delta_h^{-\alpha}(a\Delta_h^{-(m-\beta)}\Delta_h^m x) = a\Delta_h^{-(\alpha+m-\beta)}\Delta_h^m x = a\Delta_h^{-(\alpha-\beta)}x.$$

□

Remark. For integer-order $\alpha = m$, $a^C\Delta_h^m x(t) = \Delta_h^m x(t)$. The operator preserves the memory effect and allows specifying initial conditions in integer differences, useful in discrete fractional neural network models.

1.5 Notion of Stability

This section presents stability concepts for discrete systems, including Lyapunov and Mittag–Leffler stability, forming the theoretical backbone for later analysis.

1.5.1 Basic Definitions of Stability

Definition 1.28. [29] [Stability in the sense of Lyapunov] Consider the discrete system

$$x(t+h) = f(t, x(t)), \quad x(t) \in \mathbb{R}^n.$$

An equilibrium point x^* is *stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x(t_0) - x^*\| < \delta \quad \Rightarrow \quad \|x(t) - x^*\| < \epsilon, \quad \forall t \geq t_0.$$

Definition 1.29 (Asymptotic stability). The equilibrium x^* is *asymptotically stable* if it is stable and

$$\lim_{t \rightarrow \infty} x(t) = x^*.$$

Definition 1.30 (Exponential stability). The equilibrium x^* is *exponentially stable* if there exist constants $M > 0$ and $\lambda > 0$ such that

$$\|x(t) - x^*\| \leq M e^{-\lambda(t-t_0)} \|x(t_0) - x^*\|, \quad t \geq t_0.$$

Theorem 1.9 (Lyapunov direct method – discrete version). [10] Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a positive definite function and

$$\Delta_h V(x(t)) = V(x(t+h)) - V(x(t))$$

satisfy $\Delta_h V(x) \leq 0$. Then x^* is stable. If $\Delta_h V(x) < 0$ for all $x \neq x^*$, then x^* is asymptotically stable.

Proof. [10] Follows from discrete Lyapunov theory: the non-increasing nature of $V(x(t))$ guarantees boundedness within a neighborhood of x^* , and strict decrease ensures convergence to x^* . □

Remark. In discrete fractional systems, the Lyapunov function can be adapted to account for memory effects.

For linear systems $x(t+h) = Ax(t)$, eigenvalues $|\lambda_i| < 1$ ensure asymptotic stability.

1.5.2 Mittag–Leffler Stability in Discrete Fractional Systems

Definition 1.31. [16] [Mittag–Leffler function] The discrete Mittag–Leffler function $E_\alpha(\lambda, t)$ is defined by

$$E_\alpha(\lambda, t) = \sum_{k=0}^{\infty} \frac{\lambda^k (t_h)^{\alpha k}}{\Gamma(\alpha k + 1)}, \quad t_h = t/h, \quad \alpha > 0.$$

Definition 1.32 (Mittag–Leffler stability). An equilibrium x^* of a discrete fractional system

$$a^C \Delta_h^\alpha x(t) = f(t, x(t))$$

is *Mittag–Leffler stable* if there exist constants $C > 0$, $\lambda > 0$ such that

$$\|x(t) - x^*\| \leq C E_\alpha(-\lambda, t - t_0) \|x(t_0) - x^*\|, \quad t \geq t_0.$$

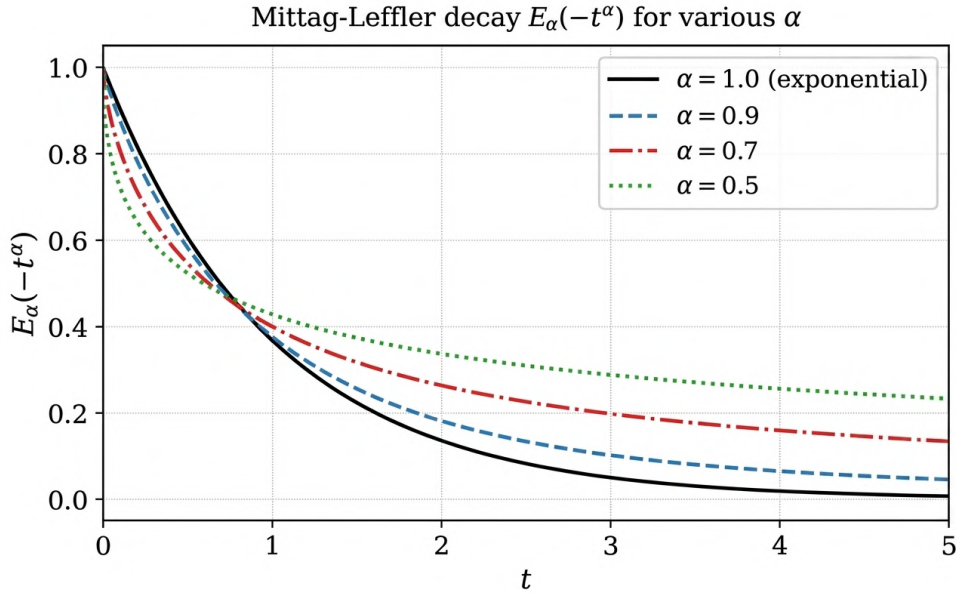


Figure 1.1: Decay of the Mittag-Leffler function $E_\alpha(-t^\alpha)$ for $\alpha \in \{0.5, 0.7, 0.9, 1.0\}$.

Theorem 1.10 (Stability criterion for linear Caputo fractional discrete systems). *Consider*

$${}_a^C\Delta_h^\alpha x(t) = Ax(t), \quad x(t_0) = x_0.$$

If all eigenvalues λ_i of A satisfy $|\arg(\lambda_i)| > \alpha\pi/2$, then the system is Mittag-Leffler stable.

Proof. The solution can be expressed as

$$x(t) = E_\alpha(A, t - t_0)x_0,$$

where $E_\alpha(A, t - t_0)$ is the matrix Mittag-Leffler function. Convergence of this series requires $|\arg(\lambda_i)| > \alpha\pi/2$, ensuring $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$. $\square$

Remark. Mittag-Leffler stability generalizes exponential stability to fractional-order discrete systems.

For $\alpha = 1$, it reduces to classical exponential stability.

Lyapunov methods can also be extended for nonlinear fractional discrete systems using fractional differences of Lyapunov functions.

1.6 Stability of Integer-Order Difference Systems

We review classical stability results for linear and nonlinear integer-order discrete systems, providing a reference for fractional extensions.

1.6.1 Stability of Linear Difference Systems

Definition 1.33. [29] [Linear discrete system] A linear discrete system is given by

$$x(t+h) = Ax(t), \quad x(t) \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}.$$

Definition 1.34 (Equilibrium point). An equilibrium point x^* satisfies $x^* = Ax^*$. For homogeneous linear systems, $x^* = 0$.

Theorem 1.11 (Stability criterion for linear systems). *The zero equilibrium of $x(t+h) = Ax(t)$ is asymptotically stable if and only if all eigenvalues λ_i of A satisfy*

$$|\lambda_i| < 1.$$

Proof. Let x_0 be the initial state. The solution is

$$x(t) = A^n x_0, \quad t = t_0 + nh.$$

If $|\lambda_i| < 1$, then $A^n \rightarrow 0$ as $n \rightarrow \infty$, hence $x(t) \rightarrow 0$. Conversely, if any $|\lambda_i| \geq 1$, $x(t)$ does not converge to zero. $\square$

Remark. If all $|\lambda_i| \leq 1$ and repeated eigenvalues have full rank, the system is stable but not asymptotically stable.

Linear systems form the foundation for analyzing stability of fractional and nonlinear systems.

1.6.2 Stability of Nonlinear Difference Systems

Definition 1.35. [21] [Nonlinear discrete system] A nonlinear discrete system is given by

$$x(t+h) = f(x(t)), \quad f: \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

Definition 1.36 (Equilibrium point). An equilibrium x^* satisfies $x^* = f(x^*)$.

Theorem 1.12 (Linearization stability – discrete Lyapunov). *Let x^* be an equilibrium of $x(t+h) = f(x(t))$ and $J = \left. \frac{\partial f}{\partial x} \right|_{x=x^*}$ the Jacobian. If all eigenvalues λ_i of J satisfy $|\lambda_i| < 1$, then x^* is locally asymptotically stable.*

Proof. The system can be linearized near x^* :

$$x(t+h) - x^* = J(x(t) - x^*) + o(\|x(t) - x^*\|).$$

For small $\|x(t) - x^*\|$, the higher-order term is negligible. By the linear system theorem, $|\lambda_i| < 1 \implies x(t) \rightarrow x^*$. $\square$

Theorem 1.13 (Lyapunov direct method – nonlinear case). [22] *Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be a positive definite function. If the discrete difference*

$$\Delta_h V(x(t)) = V(f(x(t))) - V(x(t)) \leq 0,$$

then x^* is stable. If $\Delta_h V(x(t)) < 0$ for $x \neq x^*$, then x^* is asymptotically stable.

Proof. Analogous to the linear case in Section 1.6.1: the non-increasing Lyapunov function guarantees boundedness near x^* , and strict decrease ensures convergence to x^* . $\square$

Remark. For nonlinear discrete systems, stability analysis often requires constructing suitable Lyapunov functions.

Linearization provides local stability, whereas Lyapunov methods can provide global results if a global Lyapunov function exists.

1.7 Stability of Caputo Fractional-Order Difference Systems

This section extends stability analysis to Caputo fractional-order discrete systems.

1.7.1 Linear Caputo Fractional-Order Discrete Systems

Definition 1.37. [14] [Linear Caputo fractional discrete system] A linear Caputo fractional discrete system of order $\alpha \in (0, 1]$ is defined as

$$a^C \Delta_h^\alpha x(t) = Ax(t), \quad x(t_0) = x_0, \quad t \in [a, \infty)_h,$$

where $A \in \mathbb{R}^{n \times n}$ and $a^C \Delta_h^\alpha$ is the Caputo h -difference operator.

Theorem 1.14 (Solution of linear Caputo fractional discrete system). *The solution is given by*

$$x(t) = E_\alpha(A, t - t_0)x_0,$$

where $E_\alpha(A, t - t_0)$ is the matrix discrete Mittag-Leffler function.

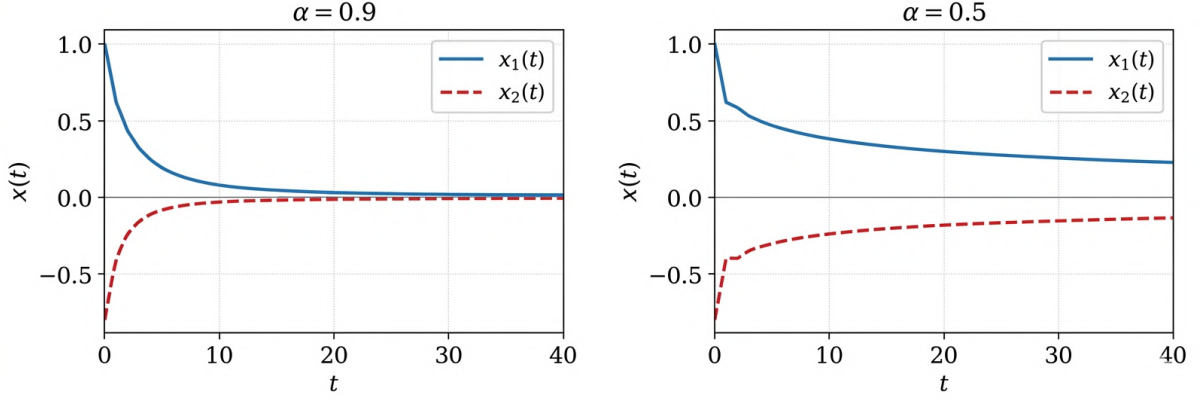


Figure 1.2: Solution of ${}^C_a\Delta_h^\alpha x(t) = Ax(t)$, with initial condition $x(0) = (1, -0.8)^\top$.

Proof. Applying the discrete Laplace transform for Caputo h -differences and using the definition of the Mittag-Leffler function, the solution series is

$$x(t) = \sum_{k=0}^{\infty} A^k \frac{(t - t_0)_h^{\alpha k}}{\Gamma(\alpha k + 1)} x_0.$$

□

Theorem 1.15 (Stability criterion). *The zero equilibrium $x^* = 0$ is Mittag-Leffler stable if all eigenvalues λ_i of A satisfy*

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}.$$

Proof. Each term in $x(t) = E_\alpha(A, t - t_0)x_0$ decays to zero as $t \rightarrow \infty$ under the eigenvalue condition, ensuring asymptotic convergence. □

Remark. For $\alpha = 1$, the criterion reduces to $|\lambda_i| < 1$, recovering classical linear discrete stability.

The Mittag-Leffler function generalizes exponential decay for integer-order systems.

1.7.2 Nonlinear Caputo Fractional-Order Discrete Systems

Definition 1.38. [14] [Nonlinear Caputo fractional discrete system] A nonlinear Caputo fractional discrete system is given by

$${}^aC\Delta_h^\alpha x(t) = f(x(t)), \quad x(t_0) = x_0,$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is nonlinear and $\alpha \in (0, 1]$.

Definition 1.39 (Equilibrium point). An equilibrium x^* satisfies

$$f(x^*) = 0.$$

Theorem 1.16 (Linearization for local stability). *Let $J = \frac{\partial f}{\partial x}|_{x=x^*}$ be the Jacobian at x^* . If all eigenvalues λ_i of J satisfy*

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2},$$

then x^* is locally Mittag-Leffler stable.

Proof. Linearize near x^* :

$$a^C \Delta_h^\alpha e(t) = J e(t), \quad e(t) = x(t) - x^*.$$

By Theorem 1.8.1, the linear system is Mittag-Leffler stable, and nonlinear terms are negligible near the equilibrium. $\square$

Theorem 1.17 (Lyapunov method for nonlinear Caputo fractional discrete systems). *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be positive definite. If the Caputo fractional difference satisfies*

$$a^C \Delta_h^\alpha V(x(t)) \leq 0, \quad x \neq x^*,$$

then x^* is stable. If

$$a^C \Delta_h^\alpha V(x(t)) < 0,$$

then x^* is asymptotically stable.

Proof. The discrete fractional Lyapunov theorem guarantees that $V(x(t))$ decreases along trajectories, ensuring convergence to x^* . $\square$

Remark. Lyapunov functions in fractional discrete systems must account for the memory effect of the Caputo difference.

For $\alpha \rightarrow 1$, results recover classical nonlinear discrete system stability.

1.8 Stability of h -Discrete Fractional-Order Systems

We generalize previous results to Delta h -fractional systems, providing definitions and stability criteria suitable for uniform-step discrete models.

1.8.1 Delta h -Fractional Difference Systems

Definition 1.40. [15] [Delta h -fractional difference] Let $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$, $\alpha > 0$, $m - 1 < \alpha \leq m$. The left Delta h -fractional difference is defined by

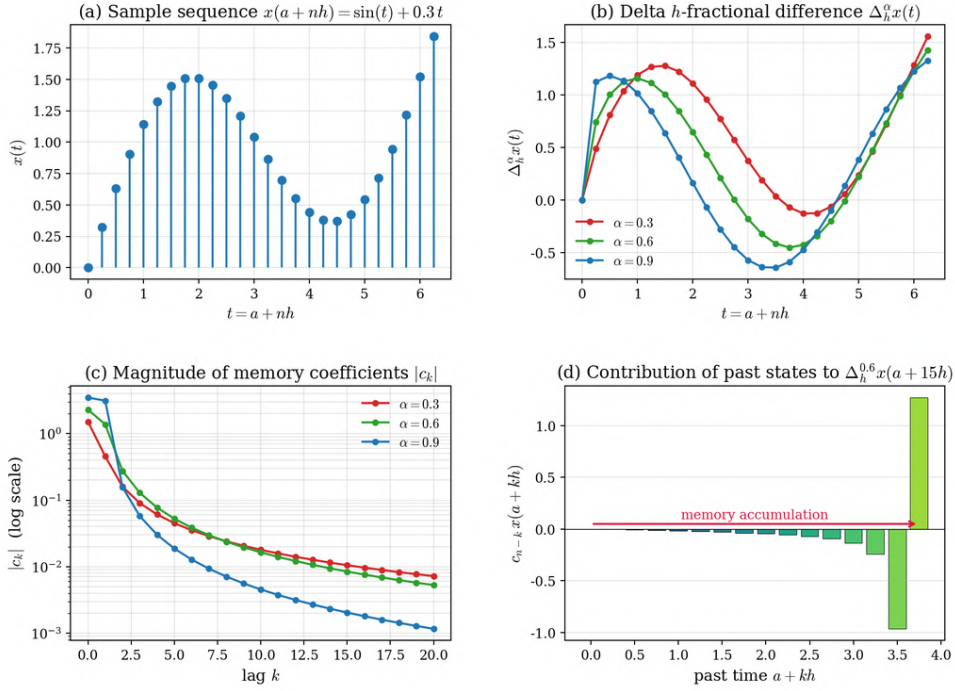
$$\Delta_h^\alpha x(t) = \Delta_h^m (a \Delta_h^{-(m-\alpha)} x)(t), \quad t \in [a + (m - \alpha)h, \infty)_h,$$

where Δ_h^m is the standard integer-order forward difference and $a \Delta_h^{-(m-\alpha)}$ is the fractional h -sum.

Remark (Memory effect). $\Delta_h^\alpha x(t)$ depends on all previous states:

$$\Delta_h^\alpha x(t) = \sum_{k=0}^n c_k x(a + kh), \quad t = a + nh, \quad c_k \neq 0.$$

This captures the long-term memory effect inherent to fractional dynamics.


 Figure 1.3: Illustrative figure: Delta h -fractional difference and the memory effect

1.8.2 Stability Definitions

Definition 1.41 (Equilibrium point). An equilibrium x^* of the Delta h -fractional system

$$\Delta_h^\alpha x(t) = f(x(t))$$

satisfies

$$f(x^*) = 0.$$

Definition 1.42 (Mittag–Leffler stability). An equilibrium x^* is Mittag–Leffler stable if there exist constants $C > 0$, $\lambda > 0$ such that

$$\|x(t) - x^*\| \leq CE_\alpha(-\lambda, t - t_0)\|x(t_0) - x^*\|, \quad t \geq t_0,$$

where E_α is the discrete Mittag–Leffler function.

1.8.3 Linear Delta h -Fractional Systems

Definition 1.43. [15] [Linear Delta h -fractional system]

$$\Delta_h^\alpha x(t) = Ax(t), \quad x(t_0) = x_0.$$

Theorem 1.18 (Solution representation). *The solution is*

$$x(t) = E_\alpha(A, t - t_0)x_0.$$

Proof. Use the series expansion of the discrete Mittag–Leffler function:

$$x(t) = \sum_{k=0}^{\infty} A^k \frac{(t - t_0)_h^{\alpha k}}{\Gamma(\alpha k + 1)} x_0.$$

□

Theorem 1.19 (Stability criterion). *The zero equilibrium $x^* = 0$ is Mittag–Leffler stable if all eigenvalues λ_i of A satisfy*

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}.$$

Proof. Follows from the convergence properties of the Mittag–Leffler function, ensuring $x(t) \rightarrow 0$ as $t \rightarrow \infty$. $\square$

1.8.4 Nonlinear Delta h -Fractional Systems

Definition 1.44. [15] [Nonlinear Delta h -fractional system]

$$\Delta_h^\alpha x(t) = f(x(t)), \quad x(t_0) = x_0.$$

Theorem 1.20 (Linearization stability). *Let $J = \frac{\partial f}{\partial x}|_{x=x^*}$. If all eigenvalues of J satisfy*

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2},$$

then x^* is locally Mittag–Leffler stable.

Proof. Linearize near x^* :

$$\Delta_h^\alpha e(t) = Je(t), \quad e(t) = x(t) - x^*.$$

Apply the linear Delta h -fractional stability theorem. $\square$

Theorem 1.21 (Lyapunov method). *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be positive definite. If*

$$\Delta_h^\alpha V(x(t)) \leq 0, \quad x \neq x^*,$$

then x^* is stable. If

$$\Delta_h^\alpha V(x(t)) < 0,$$

then x^* is asymptotically stable.

Proof. The decreasing Lyapunov function ensures boundedness and convergence to the equilibrium, accounting for memory effects via the fractional difference. $\square$

Remark. Delta h -fractional systems generalize Caputo fractional systems.

They are useful in modeling discrete fractional neural networks with uniform time steps.

Mittag–Leffler stability reduces to classical exponential stability for $\alpha = 1$.

1.9 Conclusion

In this chapter, we established the mathematical foundations of discrete fractional calculus, focusing on Caputo and Delta h -fractional difference operators. We introduced fundamental definitions, linear and nonlinear stability concepts, and Mittag–Leffler stability for fractional discrete systems. The connections between integer-order, Caputo fractional, and Delta h -fractional systems were highlighted, providing a smooth transition from classical discrete dynamics to memory-dependent fractional systems.

These results form an essential toolbox for analyzing discrete fractional neural networks. Fractional differences naturally capture the memory effects of neurons, while Lyapunov and Mittag–Leffler stability criteria provide rigorous methods for assessing network stability. The properties and theorems presented in this chapter will be extensively used in subsequent chapters to study chaotic behavior, synchronization, and control strategies in fractional-order discrete neural networks.

Chapter 2

Chaos in Fractional-Order Discrete Dynamical Systems

This chapter develops a rigorous analysis of chaos in fractional-order discrete dynamical systems, emphasizing the fundamental role of memory in shaping system behavior. It begins by introducing discrete systems with memory through fractional difference equations, highlighting their non-Markovian nature and infinite-dimensional phase-space structure. The notion of chaos is then formulated in the fractional setting using sensitivity to initial histories, topological transitivity, and dense periodic orbits, followed by the study of fractional-order chaotic maps and attractors. Lyapunov exponents are extended from classical discrete systems to memory-dependent dynamics to quantify perturbation growth and characterize chaos. Finally, bifurcation phenomena and parameter sensitivity are examined, with particular attention to the influence of the fractional order α and step size h on the transition between regular and chaotic behavior.

2.1 Discrete Dynamical Systems with Memory

Classical discrete dynamical systems are Markovian: the future state depends only on the present state. In contrast, fractional-order discrete systems possess memory, meaning that the evolution at time k depends on a weighted history of previous states. This fundamental difference modifies the phase-space structure, stability properties, and the mechanisms that may lead to chaos.

Let us consider a Caputo-type fractional difference equation of order $0 < \alpha < 1$:

$${}^C\Delta^\alpha x(k) = f(k, x(k)),$$

with initial condition $x(0) = x_0$.

Using the discrete fractional sum operator, this equation can be written in the equivalent integral (summation) form:

$$x(k) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} f(j, x(j)).$$

This representation clearly shows the memory effect: the present state depends on all previous states through a power-law kernel.

2.1.1 Fractional State Representation

In integer-order systems

$$x(k+1) = F(x(k)),$$

the state at time k completely determines the future. However, for fractional systems, the dynamics are inherently nonlocal in time.

Definition 2.1. [16] [Fractional Discrete Dynamical System] A fractional discrete dynamical system of order $0 < \alpha < 1$ is defined by

$${}^C\Delta^\alpha x(k) = F(k, x(k)),$$

or equivalently

$$x(k) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} F(j, x(j)).$$

The system is said to possess memory of power-law type.

Proposition (Non-Markovian Property). A fractional discrete system of order $0 < \alpha < 1$ is non-Markovian.

Proof. From the equivalent summation form,

$$x(k) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} F(j, x(j)).$$

Thus $x(k)$ depends on all past states $(x(0), x(1), \dots, x(k-1))$. Therefore, knowledge of $x(k-1)$ alone is insufficient to determine $x(k)$. Hence, the system is non-Markovian. $\square$

Remark. When $\alpha \rightarrow 1$, the kernel $(k-1-j)^{\alpha-1}$ approaches 1, and the fractional system reduces to the classical first-order difference equation. Thus integer-order systems appear as a limiting case of fractional ones.

2.1.2 Phase Space Structure and Non-Markovian Dynamics

Because of memory, fractional discrete systems do not evolve in a finite-dimensional phase space in the classical sense.

Definition 2.2. [16] [Extended State Vector] Define the history vector

$$X_k = (x(k), x(k-1), \dots, x(0)).$$

The fractional system can be reformulated as an infinite-dimensional dynamical system

$$X_{k+1} = \mathcal{F}(X_k).$$

Thus, fractional discrete dynamics naturally evolve in an infinite-dimensional phase space.

Theorem 2.1 (Infinite-Dimensional Embedding). *Every fractional discrete dynamical system of order $0 < \alpha < 1$ can be embedded into an infinite-dimensional Markovian system.*

Proof. Since the evolution of $x(k)$ depends on all previous states, define the state vector

$$X_k = (x(k), x(k-1), x(k-2), \dots).$$

Then the evolution rule is

$$X_{k+1} = (x(k+1), x(k), x(k-1), \dots).$$

The first component is determined by the fractional summation formula, while the remaining components are shifts of previous entries. Therefore, the system becomes Markovian in the infinite-dimensional space of sequences. $\square$

Remark. This infinite-dimensional nature explains why fractional systems often exhibit richer dynamical behavior than integer-order systems of the same apparent dimension.

Example 2.1 (Fractional Logistic-Type Map)

Consider the Caputo fractional difference equation

$${}^C\Delta^\alpha x(k) = rx(k)(1 - x(k)), \quad 0 < \alpha < 1.$$

Its equivalent summation form is

$$x(k) = x_0 + \frac{r}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} x(j)(1 - x(j)).$$

For certain values of r and α , this system exhibits complex oscillations and chaotic-like behavior. The fractional order α modifies both amplitude and stability compared to the classical logistic map.

Remark. The presence of the fractional memory term alters bifurcation thresholds and can delay or suppress chaos depending on the value of α . In subsequent sections, we formalize chaos in this fractional setting and introduce appropriate Lyapunov exponents adapted to memory-dependent systems.

2.2 Chaos in Fractional Discrete Systems

Fractional discrete systems combine nonlinear dynamics with long-term memory. This interaction produces complex behaviors that differ from classical discrete chaos. In this section, chaos is defined in the fractional setting, and the existence of chaotic attractors in fractional-order maps is described.

2.2.1 Mathematical Definitions of Chaos

Consider a fractional discrete system of order $0 < \alpha < 1$:

$${}^C\Delta^\alpha x(k) = F(x(k)).$$

Let the equivalent memory form be

$$x(k) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} F(x(j)).$$

Chaos in fractional systems must account for sensitivity to initial conditions together with the influence of memory.

Definition 2.3. [16] [Sensitive Dependence on Initial Conditions] The system exhibits sensitive dependence on initial conditions if there exist two initial histories X_0 and $\tilde{X}_0$ such that

$$|X_0 - \tilde{X}_0| \text{ is small}$$

but for some $k > 0$,

$$|x(k) - \tilde{x}(k)| > \varepsilon,$$

for a fixed $\varepsilon > 0$.

Definition 2.4 (Topological Transitivity). A fractional discrete system is topologically transitive if, for any two nonempty open sets U, V in the state space, there exists an integer $k > 0$ such that

$$\Phi^k(U) \cap V \neq \emptyset,$$

where Φ^k denotes the k -step evolution operator.

Definition 2.5 (Density of Periodic Orbits). The system has dense periodic orbits if for every point x in the state space and every neighborhood $N(x)$, there exists a periodic point $p \in N(x)$.

Definition 2.6 (Chaos in the Sense of Devaney). A fractional discrete system is chaotic if:

1. It is topologically transitive,
2. Periodic points are dense,
3. It exhibits sensitive dependence on initial conditions.

Remark. In fractional systems, sensitivity to initial conditions depends not only on the initial state $x(0)$ but on the entire initial history. Therefore, chaos is influenced by both nonlinearity and memory depth.

2.2.2 Characteristics of Chaos

Chaos refers to the occurrence of behavior in a deterministic system that appears random and unpredictable. In fractional discrete systems, chaos arises from the interplay between nonlinearity and memory effects. The main characteristics of chaotic behavior can be summarized as follows[21]

1. **Sensitivity to Initial Conditions.** A chaotic system exhibits extreme sensitivity to its initial conditions. Small differences in the initial state or initial history may lead to significantly different trajectories over time. This phenomenon is commonly referred to as the *butterfly effect*, where minor perturbations can produce large and unpredictable consequences.
2. **Phase Space Representation.** The phase space is a mathematical framework used to describe the evolution of a dynamical system. Each point in the phase space represents a possible state of the system at a given time. As time evolves, the system traces a trajectory in this space. The geometry of this trajectory provides important information about the system's behavior, particularly whether it is regular or chaotic.

3. **Fractal Dimension.** Chaotic systems often exhibit fractal structures. A fractal is a geometric object that displays self-similarity at different scales. The complexity of a chaotic attractor can be quantified using its fractal dimension, which provides a measure of how detailed the structure is across scales.
4. **Lyapunov Exponents.** Lyapunov exponents measure the rate of separation of nearby trajectories. They provide a quantitative criterion for chaos. In particular:
 - If the largest Lyapunov exponent is negative, trajectories converge to a fixed point.
 - If it is zero, the system exhibits periodic behavior.
 - If it is positive, the system exhibits chaotic behavior.

For example, the classical Lorenz system exhibits chaos for certain parameter values, such as $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$, where the largest Lyapunov exponent is approximately 0.9.

5. **Strange Attractors.** Chaotic systems often possess strange attractors, which are invariant sets with complex, non-periodic structures. These attractors typically exhibit fractal properties and are highly sensitive to initial conditions, distinguishing them from simple periodic attractors.

2.2.3 Fractional-Order Chaotic Maps

Fractional discrete chaos often appears in nonlinear maps modified by fractional difference operators.

Definition 2.7. [21] [Fractional Logistic Map] A fractional logistic-type system is defined by

$${}^C\Delta^\alpha x(k) = rx(k)(1 - x(k)), \quad 0 < \alpha < 1.$$

In summation form,

$$x(k) = x_0 + \frac{r}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} x(j)(1 - x(j)).$$

Theorem 2.2 (Boundedness of Solutions). *Assume $0 < r \leq 4$ and $x_0 \in [0, 1]$. Then the solution of the fractional logistic system remains bounded for all $k \geq 0$.*

Proof. Since $x(j) \in [0, 1]$ implies

$$0 \leq x(j)(1 - x(j)) \leq \frac{1}{4},$$

the summation term is finite:

$$\left| \frac{r}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} x(j)(1 - x(j)) \right| \leq \frac{r}{4\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1}.$$

Since $\alpha - 1 > -1$, the sum is finite for each k . Hence $x(k)$ remains bounded. $\square$

Remark. Compared with the classical logistic map, the fractional version accumulates past nonlinear interactions, which can either enhance or suppress chaotic oscillations depending on the order α .

2.2.4 Chaotic Attractors

The long-term behavior of fractional chaotic systems is characterized by attractors influenced by memory.

Definition 2.8. [21] [Fractional Chaotic Attractor] A set $A \subset \mathbb{R}^n$ is called a chaotic attractor if:

1. It is invariant under the system dynamics,
2. It attracts nearby trajectories,
3. The motion on A is chaotic.

Theorem 2.3 (Existence of Attracting Sets). *Assume the nonlinear function $F(x)$ is continuous and bounded, and the fractional system is dissipative. Then there exists a compact attracting set A .*

Proof. Since the system is dissipative, trajectories eventually enter a bounded region $B \subset \mathbb{R}^n$. From the boundedness result, solutions remain bounded. The fractional evolution operator is continuous due to the continuity of F . Therefore, the limit set

$$A = \bigcap_{k \geq 0} \overline{\bigcup_{j \geq k} \Phi^j(B)}$$

is compact, invariant, and attracts all trajectories starting in B . □

Remark. In fractional systems, attractors often have smoother geometry and slower convergence due to memory effects. The approach to the attractor depends on the entire past trajectory.

Example (Fractional Chaotic Behavior). Consider

$${}^C\Delta^\alpha x(k) = rx(k)(1 - x(k)),$$

with $r = 3.9$ and $0 < \alpha < 1$.

For values of α close to 1, the system exhibits oscillatory and chaotic-like behavior similar to the classical logistic map. For smaller α , memory may delay the onset of chaos and modify the attractor structure.

Remark. The interplay between nonlinearity and memory produces new routes to chaos that do not appear in integer-order systems. This motivates the need for quantitative measures such as Lyapunov exponents, which will be introduced in the next section.

2.3 Lyapunov Exponents for Fractional Discrete Systems

Lyapunov exponents quantify the average exponential rate of divergence (or convergence) of nearby trajectories. In classical discrete maps, they provide a precise numerical indicator of chaos. In fractional discrete systems, however, the presence of memory modifies the growth of perturbations, requiring a generalized formulation.

We begin with the classical integer-order case and then extend the concept to fractional discrete dynamics.

2.3.1 Lyapunov Exponents in Integer-Order Discrete Maps

Consider the discrete system

$$x(k+1) = F(x(k)), \quad x \in \mathbb{R}^n.$$

Let $x(k)$ be a trajectory and $\delta x(k)$ a small perturbation satisfying the linearized equation

$$\delta x(k+1) = DF(x(k)) \delta x(k),$$

where $DF(x(k))$ is the Jacobian matrix.

Definition 2.9. [27] [Lyapunov Exponent] The maximal Lyapunov exponent (MLE) is defined by

$$\lambda = \lim_{k \rightarrow \infty} \frac{1}{k} \ln \frac{|\delta x(k)|}{|\delta x(0)|},$$

provided the limit exists.

- If $\lambda < 0$, trajectories converge.
- If $\lambda = 0$, neutral behavior occurs.
- If $\lambda > 0$, the system exhibits sensitive dependence and possible chaos.

Theorem 2.4 (Lyapunov Criterion for Chaos). *If a discrete map admits a positive maximal Lyapunov exponent, then it exhibits sensitive dependence on initial conditions.*

Proof. If $\lambda > 0$, then for sufficiently large k ,

$$|\delta x(k)| \approx |\delta x(0)| e^{\lambda k}.$$

Thus arbitrarily small perturbations grow exponentially, implying sensitive dependence on initial conditions. $\square$

Remark. In one-dimensional maps,

$$x(k+1) = f(x(k)),$$

the Lyapunov exponent simplifies to

$$\lambda = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^{k-1} \ln |f'(x(j))|.$$

2.3.2 Fractional Lyapunov Exponents

Consider a fractional discrete system

$${}^C\Delta^\alpha x(k) = F(x(k)), \quad 0 < \alpha < 1,$$

with equivalent memory form

$$x(k) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} F(x(j)).$$

Let $\delta x(k)$ be a perturbation. The linearized fractional system becomes

$$\delta x(k) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} DF(x(j)) \delta x(j).$$

Definition 2.10. [27] [Fractional Lyapunov Exponent] The fractional maximal Lyapunov exponent is defined as

$$\lambda_\alpha = \limsup_{k \rightarrow \infty} \frac{1}{k} \ln \frac{|\delta x(k)|}{|\delta x(0)|}.$$

If $\lambda_\alpha > 0$, the fractional system is said to exhibit chaotic behavior.

Theorem 2.5 (Memory-Modified Growth Rate). *In fractional discrete systems with $0 < \alpha < 1$, perturbation growth is sub-exponential for sufficiently small α , even when the corresponding integer-order system is chaotic.*

Proof. Since the perturbation satisfies a convolution-type relation

$$\delta x(k) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} DF(x(j)) \delta x(j),$$

the kernel $(k-1-j)^{\alpha-1}$ decreases for $\alpha < 1$. Thus, recent perturbations dominate older ones, but with a slower accumulation compared to the exponential iteration in integer-order maps. This weighted summation slows the effective divergence rate, potentially modifying or suppressing chaos. $\square$

Remark. As $\alpha \rightarrow 1$, the fractional Lyapunov exponent converges to the classical Lyapunov exponent.

Remark. Lyapunov exponents provide a quantitative bridge between theory and simulation. In the next section, we investigate how bifurcation structures and parameter variations—including the fractional order α and the step size h —affect the transition to chaos.

2.4 Fractal Dimension in Fractional Chaotic Systems

Let $A \subset \mathbb{R}^n$ be a bounded invariant set generated by a fractional-order discrete dynamical system

$$\Delta^\alpha x(k) = F(x(k + \alpha - 1)), \quad 0 < \alpha \leq 1, \quad (2.1)$$

or equivalently,

$$x_{k+1} = G(x_k, x_{k-1}, \dots, x_0), \quad (2.2)$$

where the evolution depends on the complete trajectory history through the fractional difference operator.

The attractor A is called fractal if its geometric scaling cannot be characterized by an integer-valued Euclidean dimension. The complexity of A is quantified through a fractal dimension obtained from the asymptotic scaling law

$$N(\varepsilon) \sim \varepsilon^{-D}, \quad \varepsilon \rightarrow 0, \quad (2.3)$$

where $N(\varepsilon)$ denotes the minimum number of sets of diameter at most ε required to cover A .

Hence,

$$D = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}, \quad (2.4)$$

whenever the limit exists.

The quantity D measures the scaling rate of geometric complexity under resolution refinement. For smooth manifolds,

$$D \in \mathbb{N}, \quad (2.5)$$

whereas chaotic attractors generally satisfy

$$D \notin \mathbb{N}. \quad (2.6)$$

For dissipative fractional systems, trajectories contract toward invariant compact subsets of phase space. Simultaneously, nonlinear instability produces exponential separation of nearby trajectories. The competition between contraction and stretching generates invariant sets with noninteger scaling exponents.

Let

$$\Phi_k : A \rightarrow A \quad (2.7)$$

denote the discrete fractional evolution operator. If the Jacobian matrix satisfies

$$\|D\Phi_k(x)\| > 1 \quad (2.8)$$

along unstable directions and

$$\|D\Phi_k(x)\| < 1 \quad (2.9)$$

along stable directions, repeated stretching and folding mechanisms produce a geometrically irregular attractor.

The fractal dimension is closely related to entropy production and Lyapunov instability. If

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \quad (2.10)$$

are the Lyapunov exponents of the system, the Kaplan–Yorke dimension is defined by

$$D_{KY} = j + \frac{\sum_{i=1}^j \lambda_i}{|\lambda_{j+1}|}, \quad (2.11)$$

where j is the largest integer satisfying

$$\sum_{i=1}^j \lambda_i \geq 0, \quad \sum_{i=1}^{j+1} \lambda_i < 0. \quad (2.12)$$

For fractional chaotic systems, the fractional order α modifies both the Lyapunov spectrum and the attractor geometry. Since the fractional difference operator introduces long-range temporal dependence,

$$x_{k+1} = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^k \frac{\Gamma(k-j+\alpha)}{\Gamma(k-j+1)} F(x_j), \quad (2.13)$$

the present state depends on weighted contributions of all previous states.

The memory kernel

$$\omega_{k-j}^{(\alpha)} = \frac{\Gamma(k-j+\alpha)}{\Gamma(k-j+1)} \quad (2.14)$$

changes the contraction-expansion balance of the dynamics and therefore modifies the fractal scaling properties of the attractor.

For many fractional maps, decreasing α reduces local instability while increasing trajectory correlation. Consequently, the attractor dimension may either decrease due to enhanced dissipation or increase due to additional geometric complexity induced by memory effects.

2.4.1 Geometrical Interpretation of Fractal Dimension

Consider a bounded set $A \subset \mathbb{R}^n$. Let

$$N(\varepsilon) \quad (2.15)$$

be the smallest number of balls of radius ε required to cover A . If

$$N(\varepsilon) \sim C\varepsilon^{-d}, \quad (2.16)$$

for some constant $C > 0$, then

$$d = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \quad (2.17)$$

defines the scaling dimension of A .

For Euclidean objects:

$$N(\varepsilon) \propto \varepsilon^{-1} \quad (2.18)$$

for curves,

$$N(\varepsilon) \propto \varepsilon^{-2} \quad (2.19)$$

for surfaces, and

$$N(\varepsilon) \propto \varepsilon^{-3} \quad (2.20)$$

for volumes.

Chaotic attractors satisfy intermediate scaling laws:

$$\varepsilon^{-1} < N(\varepsilon) < \varepsilon^{-2}, \quad (2.21)$$

yielding

$$1 < D < 2, \quad (2.22)$$

or more generally,

$$m < D < m + 1, \quad m \in \mathbb{N}. \quad (2.23)$$

The noninteger value of D measures partial occupation of phase space.

For self-similar sets, suppose the attractor is decomposed into N copies of itself, each scaled by a factor r . Then

$$Nr^D = 1. \quad (2.24)$$

Hence,

$$D = \frac{\ln N}{\ln(1/r)}. \quad (2.25)$$

For the Cantor set:

$$N = 2, \quad r = \frac{1}{3}, \quad (2.26)$$

thus

$$D = \frac{\ln 2}{\ln 3}. \quad (2.27)$$

For the Sierpiński triangle:

$$N = 3, \quad r = \frac{1}{2}, \quad (2.28)$$

thus

$$D = \frac{\ln 3}{\ln 2}. \quad (2.29)$$

In chaotic dynamics, the attractor is generally not exactly self-similar. However, statistical self-similarity persists asymptotically:

$$N(\varepsilon) \approx C\varepsilon^{-D}. \quad (2.30)$$

Let

$$x_{k+1} = F(x_k) \quad (2.31)$$

be a dissipative chaotic map with Jacobian matrix $J(x)$. If

$$|\det J(x)| < 1, \quad (2.32)$$

phase-space volumes contract:

$$V_{k+1} \approx |\det J(x_k)|V_k. \quad (2.33)$$

Simultaneously, positive Lyapunov exponents imply exponential divergence:

$$\|\delta x_k\| \sim e^{\lambda k} \|\delta x_0\|, \quad \lambda > 0. \quad (2.34)$$

The coexistence of contraction and stretching produces repeated folding of trajectories, generating invariant sets with arbitrarily fine geometric structure.

For fractional-order systems, the evolution operator becomes nonlocal:

$$x_{k+1} = \sum_{j=0}^k \omega_{k-j}^{(\alpha)} F(x_j), \quad (2.35)$$

where

$$\omega_{k-j}^{(\alpha)} \sim (k-j)^{\alpha-1}. \quad (2.36)$$

Therefore, the geometry of the attractor depends not only on instantaneous deformation but also on accumulated historical interactions. The resulting attractors may exhibit:

$$D(\alpha) \neq D(1), \quad (2.37)$$

indicating that the fractal dimension varies with the fractional order.

Thus, the fractal dimension acts as a quantitative invariant describing

complexity, space-filling capacity, multiscale structure,

and the influence of memory effects on fractional chaotic attractors.

2.4.2 Hausdorff and Box-Counting Dimensions

Let $A \subset \mathbb{R}^n$ be a bounded set. The Hausdorff dimension provides a rigorous measure-theoretic characterization of the geometric scaling properties of A .

For $\delta > 0$, consider a countable δ -covering

$$A \subset \bigcup_{i=1}^{\infty} U_i, \quad \text{diam}(U_i) < \delta.$$

For $s \geq 0$, define the s -dimensional Hausdorff measure by

$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam}(U_i))^s \right\}, \quad (2.38)$$

where the infimum is taken over all possible δ -coverings of A .

The Hausdorff measure is obtained by taking the limit

$$\mathcal{H}^s(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A). \quad (2.39)$$

There exists a critical value D_H such that

$$\mathcal{H}^s(A) = \infty, \quad s < D_H,$$

and

$$\mathcal{H}^s(A) = 0, \quad s > D_H.$$

The quantity D_H is called the Hausdorff dimension of A :

$$D_H = \inf \{s \geq 0 : \mathcal{H}^s(A) = 0\}. \quad (2.40)$$

The Hausdorff dimension satisfies

$$\dim_T(A) \leq D_H \leq n, \quad (2.41)$$

where $\dim_T(A)$ denotes the topological dimension.

Although the Hausdorff dimension provides a mathematically rigorous description of fractality, its direct computation is generally difficult for chaotic attractors generated numerically. Consequently, the box-counting dimension is commonly used in applications.

Let $N(\varepsilon)$ denote the minimum number of cubes of side length ε required to cover A . The box-counting dimension is defined by

$$D_B = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}, \quad (2.42)$$

provided the limit exists.

Equivalently,

$$N(\varepsilon) \sim \varepsilon^{-D_B}. \quad (2.43)$$

For self-similar sets generated by N copies scaled by a factor r ,

$$Nr^D = 1, \quad (2.44)$$

hence

$$D = \frac{\ln N}{\ln(1/r)}. \quad (2.45)$$

The Hausdorff and box-counting dimensions satisfy

$$D_H \leq D_B. \quad (2.46)$$

For smooth manifolds,

$$D_H = D_B \in \mathbb{N},$$

whereas chaotic attractors generally satisfy

$$D_H, D_B \notin \mathbb{N}.$$

Consider a discrete chaotic orbit

$$\mathcal{O} = \{x_k\}_{k=0}^N \subset \mathbb{R}^m.$$

Partition the phase space into cubes of side length ε . If $N(\varepsilon)$ denotes the number of occupied cubes, then plotting

$$\ln N(\varepsilon)$$

against

$$\ln(1/\varepsilon)$$

produces approximately a straight line whose slope estimates D_B :

$$D_B \approx \frac{\Delta \ln N(\varepsilon)}{\Delta \ln(1/\varepsilon)}. \quad (2.47)$$

For fractional-order chaotic systems, the attractor geometry depends on the fractional order α . Thus,

$$D_B = D_B(\alpha), \quad D_H = D_H(\alpha). \quad (2.48)$$

Since the fractional memory operator modifies trajectory correlations and phase-space contraction rates, the resulting dimensions may vary continuously with α .

2.4.3 Fractal Dimension of Fractional Chaotic Attractors

Consider the fractional logistic map

$$\Delta^\alpha x(k) = rx(k)(1 - x(k)), \quad 0 < \alpha \leq 1, \quad (2.49)$$

where r is the bifurcation parameter.

Using the fractional sum representation, the solution may be written as

$$x_{k+1} = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^k \frac{\Gamma(k-j+\alpha)}{\Gamma(k-j+1)} rx_j(1 - x_j). \quad (2.50)$$

The memory kernel

$$\omega_{k-j}^{(\alpha)} = \frac{\Gamma(k-j+\alpha)}{\Gamma(k-j+1)} \quad (2.51)$$

introduces long-range temporal dependence into the dynamics.

Let

$$A_\alpha$$

denote the attractor corresponding to the fractional order α . The fractal dimension of A_α is determined from the scaling law

$$N_\alpha(\varepsilon) \sim \varepsilon^{-D(\alpha)}. \quad (2.52)$$

Hence,

$$D(\alpha) = \lim_{\varepsilon \rightarrow 0} \frac{\ln N_\alpha(\varepsilon)}{\ln(1/\varepsilon)}. \quad (2.53)$$

For integer-order systems ($\alpha = 1$), the attractor dimension depends primarily on the nonlinear stretching mechanism. In contrast, fractional systems involve both nonlinear instability and memory-induced correlations.

Suppose the Lyapunov spectrum satisfies

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n. \quad (2.54)$$

Then the Kaplan–Yorke dimension of the fractional attractor is

$$D_{KY}(\alpha) = j + \frac{\sum_{i=1}^j \lambda_i(\alpha)}{|\lambda_{j+1}(\alpha)|}, \quad (2.55)$$

where

$$\sum_{i=1}^j \lambda_i(\alpha) \geq 0, \quad \sum_{i=1}^{j+1} \lambda_i(\alpha) < 0. \quad (2.56)$$

The dependence of the Lyapunov exponents on α implies that

$$D_{KY} = D_{KY}(\alpha). \quad (2.57)$$

For many fractional maps, decreasing α modifies the attractor geometry through two competing mechanisms:

1. enhanced trajectory correlation due to memory effects,

2. increased dissipation caused by fractional accumulation.

If dissipation dominates, trajectories become more localized and

$$D(\alpha) < D(1). \quad (2.58)$$

If memory-induced stretching dominates, additional geometric complexity may appear, yielding

$$D(\alpha) > D(1). \quad (2.59)$$

Let

$$\delta x_k$$

be the perturbation between nearby trajectories. In fractional systems,

$$\delta x_{k+1} = \sum_{j=0}^k \omega_{k-j}^{(\alpha)} DF(x_j) \delta x_j, \quad (2.60)$$

where $DF(x_j)$ is the Jacobian matrix of the nonlinear map.

Unlike integer-order systems, perturbation growth depends on the entire trajectory history. Consequently, the attractor geometry is determined by nonlocal interactions in time.

For chaotic fractional attractors, one generally observes:

$$D_H < D_B < n, \quad (2.61)$$

where n is the embedding dimension.

Moreover, numerical simulations often reveal continuous deformation of the attractor as α varies:

$$A_{\alpha_1} \neq A_{\alpha_2}, \quad \alpha_1 \neq \alpha_2. \quad (2.62)$$

Therefore, the fractal dimension serves as a quantitative descriptor of:

memory effects, phase-space occupation, multiscale geometry,

and the complexity of fractional chaotic attractors.

2.4.4 Relationship Between Fractal Dimension and Chaos

The emergence of chaos in nonlinear dynamical systems is characterized by sensitivity to initial conditions, exponential divergence of nearby trajectories, and the formation of geometrically irregular invariant sets. The fractal dimension provides a quantitative description of the geometric complexity of these invariant sets and establishes a direct connection between the geometry of attractors and the dynamics of chaos.

Let

$$x_{k+1} = F(x_k)$$

be a dissipative discrete dynamical system with attractor

$$A \subset \mathbb{R}^n.$$

Assume that two nearby trajectories satisfy

$$\delta x_k = x_k - \tilde{x}_k.$$

If the system is chaotic, then

$$\|\delta x_k\| \sim e^{\lambda k} \|\delta x_0\|, \quad \lambda > 0, \quad (2.63)$$

where λ denotes the maximal Lyapunov exponent.

Positive Lyapunov exponents produce exponential stretching of phase-space trajectories. Since the system is dissipative, phase-space volumes simultaneously contract:

$$V_{k+1} = |\det J(x_k)| V_k, \quad (2.64)$$

where $J(x_k)$ is the Jacobian matrix of the map.

If

$$|\det J(x_k)| < 1, \quad (2.65)$$

the dynamics remain confined to a compact invariant set. The repeated coexistence of stretching and contraction generates folding mechanisms that produce fractal attractors.

Consequently, chaotic attractors satisfy

$$D \notin \mathbb{N}, \quad (2.66)$$

where D denotes the fractal dimension.

The connection between Lyapunov instability and fractal geometry becomes explicit through the Kaplan–Yorke conjecture. Let

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad (2.67)$$

be the Lyapunov spectrum of the system. Define the integer j such that

$$\sum_{i=1}^j \lambda_i \geq 0, \quad \sum_{i=1}^{j+1} \lambda_i < 0. \quad (2.68)$$

The Kaplan–Yorke dimension is then given by

$$D_{KY} = j + \frac{\sum_{i=1}^j \lambda_i}{|\lambda_{j+1}|}. \quad (2.69)$$

This relation demonstrates that the fractal dimension is directly determined by the balance between phase-space expansion and contraction rates.

For conservative systems,

$$\sum_{i=1}^n \lambda_i = 0, \quad (2.70)$$

and no volume contraction occurs. In dissipative chaotic systems,

$$\sum_{i=1}^n \lambda_i < 0, \quad (2.71)$$

which implies the existence of lower-dimensional attractors possessing fractal geometry.

The complexity of the attractor increases as the positive Lyapunov exponents increase. Consequently,

$$\lambda_{\max} \uparrow \implies D \uparrow. \quad (2.72)$$

For periodic dynamics,

$$\lambda_{\max} < 0, \quad (2.73)$$

and trajectories converge toward regular invariant sets with integer dimension.

At the transition to chaos,

$$\lambda_{\max} = 0, \quad (2.74)$$

corresponding to bifurcation points and critical states where the attractor undergoes geometric transformation.

For fractional-order systems,

$$\Delta^\alpha x(k) = F(x(k + \alpha - 1)), \quad 0 < \alpha \leq 1, \quad (2.75)$$

the relationship between fractal dimension and chaos becomes more intricate due to memory effects.

Using the fractional summation representation,

$$x_{k+1} = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^k \frac{\Gamma(k-j+\alpha)}{\Gamma(k-j+1)} F(x_j), \quad (2.76)$$

the present state depends on the complete trajectory history.

The perturbation dynamics become nonlocal:

$$\delta x_{k+1} = \sum_{j=0}^k \omega_{k-j}^{(\alpha)} DF(x_j) \delta x_j, \quad (2.77)$$

where

$$\omega_{k-j}^{(\alpha)} = \frac{\Gamma(k-j+\alpha)}{\Gamma(k-j+1)}. \quad (2.78)$$

Therefore, both Lyapunov exponents and fractal dimensions depend explicitly on the fractional order:

$$\lambda_i = \lambda_i(\alpha), \quad D = D(\alpha). \quad (2.79)$$

The memory effect modifies the stretching-folding mechanism responsible for chaotic attractor formation. In particular, decreasing α may:

1. reduce local instability through enhanced damping,
2. increase long-range trajectory correlation,
3. alter the topology of the attractor,
4. modify the scaling properties of phase-space occupation.

As a consequence, the fractal dimension may vary continuously with the fractional order:

$$\frac{dD}{d\alpha} \neq 0. \quad (2.80)$$

In many fractional chaotic maps, one observes:

$$\alpha \downarrow \implies \lambda_{\max} \downarrow, \quad (2.81)$$

leading to reduced chaotic intensity and smaller attractor dimensions.

However, certain systems exhibit memory-induced complexity where temporal nonlocality generates additional multiscale structures:

$$D(\alpha) > D(1). \quad (2.82)$$

Hence, the fractal dimension acts as a geometric indicator of chaos intensity and reflects the combined effects of:

instability, dissipation, memory, multiscale structure.

For fractional chaotic systems, the simultaneous analysis of:

$$\lambda_{\max}, \quad D_H, \quad D_B, \quad D_{KY},$$

provides a complete characterization of both the dynamical and geometrical properties of the attractor.

2.5 Bifurcation and Parameter Sensitivity Analysis

Fractional discrete systems exhibit rich bifurcation structures due to the combination of nonlinearity and memory effects. Understanding how parameters influence the transition from regular to chaotic behavior is crucial for both analysis and control design. In this section, we examine routes to chaos and the effect of fractional order and step size (h) on system dynamics [13]

2.5.1 Routes to Chaos

Consider a fractional discrete system

$${}^C\Delta^\alpha x(k) = F(x(k), \mu),$$

where μ is a bifurcation parameter. As μ varies, the system may undergo:

1. **Period-doubling bifurcations:** Successive doubling of the period leading to chaos.
2. **Saddle-node bifurcations:** Creation or annihilation of fixed points.
3. **Hopf-like bifurcations:** Emergence of quasi-periodic orbits in higher-dimensional fractional systems.

Definition 2.11. [13] [Bifurcation Point] A value $\mu = \mu_c$ is called a bifurcation point if a small change in μ qualitatively changes the asymptotic dynamics of the system.

2.5.2 Effect of Fractional Order on Bifurcations

Theorem 2.6. [13] [Bifurcation Induced by Fractional Order] Let $0 < \alpha_1 < \alpha_2 \leq 1$. For a fixed μ , the system

$${}^C\Delta^\alpha x(k) = F(x(k), \mu)$$

may transition from periodic to chaotic behavior as α decreases.

Proof. 1. The fractional summation kernel

$$(k - 1 - j)^{\alpha-1}$$

weights past states differently depending on α .

2. Smaller α increases the memory effect of earlier states, effectively coupling more past states into the dynamics.
3. This can destabilize previously periodic orbits and produce bifurcations leading to chaos.

□

2.5.3 Effect of Fractional Order and Step Size (h)

Fractional discrete systems may also depend on the step size h if formulated using the h -difference operator:

$${}^C\Delta_h^\alpha x(k) = F(x(k), \mu).$$

- **Smaller h :** increases the resolution of memory accumulation, which may delay bifurcations and suppress chaos.
- **Larger h :** reduces memory resolution, possibly triggering early onset of irregular behavior.

Remark. [13] Both the fractional order α and the step size h serve as bifurcation parameters. Varying them allows tuning of system dynamics, which is particularly useful when designing fractional discrete neural networks with desired stability or chaotic properties.

2.6 Use of Chaos

For many scientists, chaos theory represents the first step toward the unification of sciences. Indeed, it is a theory whose applications extend to virtually all scientific fields. In this section, we present some practical applications of chaos[13]

2.6.1 In Communication Systems

The image of a chaotic model is structured around a skeleton composed of a set of unstable periodic orbits (UPOs). The set of these UPOs, dense within the attractor, as well as the orbits transitioning between them, form the chaotic attractor.

The idea of using chaos in secure multi-user communications is often based on the control and proper utilization of UPOs. The main idea is to use the skeleton of a chaotic attractor as a reservoir of potential communication waves. In this way, each user is provided with a unique code within the same channel.

The advantage of multi-scroll attractors lies in their ability to generate shorter orbits (through more complex chaos), resulting in faster message transmission and improved communication security.

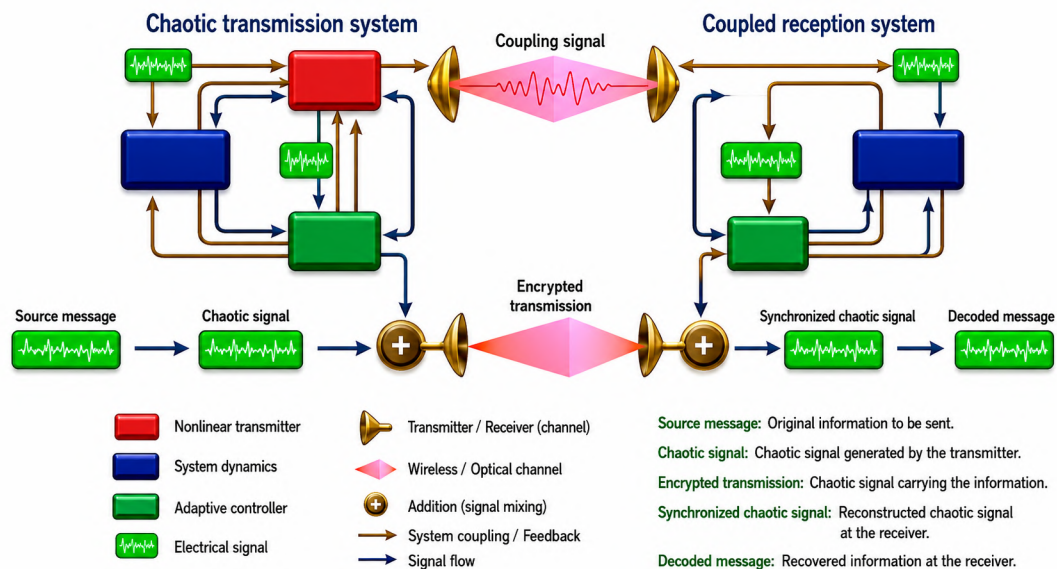


Figure 2.1: Maximal Lyapunov Exponent of the Fractional Logistic Map

2.6.2 In Biology

Chaos theory in biology helps explain differences in animal populations and brain oscillations (electroencephalograms, that is, graphical recordings of the electrical activity of the brain using electrodes placed on the scalp) through strange attractors[13]

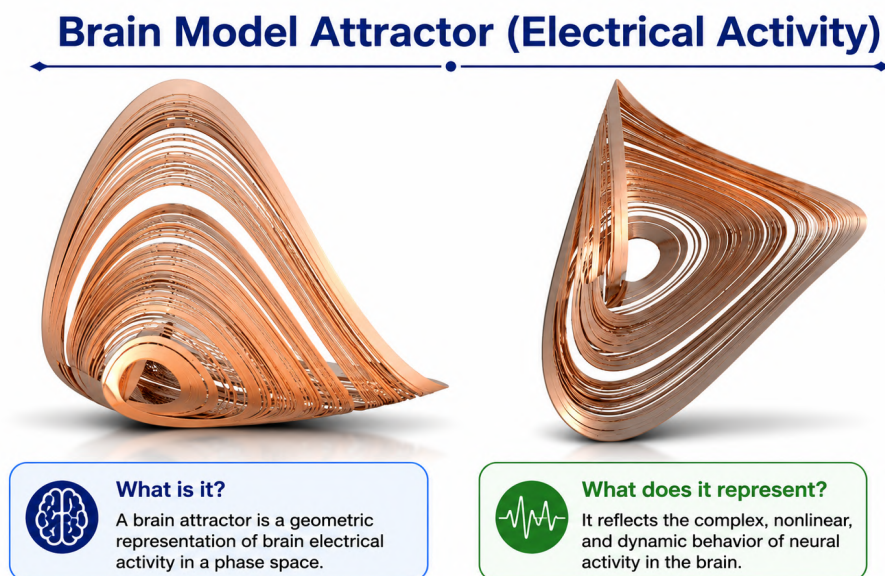


Figure 2.2: Attractor of a brain model representing electrical activity.

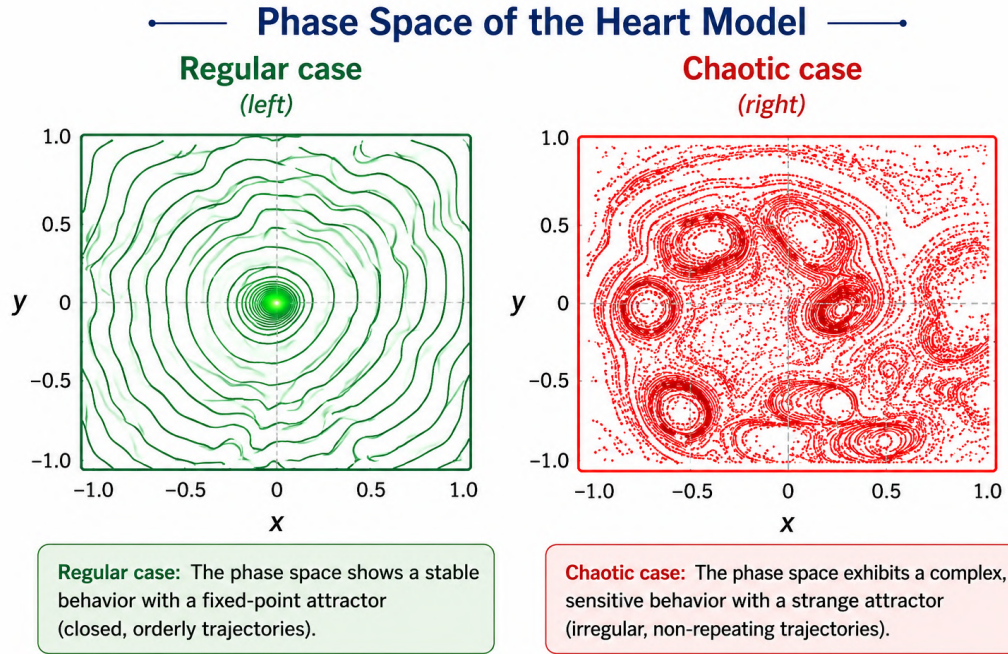


Figure 2.3: Phase space of the heart model in the regular case (left) and chaotic case (right).

Likewise, the arrhythmias typical of many heart diseases can also be explained by chaos theory. In a normal heart, electrical impulses travel uniformly through muscle fibers, forcing the cardiac ventricle to contract and pump blood. Once contracted, the fibers become insensitive to electrical signals; this is known as the refractory period.

Thus, differences in the duration of the refractory period from one region of the ventricle to another may cause spasmodic contractility, which can lead to a heart attack.

2.6.3 In Economics

In economics, commercial movements, financial markets, and economic cycles can partly be explained by chaos theory, where fractals are closely linked to randomness and therefore allow the modeling of complex random phenomena. This explains their use in finance to model fluctuations in stock market prices[13]

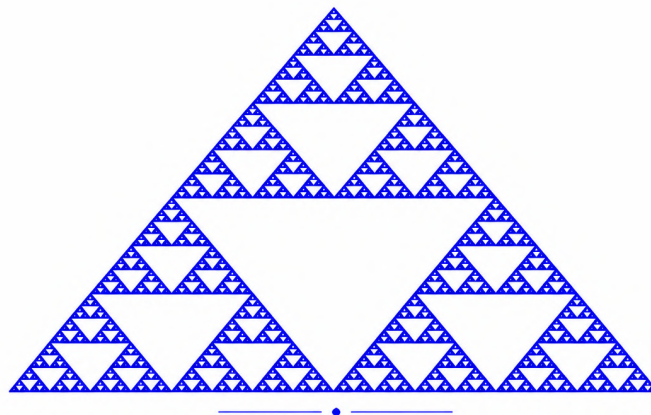
2.6.4 In Computer Science

In computer science, image compression methods have been developed using fractals. Synthetic images in cinema and video games are becoming increasingly realistic thanks to fractals. Indeed, objects provided by Euclidean geometry are unlikely to faithfully represent the real world: do we often see perfect circles or cubes in nature? No, the shapes found in nature correspond much more closely to fractal forms[13].

2.6.5 In Art

Even in the field of art, since the 1980s, the beauty of fractals has been widely appreciated and exploited, leading to a proliferation of exhibitions centered on these fascinating images. Fractal images possess a certain aesthetic appeal, but one may wonder whether they have other uses. We can observe several remarkable functions of fractals in different domains. Fractals are geometric figures with a property of self-similarity. Thus, we observe similar patterns in a fractal design regardless of the scale at which we examine it.

A famous example is the Sierpinski triangle, composed of an equilateral triangle divided into smaller equilateral triangles, which are themselves subdivided into equilateral triangles, and so on[13].



The Sierpiński triangle: an example of a fractal

*A fractal is a geometric object that is self-similar at different scales:
each part resembles the whole.*

Figure 2.4: The Sierpinski Triangle : An Example of a Fractal.

Example : Fractional Logistic-Type Map

Consider the Caputo fractional difference equation[13]

$${}^C\Delta^\alpha x(k) = r x(k)(1 - x(k)), \quad 0 < \alpha < 1.$$

- Fix $h = 1$ and vary α from 0.6 to 1.0.
- Observe period-doubling bifurcations as α decreases.
- For $\alpha \approx 1$, the system behaves similarly to the classical logistic map.
- For smaller α , long-term memory alters the bifurcation thresholds and modifies the route to chaos.

Step 1: Equivalent Summation Form.

The equivalent summation form of the equation is:

$$x(k) = x_0 + \frac{r}{\Gamma(\alpha)} \sum_{j=0}^{k-1} (k-1-j)^{\alpha-1} x(j)(1-x(j)).$$

This form explicitly reveals the *memory effect*: the current state $x(k)$ depends on the entire past trajectory $\{x(0), x(1), \dots, x(k-1)\}$, weighted by the power-law kernel $(k-1-j)^{\alpha-1}$.

Step 2: Effect of Memory on Trajectories.

We compare two cases under the same initial condition $x(0) = 0.2$, $\lambda = 3.7$:

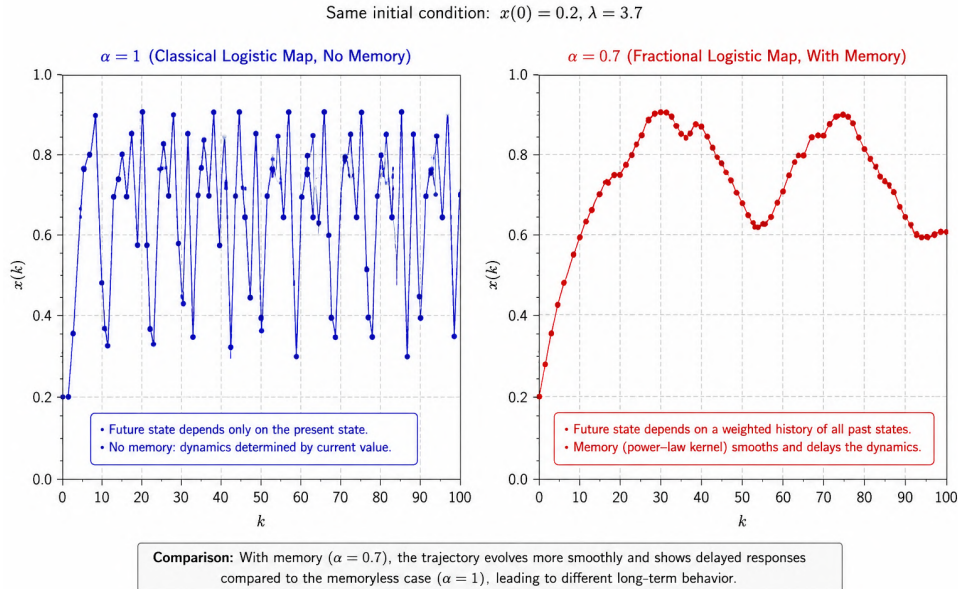


Figure 2.5: Fractional Logistic Map: Effect of Memory on Trajectories

- $\alpha = 1$ (classical logistic map, **no memory**): the future state depends only on the present state; no memory dynamics.

- $\alpha = 0.7$ (fractional logistic map, **with memory**): the future state depends on a weighted history of all past states; the power-law kernel smooths and delays the dynamics.

Remark. The presence of the fractional memory term alters bifurcation thresholds and can delay or suppress chaos depending on the value of α . In subsequent sections, we formalize chaos in this fractional setting and introduce appropriate Lyapunov exponents adapted to memory-dependent systems.

For certain values of r and α , this system exhibits complex oscillations and chaotic-like behavior. The fractional order α modifies both amplitude and stability compared to the classical logistic map.

Step 3: Sensitive Dependence on Initial Conditions.

A key signature of chaos is the *butterfly effect*: two trajectories starting from very close initial conditions $x_0 = 0.5$ and $x_0 = 0.5001$ (with $r = 3.95$, $\alpha = 0.8$) remain nearly identical for a short time, then diverge rapidly and become completely uncorrelated. The log of separation $\log_{10} |x_n - y_n|$ grows with a positive average slope, confirming sensitive dependence on initial conditions.

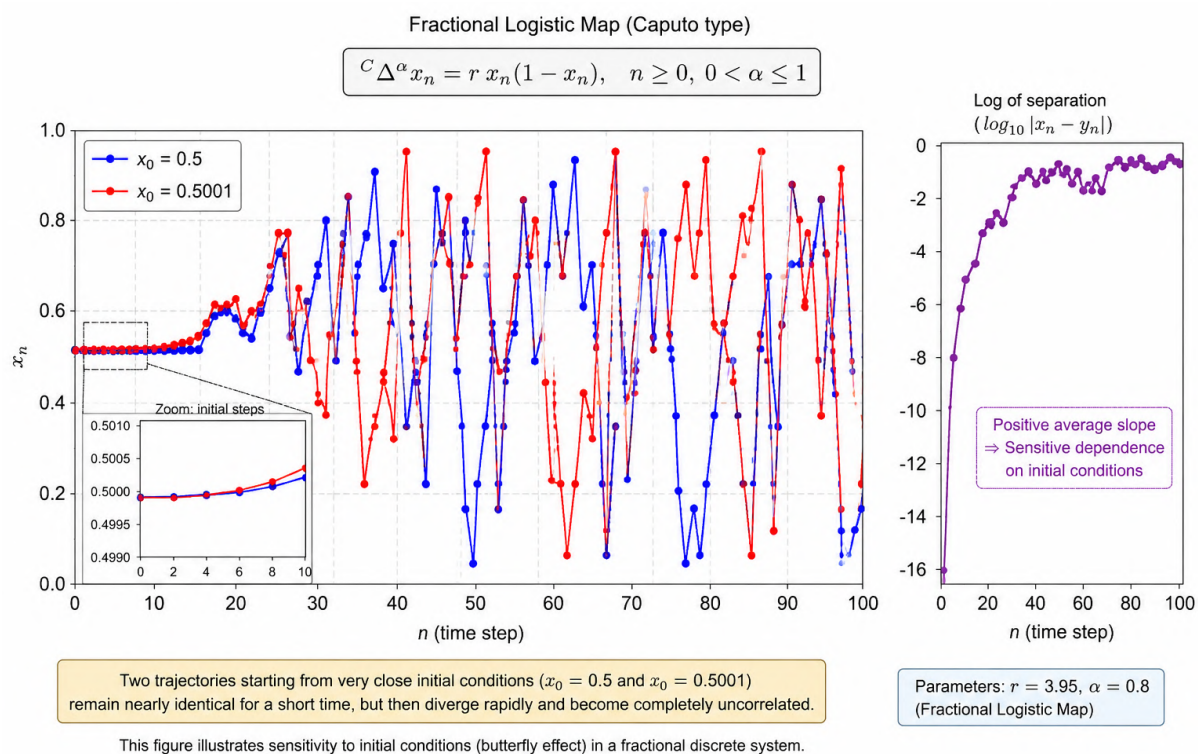


Figure 2.6: Divergence of Two Trajectories in a Fractional Discrete Map

Step 4: Chaotic Attractor and Lyapunov Exponent Analysis.

The maximal Lyapunov exponent λ_0 quantifies the average exponential rate of separation of nearby trajectories. Three regimes are identified:

- $\lambda_0 < 0$: **Convergent behavior** — fixed point or periodic orbit (stable).
- $\lambda_0 = 0$: **Onset of chaos** — transition/bifurcation point.
- $\lambda_0 > 0$: **Chaotic behavior** — sensitive dependence on initial conditions; existence of a chaotic attractor.

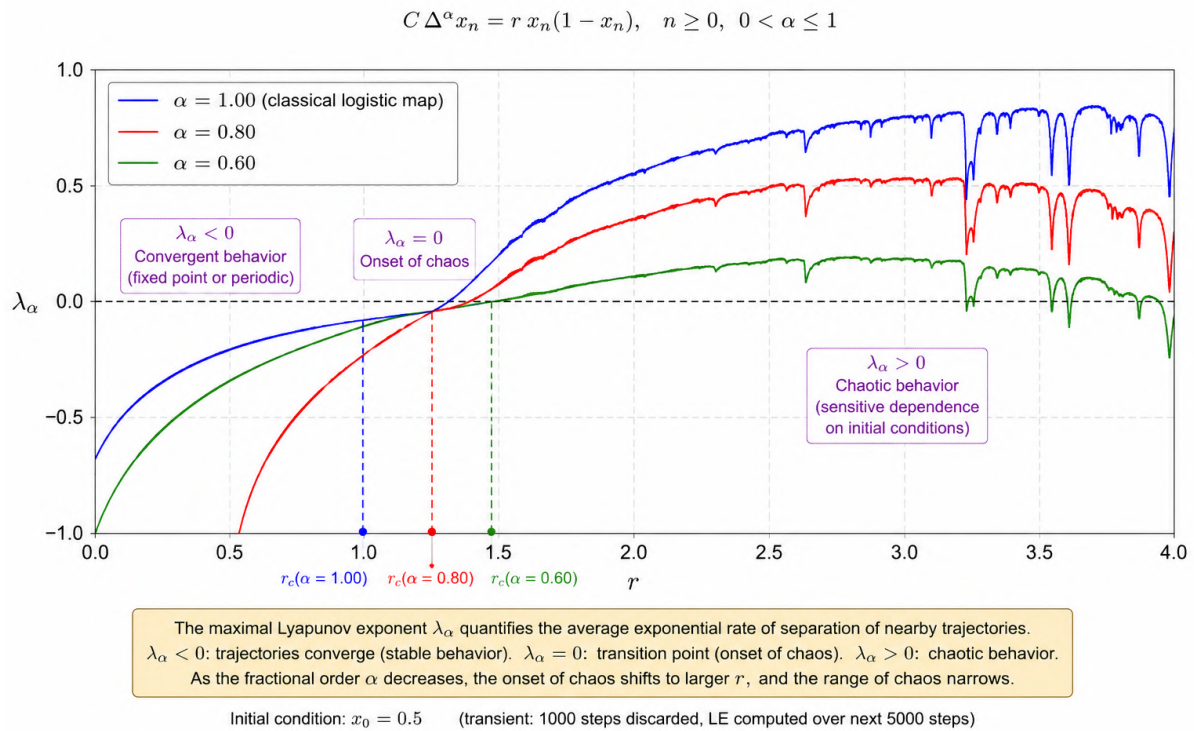


Figure 2.7: Maximal Lyapunov Exponent of the Fractional Logistic Map

As α decreases, the onset of chaos shifts to larger values of r and the range of chaotic behavior narrows, demonstrating that memory *delays* and *suppresses* the transition to chaos.

Step 5: Bifurcation Diagram and Parameter Sensitivity.

We plot the bifurcation diagram of x_k versus the fractional order α , with fixed $r = 3.9$, to trace the transition from stable to chaotic behavior:

- $\alpha \in [0.1, 0.4]$: **stable fixed point** (convergent behavior).
- $\alpha \in [0.4, 0.6]$: **period-doubling** bifurcations begin.
- $\alpha \in [0.6, 0.85]$: **complex dynamics**.
- $\alpha \rightarrow 1$: **chaos with periodic windows**; structure resembles the classical logistic map.

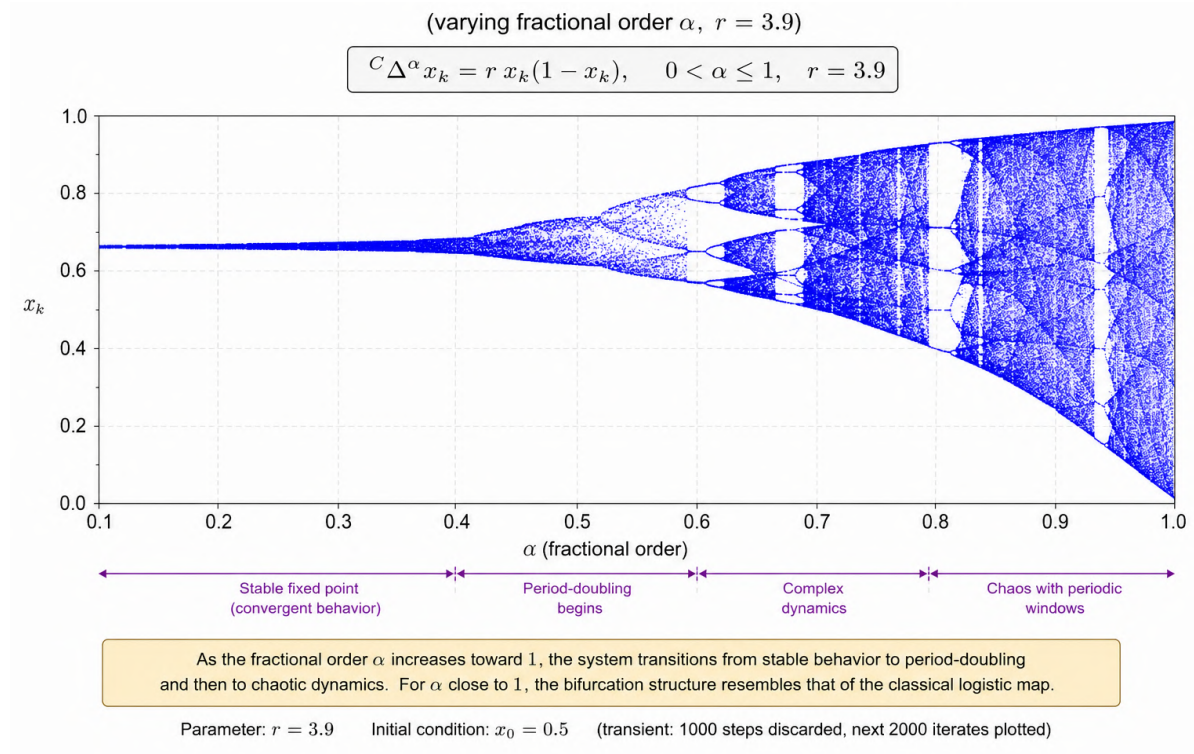


Figure 2.8: Bifurcation Diagram of the Fractional Logistic Map

Remark. Parameter sensitivity analysis highlights the importance of memory effects in fractional discrete systems. Both α and h influence attractor geometry, Lyapunov exponents, and bifurcation diagrams. These effects are critical when applying fractional discrete models to neural networks, where control and stabilization strategies must account for memory-induced delays and enhanced sensitivity.

Example: A Three-Dimensional Fractional-Order Chaotic System

To illustrate the theoretical concepts developed in this chapter — including chaotic attractors, Lyapunov exponents, fractal dimension, and bifurcation — we consider a concrete three-dimensional fractional-order dynamical system. This example demonstrates how memory effects, introduced through the Caputo fractional derivative, modify the qualitative behaviour of a nonlinear system and give rise to chaotic dynamics[24].

System Description

We consider the fractional-order system defined by the Caputo derivative:

$$\begin{cases} {}^C D^\alpha x_1(t) = a x_1 + b x_2 + c x_1 x_3, \\ {}^C D^\alpha x_2(t) = d x_2 - x_1 x_3, \\ {}^C D^\alpha x_3(t) = -e x_3 + x_1 x_2, \end{cases} \quad (2.83)$$

where $0 < \alpha \leq 1$ is the fractional order and the system parameters are

$$a = -10, \quad b = 28, \quad c = -1, \quad d = -1, \quad e = \frac{8}{3}.$$

The initial conditions are taken as $(x_1(0), x_2(0), x_3(0)) = (0.1, 0.1, 0.1)$.

This system is a fractional-order analogue of the classical Lorenz system. Using the Caputo fractional difference operator, the present state depends on the entire past trajectory through a power-law weighted kernel, which is the fundamental memory effect introduced in Section[24].

Fixed Points and Stability Analysis

The equilibrium points are found by setting the right-hand side of (2.83) equal to zero. A straightforward calculation yields three fixed points:

$$E_0 = (0, 0, 0),$$

$$E_+ = (\sqrt{be}, \sqrt{be}, d), \quad E_- = (-\sqrt{be}, -\sqrt{be}, d).$$

To study the local stability of E_+ and E_- , we compute the Jacobian matrix of system (2.83) at a generic equilibrium point $x^* = (x_1^*, x_2^*, x_3^*)$:

$$J(x^*) = \begin{pmatrix} a & b & c x_3^* \\ -x_3^* & d & -x_1^* \\ x_2^* & x_1^* & -e \end{pmatrix}.$$

According to the fractional stability criterion, an equilibrium is unstable whenever at least one eigenvalue λ_i of J satisfies

$$|\arg(\lambda_i)| < \frac{\alpha\pi}{2}.$$

Numerical computation confirms that this condition holds for all three fixed points with the given parameters and α close to 1, so all equilibria are unstable. This instability is a necessary condition for the existence of a chaotic attractor.

Dissipativity

For system (2.83), the divergence of the vector field is

$$\nabla \cdot F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = a + d - e = -10 + (-1) - \frac{8}{3} = -\frac{41}{3} < 0.$$

Therefore, the system is dissipative: phase-space volumes contract exponentially at the rate $\nabla \cdot F$. This guarantees that all trajectories eventually enter a compact invariant set — the chaotic attractor — as formalised in Theorem 2.3.

Minimum Fractional Order for Chaos

A necessary condition for a fractional-order system to exhibit chaotic behaviour is that the eigenvalues of the Jacobian at an unstable equilibrium must lie in the unstable sector of the fractional complex plane, i.e.

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}.$$

Using the eigenvalues computed at E_+ , one determines that chaotic behaviour is sustained for

$$\alpha > \alpha_{\min} \approx 0.89.$$

For α below this threshold, the memory effect suppresses the instability and the system transitions to periodic or quasi-periodic behaviour. This is consistent with Theorem 2.6 and the bifurcation analysis of Section 2.5.2, where decreasing α was shown to shift the onset of chaos to larger parameter values.

Lyapunov Exponents and Kaplan–Yorke Dimension

For $\alpha = 0.95$ and the parameter values above, numerical computation of the fractional Lyapunov spectrum yields:

$$\lambda_1 \approx 0.91, \quad \lambda_2 \approx 0, \quad \lambda_3 \approx -14.57.$$

Since the largest Lyapunov exponent $\lambda_1 > 0$, the system exhibits sensitive dependence on initial conditions and chaotic behaviour (Theorem 2.4). The zero exponent λ_2 reflects the neutral direction along the flow, while the large negative exponent λ_3 confirms strong dissipation.

The Kaplan–Yorke dimension of the attractor is computed using formula[24]:

$$D_{KY} = 2 + \frac{\lambda_1 + \lambda_2}{|\lambda_3|} = 2 + \frac{0.91}{14.57} \approx 2.062.$$

This non-integer value confirms the fractal nature of the chaotic attractor, as discussed in Section 2.4. The attractor occupies slightly more than a two-dimensional surface in phase space, which is characteristic of dissipative chaotic systems.

Bifurcation Analysis

To analyse the effect of the fractional order on the dynamics, we vary α from 0.5 to 1.0 while keeping all other parameters fixed. The bifurcation diagram of x_1 as a function of α reveals the following regimes:

- $\alpha \in (0.50, 0.75)$: the system converges to a stable fixed point.
- $\alpha \in (0.75, 0.89)$: period-doubling bifurcations appear and quasi-periodic windows emerge.
- $\alpha \in (0.89, 1.00)$: fully developed chaotic attractor, consistent with a positive maximal Lyapunov exponent.

This transition from order to chaos as α increases is a direct manifestation of the role of memory in fractional systems: a smaller fractional order introduces stronger historical damping, which stabilises the trajectories, while $\alpha \rightarrow 1$ recovers the integer-order dynamics where chaos is fully developed.

Remark. This example illustrates the complete interplay between the theoretical concepts of this chapter. The non-integer dimension of the attractor reflects its fractal geometry (Section 2.4); the positive Lyapunov exponent quantifies chaos (Section 2.3); the bifurcation diagram shows how α controls the transition to chaos (Section 2.5); and the power-law memory kernel connects all these phenomena to the non-Markovian dynamics introduced in Section 2.1.

2.7 Conclusion

In this chapter, we explored the dynamics of fractional-order discrete systems with a focus on chaos. Unlike classical discrete systems, fractional systems possess memory effects, meaning the current state depends on the entire history of the system. This non-Markovian property significantly enriches the dynamics and leads to new behaviors not observed in integer-order systems. We introduced fractional discrete dynamical systems, formalized the concept of memory via the Caputo fractional difference operator, and discussed the extended phase-space representation to account for the infinite-dimensional nature of these systems. The section on chaos provided rigorous definitions adapted to fractional systems, including sensitive dependence on initial conditions, topological transitivity, and the existence of chaotic attractors. Examples, such as the fractional logistic map, illustrated how memory alters the onset and structure of chaos. The Lyapunov exponent analysis was extended to fractional systems, demonstrating how the accumulation of past states modifies perturbation growth and can suppress or enhance chaotic behavior. Finally, the bifurcation and parameter sensitivity analysis highlighted the influence of the fractional order (α) and step size (h) on the transition from periodic to chaotic dynamics. These parameters serve as powerful tools for controlling or tuning system behavior and are particularly relevant for the design and stabilization of fractional discrete neural networks, which will be the focus of the next chapter.

Chapter 3

An Overview on Fractional-Order Neural Networks

This chapter presents an overview of fractional-order neural networks and their role in modeling nonlinear dynamical systems with memory and hereditary effects. It begins with biological and artificial neural networks, then introduces continuous-time fractional-order neural network models, their dynamical behavior, and their main applications. Finally, it discusses recent advances in different fractional neural network models and the discrete fractional-order neural networks that will be used in later chapters.

3.1 Biological Neural Networks

This section introduces biological neural networks and highlights how fractional-order models naturally capture memory effects observed in neural activity[24]

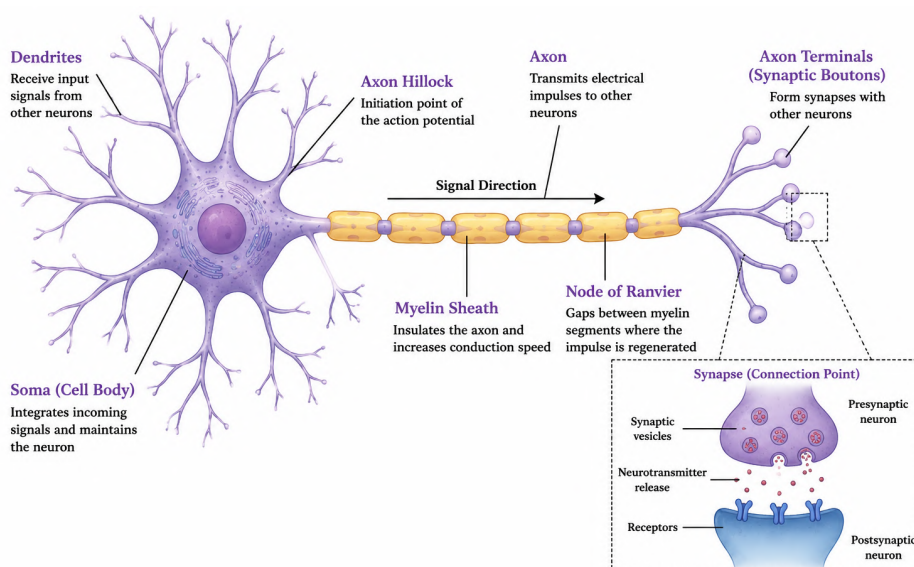


Figure 3.1: Biological Neuron and Signal Transmission

Definition 3.1 (Continuous-time biological neuron model). [24] A biological neuron can be abstracted as a dynamical system whose state variable represents the membrane potential or firing rate. A standard continuous-time rate model is given by

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^n w_{ij} \phi(x_j(t)) + I_i(t), \quad (3.1)$$

where $x_i(t)$ denotes the state of neuron i , $a_i > 0$ is the decay rate, w_{ij} is the synaptic weight from neuron j to neuron i , $\phi(\cdot)$ is a nonlinear activation function, and $I_i(t)$ represents an external stimulus.

Remark. [24] This model captures the instantaneous dynamics of the neuron, but it does not explicitly incorporate long-term memory or hereditary effects that are often observed in biological neural systems.

Definition 3.2 (Fractional-order biological neuron model). [24] To account for memory effects, the integer-order derivative can be replaced by a Caputo fractional derivative. The resulting fractional-order neuron model is

$$D_t^\alpha x_i(t) = -a_i x_i(t) + \sum_{j=1}^n w_{ij} \phi(x_j(t)) + I_i(t), \quad 0 < \alpha \leq 1, \quad (3.2)$$

where D_t^α denotes the Caputo fractional derivative of order α .

Remark. The parameter α controls the memory intensity of the model. When $\alpha = 1$, the classical neuron model is recovered. For $0 < \alpha < 1$, the current state depends on the whole past history of the neuron.

Definition 3.3 (Hebbian learning rule). [24] A general Hebbian learning rule for synaptic adaptation is given by

$$\frac{dw_{ij}(t)}{dt} = \eta x_i(t)x_j(t) - \gamma w_{ij}(t), \quad (3.3)$$

where η is the learning rate and γ represents the decay of synaptic strength.

Definition 3.4 (Fractional-order Hebbian learning rule). [18] A fractional-order extension of the Hebbian rule is defined by

$$D_t^\beta w_{ij}(t) = \eta x_i(t)x_j(t) - \gamma w_{ij}(t), \quad 0 < \beta \leq 1, \quad (3.4)$$

where D_t^β denotes a fractional derivative of order β .

Remark. [18] The fractional-order learning rule introduces memory-dependent synaptic adaptation and provides a mathematical framework consistent with long-term potentiation and long-term depression phenomena observed in neural systems.

Definition 3.5 (Matrix form of biological neural dynamics). [19] Let $\mathbf{x}(t) = [x_1(t), \dots, x_n(t)]^\top$ be the state vector of a biological neural network. Its dynamics can be written as

$$\dot{\mathbf{x}}(t) = -A\mathbf{x}(t) + W\Phi(\mathbf{x}(t)) + \mathbf{I}(t), \quad (3.5)$$

where $A = \text{diag}(a_1, \dots, a_n)$, $W = [w_{ij}]$ is the adjacency or synaptic weight matrix, $\Phi(\mathbf{x})$ applies the activation function componentwise, and $\mathbf{I}(t)$ is the external input vector.

Remark. [19] Fractional-order extensions replace $\dot{\mathbf{x}}(t)$ by $D_t^\alpha \mathbf{x}(t)$, thereby allowing the network topology and past states to jointly influence the present dynamics. This leads to richer transient and oscillatory responses.

Remark. [19] Biological observations motivating fractional-order neural modeling include:

1. **Power-law adaptation:** neural responses often decay according to power laws rather than exponential laws.
2. **Persistent memory effects:** neural activity depends on inputs over long temporal intervals.
3. **Robustness to perturbations:** fractional-order systems often exhibit smoother transients and intrinsic damping.

These features motivate the use of fractional-order neural networks in computational neuroscience and in the design of artificial models with memory.

3.2 Artificial Neural Networks

This section reviews artificial neural networks as computational systems inspired by biological neurons and introduces their basic mathematical models.

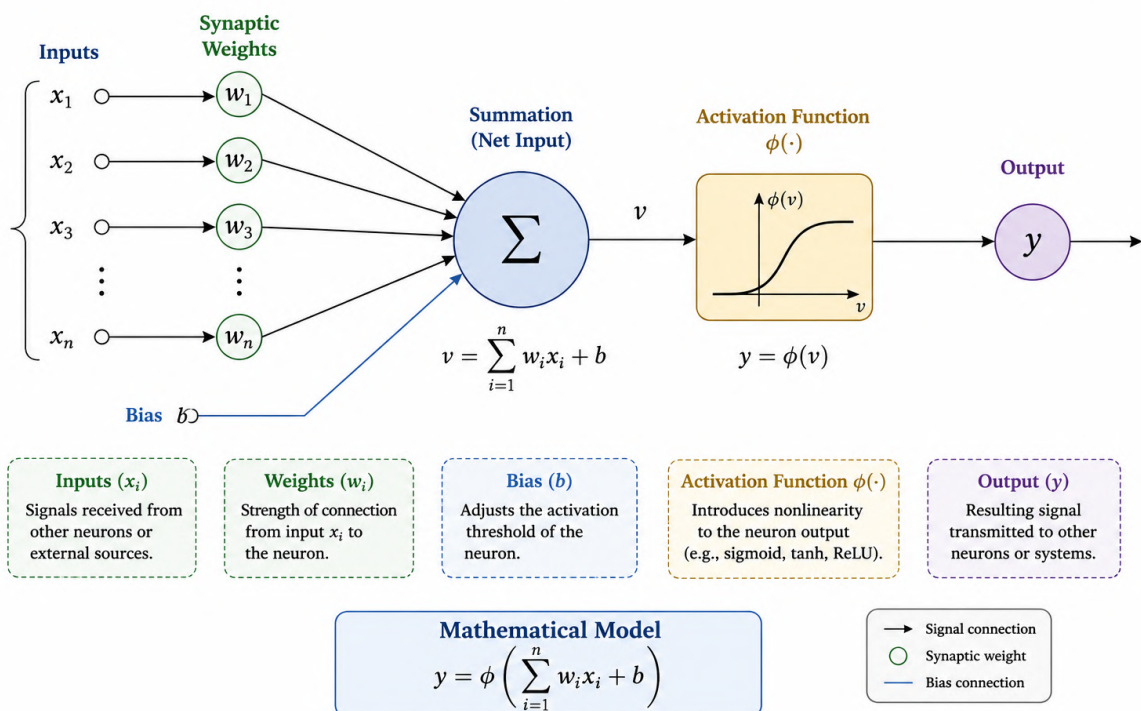


Figure 3.2: Artificial Neuron Model

3.2.1 The Neuron Models

Definition 3.6 (Artificial neuron). An artificial neuron receives an input vector $\mathbf{x} = [x_1, x_2, \dots, x_n]^\top$, combines it with synaptic weights, and produces an output through a nonlinear activation function:

$$y = \phi \left(\sum_{i=1}^n w_i x_i + b \right), \quad (3.6)$$

where w_i are the synaptic weights, b is a bias term, and $\phi(\cdot)$ is the activation function.

Definition 3.7 (Continuous-time dynamic neuron). A dynamic continuous-time neuron can be modeled as

$$\tau \frac{dy(t)}{dt} = -y(t) + \phi \left(\sum_{i=1}^n w_i x_i(t) + b \right), \quad (3.7)$$

where τ is a time constant describing the response speed of the neuron.

Definition 3.8 (Discrete-time dynamic neuron). A discrete-time neuron updates its state iteratively according to

$$y[k+1] = \phi \left(\sum_{i=1}^n w_i x_i[k] + b \right). \quad (3.8)$$

Remark. The discrete-time formulation is particularly suitable for digital implementation and numerical simulation, while the continuous-time formulation is often preferred for dynamical systems analysis.

3.2.2 Activation Function

Definition 3.9 (Common activation functions [20]). Typical activation functions used in artificial neural networks include:

1. **Sigmoid function:**

$$\phi(x) = \frac{1}{1 + e^{-x}}. \quad (3.9)$$

2. **Hyperbolic tangent:**

$$\phi(x) = \tanh(x). \quad (3.10)$$

3. **Rectified Linear Unit (ReLU):**

$$\phi(x) = \max(0, x). \quad (3.11)$$

4. **Leaky ReLU:**

$$\phi(x) = \begin{cases} x, & x \geq 0, \\ \alpha x, & x < 0. \end{cases} \quad (3.12)$$

5. **Softmax function:**

$$\phi_i(\mathbf{x}) = \frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}}, \quad i = 1, \dots, n. \quad (3.13)$$

Remark. Activation functions introduce nonlinearity into the network and determine, to a large extent, the expressive power and training behavior of the model.

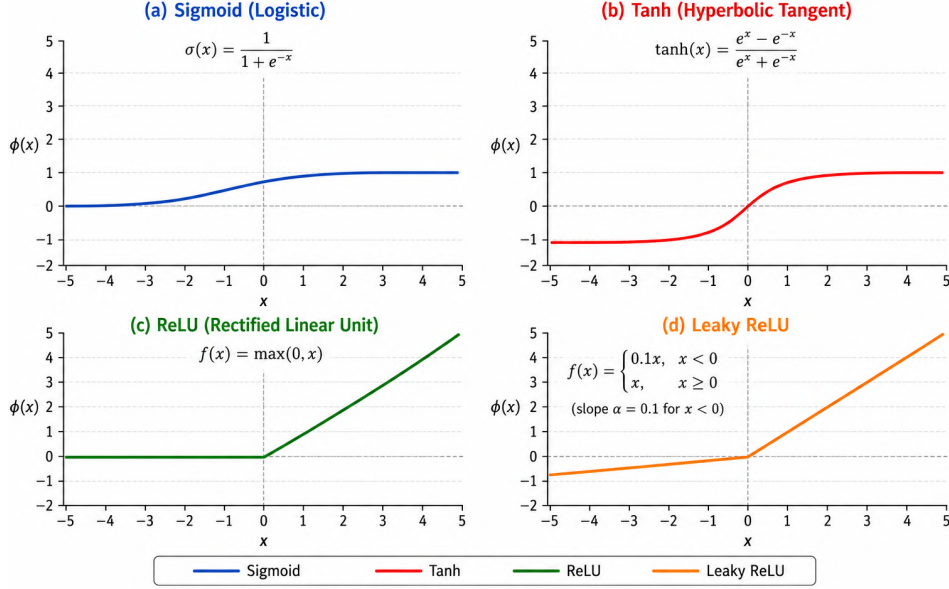


Figure 3.3: Common Activation Functions

3.2.3 General Properties of The Neural Networks

Definition 3.10. [20] **[Feedforward artificial neural network]** A feedforward neural network with L layers is described by

$$\mathbf{y}^{(l)} = \phi^{(l)}\left(W^{(l)}\mathbf{y}^{(l-1)} + \mathbf{b}^{(l)}\right), \quad l = 1, 2, \dots, L, \quad (3.14)$$

where $\mathbf{y}^{(0)} = \mathbf{x}$ is the input vector, $W^{(l)}$ is the weight matrix, $\mathbf{b}^{(l)}$ is the bias vector, and $\phi^{(l)}$ is the activation function of layer l .

Remark. Artificial neural networks are usually organized into input, hidden, and output layers. Hidden layers perform nonlinear transformations, while the output layer produces the final prediction or decision.

Definition 3.11 (Gradient-based learning rule). The weights of an artificial neural network are adjusted to minimize a loss function $\mathcal{L}(\mathbf{y}^{(L)}, \mathbf{t})$, where $\mathbf{t}$ denotes the target output. A standard gradient-based update is

$$W^{(l)} \leftarrow W^{(l)} - \eta \frac{\partial \mathcal{L}}{\partial W^{(l)}}, \quad (3.15)$$

where η is the learning rate.

Definition 3.12 (Fractional-order learning rule). A fractional-order generalization of the learning dynamics can be written as

$$D_t^\alpha W^{(l)}(t) = -\frac{\partial \mathcal{L}}{\partial W^{(l)}}, \quad 0 < \alpha \leq 1. \quad (3.16)$$

Remark. Fractional-order learning rules allow the training dynamics to depend on past gradients and past updates, thereby introducing memory effects that may improve convergence patterns and learning robustness.

3.3 Fractional-Order Neural Networks

This section introduces continuous-time fractional-order neural networks and describes their main dynamical properties, including chaos, bifurcation, and stability[31].

3.3.1 Dynamic Analysis of Fractional Order Neural Networks

Definition 3.13. [31] [**Continuous-time fractional-order neural network**] An n -dimensional continuous-time fractional-order neural network is defined by

$${}^C D_t^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + I_i, \quad i = 1, 2, \dots, n, \quad (3.17)$$

where ${}^C D_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0, 1]$, $x_i(t)$ is the state of the i -th neuron, $c_i > 0$ is the self-regulation coefficient, a_{ij} are the connection weights, f_j is the activation function, and I_i is a constant external input.

Remark. Compared with integer-order neural networks, fractional-order neural networks possess nonlocal dynamics and can capture long-term dependencies, slow transients, and complex oscillatory behavior more effectively.

3.3.1.1 Chaos of Fractional Order Neural Networks

Definition 3.14. [30] [**Fractional-order three-neuron ring network**] A representative fractional-order neural network exhibiting chaotic behavior is the three-neuron ring system

$$\begin{cases} {}^C D_t^\alpha x_1(t) = -ax_1(t) + T_0 \sin(x_1(t)) + T_1 \sin(x_2(t)) + T_2 \sin(x_3(t)), \\ {}^C D_t^\alpha x_2(t) = -ax_2(t) + T_2 \sin(x_1(t)) + T_0 \sin(x_2(t)) + T_1 \sin(x_3(t)), \\ {}^C D_t^\alpha x_3(t) = -ax_3(t) + T_1 \sin(x_1(t)) + T_2 \sin(x_2(t)) + T_0 \sin(x_3(t)). \end{cases} \quad (3.18)$$

Remark. The above system may exhibit chaotic oscillations depending on the fractional order α and the coupling parameters T_0, T_1, T_2 . Chaos is usually investigated numerically by using phase portraits, Lyapunov exponents, and bifurcation diagrams.

3.3.1.2 Limit Cycle and Hopf Bifurcation of Fractional Order Neural Networks

Definition 3.15. [30] [**Linearized fractional-order system**] Let $\bar{\mathbf{x}}$ be an equilibrium point of a fractional-order neural network. Its linearization is

$${}^C D_t^\alpha \mathbf{x}(t) = J(\bar{\mathbf{x}}) \mathbf{x}(t), \quad (3.19)$$

where $J(\bar{\mathbf{x}})$ is the Jacobian matrix evaluated at the equilibrium.

Theorem 3.1 (Fractional Hopf-type bifurcation condition). [30] *A Hopf-type bifurcation in a fractional-order neural network occurs when a pair of complex conjugate eigenvalues λ_i of the Jacobian satisfies*

$$\arg(\lambda_i) = \frac{\alpha\pi}{2}, \quad |\lambda_i| = 1, \quad \frac{d\lambda_i}{d\mu} \neq 0, \quad (3.20)$$

where μ is the bifurcation parameter.

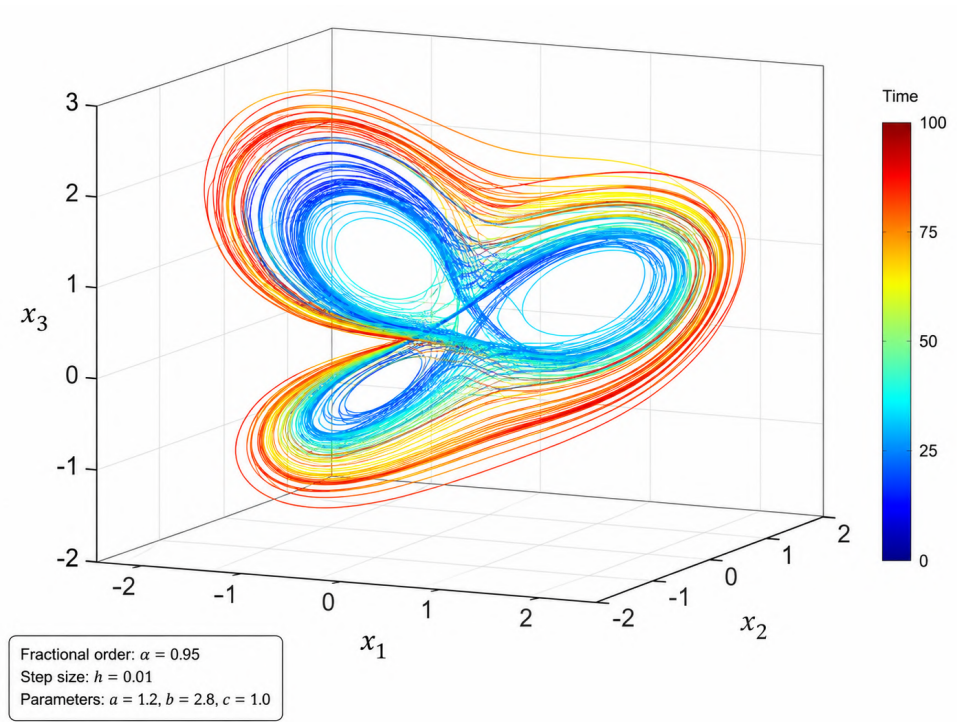


Figure 3.4: Chaotic Attractor of Fractional-Order three-Neuron Ring Network

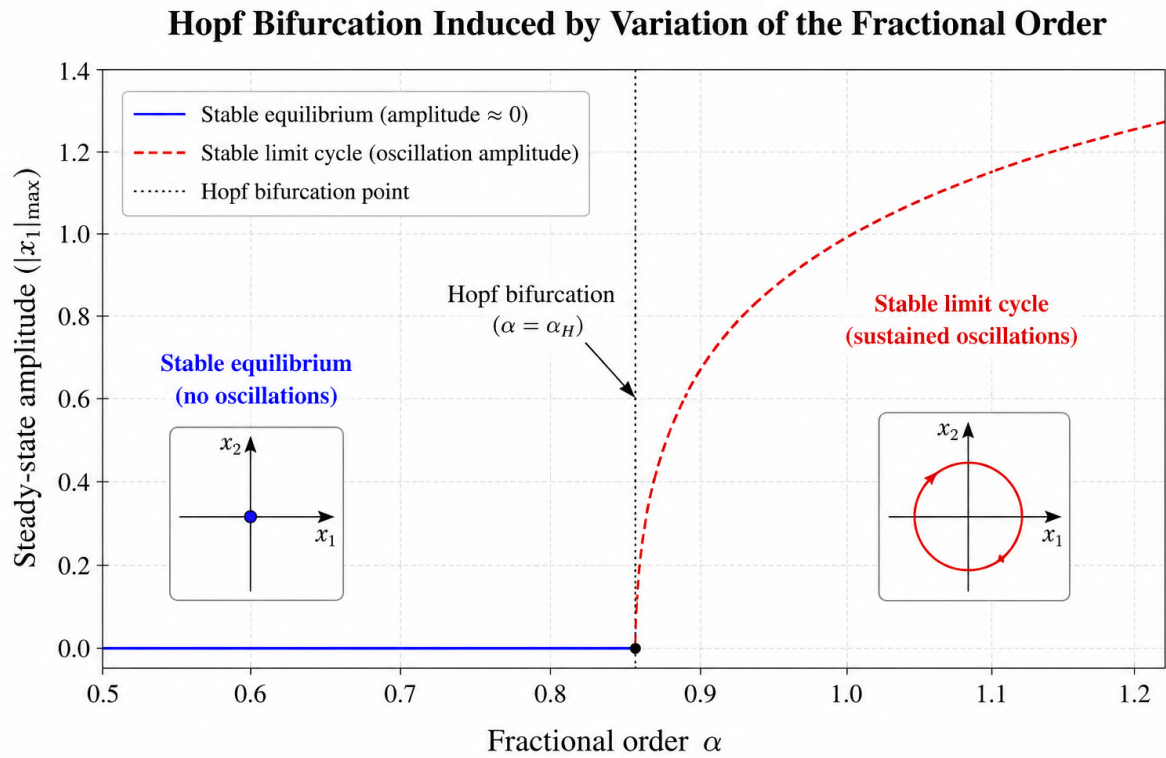


Figure 3.5: Hopf Bifurcation Induced by Variation of the Fractional Order.

Proof. [30] The proof follows from the fractional-order linear stability boundary. In fractional systems, the stability sector is determined by the angle $\alpha\pi/2$. When a pair of eigenvalues crosses this critical boundary transversally as the parameter μ varies, oscillatory solutions emerge, giving rise to a Hopf-type bifurcation. $\square$

Remark. In autonomous fractional-order neural networks, exact periodic solutions may not always exist in the classical sense. Consequently, the observed oscillations in simulations are often quasi-periodic, approximate, or pseudo-limit cycles.

3.3.1.3 Stability of Fractional Order Neural Networks

This subsection presents the two main approaches used in the stability analysis of fractional-order neural networks[29].

Theorem 3.2 (Linearized stability criterion). [29] Consider the linear fractional-order system

$${}^C D_t^\alpha \mathbf{x}(t) = D\mathbf{x}(t), \quad 0 < \alpha < 2, \quad D \in \mathbb{R}^{n \times n}. \quad (3.21)$$

Then the system is asymptotically stable if and only if

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \quad \forall \lambda_i \in \sigma(D), \quad (3.22)$$

where $\sigma(D)$ denotes the set of eigenvalues of D .

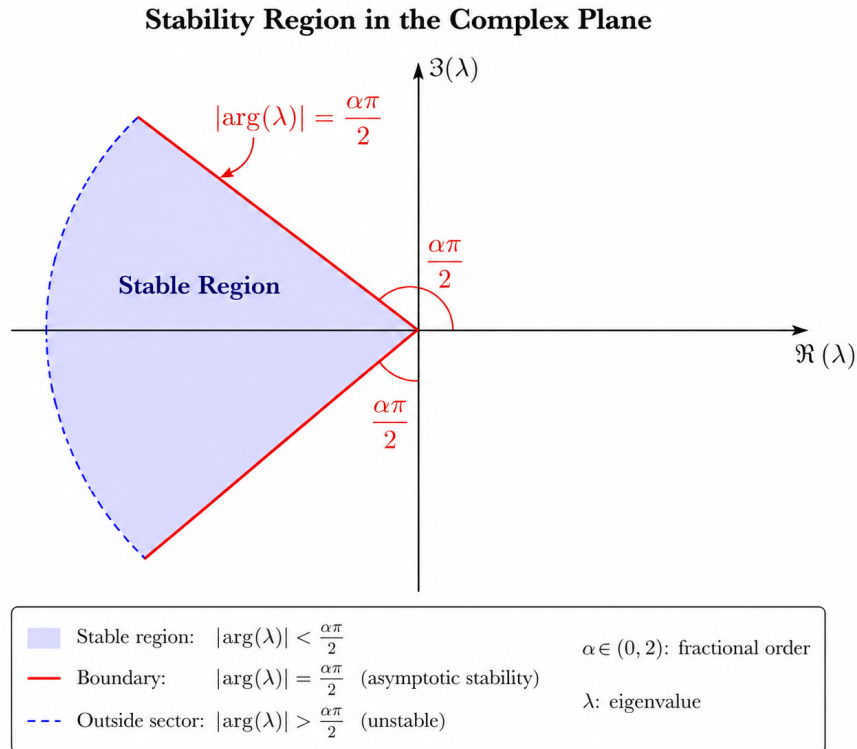


Figure 3.6: Stability Region in the Complex Plane

Proof. The solution of the linear fractional-order system is expressed in terms of Mittag–Leffler functions. Each modal component evolves like $E_\alpha(\lambda_i t^\alpha)$. The decay of these terms is guaranteed precisely when all eigenvalues lie outside the sector $|\arg(\lambda_i)| \leq \alpha\pi/2$. Hence, the stated spectral condition is necessary and sufficient for asymptotic stability. $\square$

Theorem 3.3 (Mittag–Leffler stability criterion). [31] *Consider the nonlinear fractional-order system*

$${}^C D_t^\alpha \mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t)) \quad (3.23)$$

with equilibrium at $\bar{\mathbf{x}} = 0$. Let $V(\mathbf{x})$ be a continuously differentiable Lyapunov function. If there exist constants $\gamma_1, \gamma_2, \gamma_3 > 0$ and $p > 0$ such that

$$\gamma_1 \|\mathbf{x}\|^p \leq V(\mathbf{x}) \leq \gamma_2 \|\mathbf{x}\|^p, \quad {}^C D_t^\alpha V(\mathbf{x}) \leq -\gamma_3 \|\mathbf{x}\|^p, \quad (3.24)$$

then the system is globally Mittag–Leffler stable, and its trajectories satisfy

$$\|\mathbf{x}(t)\| \leq m(\mathbf{x}_0) E_\alpha(-\lambda t^\alpha), \quad (3.25)$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (3.26)$$

$m(\mathbf{x}_0)$ is locally Lipschitz, and $\lambda > 0$.

Proof. [31] The proof is based on the fractional Lyapunov direct method. The inequalities imposed on $V(\mathbf{x})$ provide lower and upper bounds in terms of $\|\mathbf{x}\|^p$, while the negativity of the fractional derivative ensures decay along trajectories. By applying a comparison principle for fractional systems, one obtains the Mittag–Leffler estimate. $\square$

Remark. The fractional order α strongly affects both local and global stability. As α decreases from 1, convergence becomes slower and memory effects become more pronounced, often resulting in delayed chaos, pseudo-limit cycles, or long transient oscillations.

3.3.2 Application of Fractional Neural Networks

This subsection highlights some important applications of fractional-order neural networks in science and engineering[29].

3.3.2.1 Digital Image Processing

Definition 3.16. [29] **[Fractional-order image processing model]** Let an image be represented by a matrix $I \in \mathbb{R}^{M \times N}$, and let x_{ij} denote the pixel intensity at location (i, j) . A fractional-order neural model for image processing is

$${}^C D_t^\alpha x_{ij}(t) = -c x_{ij}(t) + \sum_{(k,l) \in \mathcal{N}_{ij}} w_{ijkl} f(x_{kl}(t)) + I_{ij}, \quad (3.27)$$

where $\mathcal{N}_{ij}$ denotes the neighborhood of pixel (i, j) .

Definition 3.17 (Fractional-order denoising cost functional). A cost functional for image denoising can be defined as

$$J = \frac{1}{2} \sum_{i,j} \left(x_{ij}^{\text{noisy}} - x_{ij}(t) \right)^2 + \lambda \sum_{i,j} \sum_{(k,l) \in \mathcal{N}_{ij}} \left| {}^C D_t^\alpha x_{ij}(t) - {}^C D_t^\alpha x_{kl}(t) \right|, \quad (3.28)$$

where λ is a regularization parameter balancing fidelity and smoothness.

Remark. Fractional-order models in image processing are particularly effective for adaptive smoothing and edge preservation, often outperforming integer-order approaches in nonstationary or noisy settings.

3.3.2.2 Biomedical Engineering

Definition 3.18. [29] **[Fractional-order biomedical signal model]** Let $\mathbf{s}(t)$ be a biomedical input signal such as an ECG. A fractional-order recurrent neural network for biomedical signal processing can be written as

$${}^C D_t^\alpha \mathbf{x}(t) = -C\mathbf{x}(t) + Wf(\mathbf{x}(t)) + V\mathbf{s}(t), \quad (3.29)$$

where C is a diagonal self-feedback matrix, W is the recurrent weight matrix, and V is the input connection matrix.

Remark. The memory property of fractional derivatives makes these models well suited for biomedical time series, since physiological data often contain long-range temporal correlations.

3.3.2.3 Finance

Definition 3.19. [29] **[Fractional-order financial forecasting model]** A fractional-order neural network for forecasting a financial variable $y(t)$ may be defined by

$${}^C D_t^\alpha y(t) = -cy(t) + \sum_{i=1}^n w_i f(y(t - \tau_i)) + \epsilon(t), \quad (3.30)$$

where τ_i are time delays and $\epsilon(t)$ is a noise term.

Remark. Fractional-order dynamics are useful in financial modeling because many financial time series exhibit persistence, long memory, and heavy-tailed fluctuations.

3.3.2.4 Controller

Definition 3.20. [29] **[Fractional-order plant]** A fractional-order plant can be represented as

$${}^C D_t^\beta y(t) = g(y(t), u(t)), \quad 0 < \beta \leq 1, \quad (3.31)$$

where $y(t)$ is the plant output and $u(t)$ is the control input.

Definition 3.21 (Fractional-order neural controller). A fractional-order neural controller can be written as

$$u(t) = \sum_{i=1}^n w_i f(x_i(t)), \quad {}^C D_t^\alpha x_i(t) = -c_i x_i(t) + e(t), \quad (3.32)$$

where $e(t) = y_{\text{ref}}(t) - y(t)$ is the tracking error.

Remark. Fractional-order neural controllers often provide smoother responses and enhanced robustness in the presence of uncertainty, delays, and nonlinearities.

3.3.2.5 System Identification

Definition 3.22. [29] **[Fractional-order system identification model]** Suppose the original system is described by

$${}^C D_t^\beta y(t) = h(\mathbf{y}(t), \mathbf{u}(t)). \quad (3.33)$$

A fractional-order neural approximation of this system is

$${}^C D_t^\alpha \hat{y}(t) = -c\hat{y}(t) + \sum_{i=1}^n w_i f_i(\mathbf{y}(t), \mathbf{u}(t)), \quad (3.34)$$

where $\hat{y}(t)$ is the network output approximating the true system output $y(t)$.

Definition 3.23 (Identification cost functional). A corresponding training objective is

$$J = \frac{1}{2} \int_0^T (y(t) - \hat{y}(t))^2 dt. \quad (3.35)$$

Remark. Fractional-order neural models can represent hereditary and memory-dependent behavior more accurately than classical integer-order models, which makes them attractive in system identification.

3.3.3 Recent Advances in Different Fractional Neural Network Models

This subsection summarizes some recent architectures that extend classical neural models by incorporating fractional-order dynamics.

3.3.3.1 Fractional-Order Hopfield Neural Networks

Definition 3.24. [26] **[Fractional-order Hopfield neural network]** A fractional-order Hopfield neural network with n neurons is described by

$${}^C D_t^\alpha x_i(t) = -a_i x_i(t) + \sum_{j=1}^n w_{ij} f(x_j(t)) + I_i, \quad i = 1, 2, \dots, n, \quad (3.36)$$

where $a_i > 0$ is the self-inhibition coefficient, w_{ij} are synaptic weights, and I_i is an external input.

Definition 3.25 (Energy function of a fractional-order Hopfield network). An associated energy function can be written as

$$E(t) = -\frac{1}{2} \sum_{i,j} w_{ij} f(x_i(t)) f(x_j(t)) + \sum_i \int_0^{x_i(t)} a_i f(s) ds - \sum_i I_i f(x_i(t)). \quad (3.37)$$

Remark. The fractional-order extension enriches the convergence behavior of Hopfield networks and may reduce the occurrence of spurious minima in associative memory tasks.

3.3.3.2 Fractional-Order Cellular Neural Network

Definition 3.26. [26] [**Fractional-order cellular neural network**] Let (i, j) denote a cell in an $M \times N$ grid. A fractional-order cellular neural network is defined by

$${}^C D_t^\alpha x_{ij}(t) = -x_{ij}(t) + \sum_{(k,l) \in \mathcal{N}_{ij}} a_{ijkl} f(x_{kl}(t)) + \sum_{(k,l) \in \mathcal{N}_{ij}} b_{ijkl} u_{kl}(t) + I_{ij}, \quad (3.38)$$

where $\mathcal{N}_{ij}$ is the neighborhood of cell (i, j) , a_{ijkl} and b_{ijkl} are feedback and control templates, and $u_{kl}(t)$ is the input.

Remark. Fractional-order cellular neural networks are effective in image processing, dynamic filtering, and pattern recognition because they combine local interactions with global memory effects.

3.3.3.3 Fractional-Order Memristor-Based Neural Networks

Definition 3.27. [26] [**Fractional-order memristor-based neural network**] A fractional-order memristor-based neural network is modeled by

$${}^C D_t^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^n G_{ij}(t, x_j(t)) f(x_j(t)) + I_i(t), \quad (3.39)$$

where $G_{ij}(t, x_j(t))$ denotes the memductance of the memristive synapse.

Remark. These networks combine the physical memory of memristive devices with the hereditary effects of fractional dynamics, which can improve memory retention and transient behavior.

3.3.3.4 Fractional-Order Complex Valued Neural Networks

Definition 3.28. [26] [**Fractional-order complex-valued neural network**] A complex-valued fractional-order neural network is described by

$${}^C D_t^\alpha z_i(t) = -c_i z_i(t) + \sum_{j=1}^n w_{ij} f(z_j(t)) + I_i, \quad z_i(t) \in \mathbb{C}, \quad (3.40)$$

where $z_i(t) = x_i(t) + iy_i(t)$ and $f : \mathbb{C} \rightarrow \mathbb{C}$ is a complex-valued activation function.

Remark. Fractional-order complex-valued neural networks are especially useful for processing oscillatory, wave-based, or phase-sensitive signals such as those arising in communications, radar, and image analysis.

3.4 Discrete Fractional-Order Neural Networks

This section introduces discrete fractional-order neural networks and discusses their stability and chaotic behavior.

3.4.1 Stability in Discrete Fractional-Order Neural Networks

Definition 3.29. [23] [**Discrete fractional-order neural network**] Let $x[k] \in \mathbb{R}^n$ denote the state vector of a discrete-time fractional-order neural network with order $0 < \alpha \leq 1$. A general discrete fractional-order neural network is written as

$$\Delta^\alpha x[k+1] = -Ax[k] + Wf(x[k]) + I[k], \quad (3.41)$$

where Δ^α is a Caputo-like fractional difference operator, $A \in \mathbb{R}^{n \times n}$ is a diagonal matrix of self-inhibition coefficients, $W \in \mathbb{R}^{n \times n}$ is the synaptic weight matrix, $f(\cdot)$ is the activation function, and $I[k]$ is an external input vector.

Remark (Memory effect). The fractional difference operator makes the current state depend on the whole previous trajectory. Hence, discrete fractional-order neural networks inherit the long-memory effect characteristic of fractional dynamical systems.

3.4.1.1 Stability via Mittag–Leffler Functions

Definition 3.30. [23] [**Linear discrete fractional-order neural network**] A linear discrete fractional-order neural network is given by

$$\Delta^\alpha x[k+1] = Ax[k]. \quad (3.42)$$

Theorem 3.4 (Formal solution representation). [22] *The solution of the linear discrete fractional-order neural network can be formally expressed as*

$$x[k] = \sum_{j=0}^k \binom{k}{j}_\alpha A^j x[0] = E_\alpha(Ak^\alpha)x[0], \quad (3.43)$$

where $\binom{k}{j}_\alpha$ denotes the fractional binomial coefficient and $E_\alpha(\cdot)$ is the Mittag–Leffler function.

Proof. [22] The representation follows from the discrete fractional analogue of the variation-of-constants formula and the power-series expansion of the Mittag–Leffler function. The fractional binomial coefficients encode the memory contribution of all previous states. $\square$

Theorem 3.5 (Spectral stability criterion). *The linear discrete fractional-order neural network is stable if*

$$\lim_{k \rightarrow \infty} \|E_\alpha(Ak^\alpha)\| = 0, \quad (3.44)$$

which is ensured when the eigenvalues λ_i of A satisfy

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}. \quad (3.45)$$

Proof. The asymptotic behavior of the solution is governed by the discrete Mittag–Leffler matrix function. Under the stated spectral condition, this function decays to zero, which implies convergence of all trajectories to the equilibrium. $\square$

3.4.1.2 Uniform Stability

Definition 3.31. [23] [**Uniform stability**] A discrete fractional-order neural network is uniformly stable if for every initial perturbation $x[0]$ there exists a constant $M > 0$ such that

$$\|x[k]\| \leq M\|x[0]\|, \quad \forall k \geq 0. \quad (3.46)$$

Remark. Uniform stability guarantees boundedness of all trajectories independently of the initial time and is important in digital implementations and signal processing applications.

3.4.1.3 Asymptotic Stability

Definition 3.32. [27] [**Asymptotic stability**] A discrete fractional-order neural network is asymptotically stable if

$$\lim_{k \rightarrow \infty} x[k] = 0 \quad (3.47)$$

for zero external input.

Definition 3.33 (Discrete fractional Lyapunov condition). If there exists a Lyapunov function $V[k]$ such that

$$\Delta^\alpha V[k] \leq 0, \quad V[k] > 0, \quad (3.48)$$

then the trajectories remain bounded and may converge to the equilibrium under suitable additional conditions.

3.4.1.4 Stability within Finite Time

Definition 3.34 (Finite-time stability [27]). A discrete fractional-order neural network is finite-time stable if there exists $K > 0$ such that

$$\|x[k]\| \leq \varepsilon, \quad \forall 0 \leq k \leq K, \quad (3.49)$$

where $\varepsilon > 0$ is a prescribed tolerance.

Remark. Finite-time stability is especially relevant in real-time control and computation, where convergence within a limited number of steps is required.

3.4.1.5 Exponential Stability

Definition 3.35. [27] [**Fractional exponential stability**] A discrete fractional-order neural network is exponentially stable in the fractional sense if

$$\|x[k]\| \leq C\rho^{k^\alpha}\|x[0]\|, \quad 0 < \rho < 1, \quad C > 0. \quad (3.50)$$

Remark. This notion generalizes the classical exponential decay of integer-order discrete systems by incorporating the fractional-order parameter α into the decay rate.

3.4.2 Chaos in Discrete Fractional-Order Neural Networks

This subsection illustrates how discrete fractional-order neural networks can generate chaotic behavior.

Definition 3.36. [24] **[Scalar chaotic discrete fractional model]** A scalar fractional neural-type map can be written as

$$\Delta^\alpha x[k+1] = rx[k](1-x[k]), \quad (3.51)$$

which is a fractional-order generalization of the logistic map.

Remark. For suitable values of the parameter r and the fractional order α , the above system exhibits aperiodic oscillations, bifurcations, and attractors with fractal-like structure.

Remark. Chaotic behavior in discrete fractional-order neural networks can be detected through:

- positive Lyapunov exponents,
- bifurcation diagrams under parameter variation,
- phase-space reconstruction and strange attractors.

The coexistence of nonlinearity, discretization, and memory makes these systems rich and suitable for applications such as signal processing, secure communications, and chaos-based computation.

Example : (Ulam–Hyers Stability of Discrete Fractional Multiple-Order Neural Networks)

Definition 3.37 (Ulam–Hyers Stability). [24] We define the system to be Ulam–Hyers stable if there exists a constant $c > 0$ such that for every $\epsilon > 0$, whenever $x \in \mathbb{R}$ satisfies

$$\left\| {}_h^C \Delta_{t_{kl}}^{\alpha_k} x(t) + Ax(t + \alpha_k h) - Dg(x(t + \alpha_k h)) \right\| \leq \epsilon, \quad (3.52)$$

for

$$t \in (h\mathbb{N})_{t_{kl} + (1-\alpha_k)h}, \quad 0 < \alpha_k \leq 1, \quad k = 0, \dots, m-1,$$

then there exists a solution y of system (6.1) such that

$$\|x(t) - y(t)\| \leq c\epsilon.$$

Initially, we establish the following lemma.

Theorem 3.6. [24] *If x satisfies Equation (3.52), then*

$$\begin{aligned} & \left\| x(t) - x(t_0) + \sum_{n=1}^{m-1} \frac{h}{\Gamma(\alpha_{n-1})} \sum_{s=\frac{t_{(n-1)l}}{h} + 1 - \alpha_{n-1}}^{\frac{t}{h} - \alpha_{n-1}} (t - \sigma(sh))_h^{\alpha_{n-1}-1} [Ax(sh) - Dg(x(sh))] \right. \\ & \quad \left. + \frac{h}{\Gamma(\alpha_{m-1})} \sum_{s=\frac{t_{(m-1)l}}{h} + 1 - \alpha_{m-1}}^{\frac{t}{h} - \alpha_{m-1}} (t - \sigma(sh))_h^{\alpha_{m-1}-1} [Ax(sh) - Dg(x(sh))] \right\| \leq d\epsilon, \quad (3.53) \end{aligned}$$

where

$$d = \left(\sum_{n=1}^{m-1} \frac{(t + (\alpha_{n-1} - 1)h + t_{(n-1)l})_h^{\alpha_{n-1}}}{\Gamma(1 + \alpha_{n-1})} + \sup_{t \in \{t_{(m-1)l} + h, \dots, t_{ml}\}} \frac{(t + (\alpha_{m-1} - 1)h + t_{(m-1)l})_h^{\alpha_{m-1}}}{\Gamma(1 + \alpha_{m-1})} \right). \quad (3.54)$$

Proof. [24] Suppose that $x(t)$ satisfies Equation (3.52). Then there exists a function $f(t)$ such that

$${}_h^C \Delta_{t_{kl}}^{\alpha_k} x(t) + Ax(t + \alpha_k h) - Dg(x(t + \alpha_k h)) = f(t),$$

for

$$t \in \{t_{kl} + h, \dots, t_{(k+1)l}\}, \quad k = 0, 1, \dots, m-1.$$

This implies

$$\begin{aligned} {}_h^C \Delta_{t_0}^{\alpha_0} x(t) + Ax(t + \alpha_0 h) - Dg(x(t + \alpha_0 h)) &= f(t), \quad t \in \{t_0 + h, \dots, t_l\}, \\ {}_h^C \Delta_{t_l}^{\alpha_1} x(t) + Ax(t + \alpha_1 h) - Dg(x(t + \alpha_1 h)) &= f(t), \quad t \in \{t_l + h, \dots, t_{2l}\}, \\ &\vdots \end{aligned}$$

$${}_h^C \Delta_{t_{(m-1)l}}^{\alpha_{m-1}} x(t) + Ax(t + \alpha_{m-1} h) - Dg(x(t + \alpha_{m-1} h)) = f(t), \quad t \in \{t_{(m-1)l} + h, \dots, t_{ml}\}.$$

Using the fractional summation formula, we obtain

$$\begin{aligned} x(t) - x(t_0) &+ \sum_{n=1}^{m-1} \frac{h}{\Gamma(\alpha_{n-1})} \sum_{s=\frac{t_{(n-1)l}}{h} + 1 - \alpha_{n-1}}^{\frac{t}{h} - \alpha_{n-1}} (t - \sigma(sh))_h^{\alpha_{n-1}-1} [Ax(sh) - Dg(x(sh))] \\ &+ \frac{h}{\Gamma(\alpha_{m-1})} \sum_{s=\frac{t_{(m-1)l}}{h} + 1 - \alpha_{m-1}}^{\frac{t}{h} - \alpha_{m-1}} (t - \sigma(sh))_h^{\alpha_{m-1}-1} [Ax(sh) - Dg(x(sh))] \\ &= \sum_{n=1}^{m-1} \frac{h}{\Gamma(\alpha_{n-1})} \sum_{s=\frac{t_{(n-1)l}}{h} + 1 - \alpha_{n-1}}^{\frac{t}{h} - \alpha_{n-1}} (t - \sigma(sh))_h^{\alpha_{n-1}-1} f(sh) \\ &+ \frac{h}{\Gamma(\alpha_{m-1})} \sum_{s=\frac{t_{(m-1)l}}{h} + 1 - \alpha_{m-1}}^{\frac{t}{h} - \alpha_{m-1}} (t - \sigma(sh))_h^{\alpha_{m-1}-1} f(sh). \end{aligned} \quad (3.55)$$

Taking norms on both sides yields

$$\left\| x(t) - x(t_0) + \dots \right\| \leq d \|f\|.$$

Since $\|f\| \leq \epsilon$, it follows that

$$\left\| x(t) - x(t_0) + \dots \right\| \leq d\epsilon.$$

Hence, the proof is complete. $\square$

A novel theorem is now presented to establish the Ulam–Hyers stability of the variable-order fractional discrete neural network.

Theorem 3.7. [23] *Under assumptions (A_1) and (A_2) , the system exhibits Ulam–Hyers stability.*

Proof. [23] Let y be a solution of system. Then

$$\begin{aligned} y(t) = & x(t_0) + \sum_{n=1}^{m-1} \frac{h}{\Gamma(\alpha_{n-1})} \sum_{s=\frac{t_{(n-1)l}}{h}+1-\alpha_{n-1}}^{\frac{t}{h}-\alpha_{n-1}} (t - \sigma(sh))_h^{\alpha_{n-1}-1} [Ay(sh) - Dg(y(sh))] \\ & + \frac{h}{\Gamma(\alpha_{m-1})} \sum_{s=\frac{t_{(m-1)l}}{h}+1-\alpha_{m-1}}^{\frac{t}{h}-\alpha_{m-1}} (t - \sigma(sh))_h^{\alpha_{m-1}-1} [Ay(sh) - Dg(y(sh))]. \end{aligned} \quad (3.56)$$

Therefore,

$$\begin{aligned} \|x(t) - y(t)\| \leq & d\epsilon \\ & + [MA + LMD] \left(\sum_{n=1}^{m-1} \frac{(t + (\alpha_{n-1} - 1)h + t_{(n-1)l})_h^{\alpha_{n-1}}}{\Gamma(1 + \alpha_{n-1})} \right. \\ & \left. + \sup_{t \in \{t_{(m-1)l}+h, \dots, t_{ml}\}} \frac{(t + (\alpha_{m-1} - 1)h + t_{(m-1)l})_h^{\alpha_{m-1}}}{\Gamma(1 + \alpha_{m-1})} \right) \|x(t) - y(t)\|. \end{aligned} \quad (3.57)$$

Consequently,

$$\|x(t) - y(t)\| \leq \frac{d}{1 - M} \epsilon \leq c\epsilon,$$

where

$$c = \frac{d}{1 - M}.$$

Thus, the system satisfies the Ulam–Hyers stability property. $\square$

Exercise [24]

Consider the discrete fractional multiple-order neural network described by system (6.1).

1. Show that system (6.1) is Ulam–Hyers stable if there exists a constant $c > 0$ such that for every $\epsilon > 0$, whenever a function $x(t)$ satisfies

$$\left\| {}^C_h \Delta_{t_{kl}}^{\alpha_k} x(t) + Ax(t + \alpha_k h) - Dg(x(t + \alpha_k h)) \right\| \leq \epsilon, \quad (3.58)$$

for

$$t \in (h\mathbb{N})_{t_{kl} + (1-\alpha_k)h}, \quad 0 < \alpha_k \leq 1, \quad k = 0, \dots, m-1,$$

then there exists an exact solution $y(t)$ of system (6.1) such that

$$\|x(t) - y(t)\| \leq c\epsilon.$$

2. Assume that $x(t)$ satisfies Equation (3.58). Prove that there exists a perturbation function $f(t)$ satisfying

$${}^C_h \Delta_{t_{kl}}^{\alpha_k} x(t) + Ax(t + \alpha_k h) - Dg(x(t + \alpha_k h)) = f(t),$$

with

$$\|f(t)\| \leq \epsilon.$$

3. Using the fractional summation formula, derive the relation

$$\begin{aligned} x(t) - x(t_0) &+ \sum_{n=1}^{m-1} \frac{h}{\Gamma(\alpha_{n-1})} \sum (t - \sigma(sh))_h^{\alpha_{n-1}-1} [Ax(sh) - Dg(x(sh))] \\ &+ \frac{h}{\Gamma(\alpha_{m-1})} \sum (t - \sigma(sh))_h^{\alpha_{m-1}-1} [Ax(sh) - Dg(x(sh))] \\ &= \sum_{n=1}^{m-1} \frac{h}{\Gamma(\alpha_{n-1})} \sum (t - \sigma(sh))_h^{\alpha_{n-1}-1} f(sh) + \frac{h}{\Gamma(\alpha_{m-1})} \sum (t - \sigma(sh))_h^{\alpha_{m-1}-1} f(sh). \end{aligned} \quad (3.59)$$

4. Show that

$$\left\| x(t) - x(t_0) + \dots \right\| \leq d\epsilon,$$

where

$$d = \left(\sum_{n=1}^{m-1} \frac{(t + (\alpha_{n-1} - 1)h + t_{(n-1)l})_h^{\alpha_{n-1}}}{\Gamma(1 + \alpha_{n-1})} + \sup_t \frac{(t + (\alpha_{m-1} - 1)h + t_{(m-1)l})_h^{\alpha_{m-1}}}{\Gamma(1 + \alpha_{m-1})} \right).$$

5. Assume that conditions (A_1) and (A_2) hold. Prove that system (6.1) is Ulam–Hyers stable.

Solution [24]

Step 1. Definition of Ulam–Hyers Stability

System (6.1) is said to be Ulam–Hyers stable if there exists a constant $c > 0$ such that for every $\epsilon > 0$, whenever a function $x(t)$ satisfies

$$\left\| {}^C_h \Delta_{t_{kl}}^{\alpha_k} x(t) + Ax(t + \alpha_k h) - Dg(x(t + \alpha_k h)) \right\| \leq \epsilon,$$

then there exists an exact solution $y(t)$ satisfying

$$\|x(t) - y(t)\| \leq c\epsilon.$$

Step 2. Introduction of the Perturbation Function

Since $x(t)$ satisfies Equation (3.58), there exists a function $f(t)$ such that

$${}^C_h \Delta_{t_{kl}}^{\alpha_k} x(t) + Ax(t + \alpha_k h) - Dg(x(t + \alpha_k h)) = f(t),$$

where

$$\|f(t)\| \leq \epsilon.$$

Step 3. Fractional Summation Representation

Applying the discrete fractional summation operator yields

$$\begin{aligned} x(t) - x(t_0) &+ \sum_{n=1}^{m-1} \frac{h}{\Gamma(\alpha_{n-1})} \sum (t - \sigma(sh))_h^{\alpha_{n-1}-1} [Ax(sh) - Dg(x(sh))] \\ &+ \frac{h}{\Gamma(\alpha_{m-1})} \sum (t - \sigma(sh))_h^{\alpha_{m-1}-1} [Ax(sh) - Dg(x(sh))] \\ &= \sum_{n=1}^{m-1} \frac{h}{\Gamma(\alpha_{n-1})} \sum (t - \sigma(sh))_h^{\alpha_{n-1}-1} f(sh) + \frac{h}{\Gamma(\alpha_{m-1})} \sum (t - \sigma(sh))_h^{\alpha_{m-1}-1} f(sh). \end{aligned} \tag{3.60}$$

Step 4. Norm Estimation

Taking norms on both sides, we obtain

$$\begin{aligned} \left\| x(t) - x(t_0) + \cdots \right\| &\leq \left(\sum_{n=1}^{m-1} \frac{(t + (\alpha_{n-1} - 1)h + t_{(n-1)l})_h^{\alpha_{n-1}}}{\Gamma(1 + \alpha_{n-1})} \right. \\ &\quad \left. + \sup_t \frac{(t + (\alpha_{m-1} - 1)h + t_{(m-1)l})_h^{\alpha_{m-1}}}{\Gamma(1 + \alpha_{m-1})} \right) \|f\|. \end{aligned} \tag{3.61}$$

Since

$$\|f\| \leq \epsilon,$$

we conclude that

$$\left\| x(t) - x(t_0) + \cdots \right\| \leq d\epsilon.$$

Step 5. Proof of Ulam–Hyers Stability

Let $y(t)$ be the exact solution of system (6.1). Then

$$\begin{aligned} y(t) = & x(t_0) + \sum_{n=1}^{m-1} \frac{h}{\Gamma(\alpha_{n-1})} \sum (t - \sigma(sh))_h^{\alpha_{n-1}-1} [Ay(sh) - Dg(y(sh))] \\ & + \frac{h}{\Gamma(\alpha_{m-1})} \sum (t - \sigma(sh))_h^{\alpha_{m-1}-1} [Ay(sh) - Dg(y(sh))]. \end{aligned} \quad (3.62)$$

Subtracting the equations corresponding to $x(t)$ and $y(t)$ gives

$$\|x(t) - y(t)\| \leq d\epsilon + \left\| \sum [-A(x(sh) - y(sh))] \right\| + \left\| \sum D(g(x(sh)) - g(y(sh))) \right\|.$$

Using the Lipschitz condition

$$\|g(x) - g(y)\| \leq L\|x - y\|,$$

we obtain

$$\|x(t) - y(t)\| \leq d\epsilon + [MA + LMD]\|x(t) - y(t)\|.$$

Let

$$M = [MA + LMD].$$

If $M < 1$, then

$$(1 - M)\|x(t) - y(t)\| \leq d\epsilon.$$

Hence,

$$\|x(t) - y(t)\| \leq \frac{d}{1 - M}\epsilon.$$

Setting

$$c = \frac{d}{1 - M},$$

we conclude that

$$\|x(t) - y(t)\| \leq c\epsilon.$$

Therefore, system (6.1) is Ulam–Hyers stable

Numerical Simulation [24]

Example. Let us explore a two-dimensional fractional discrete-time neural network given by

$${}^C_h\Delta_{t_k}^{\alpha_k} r(t) = -Ar(t+\alpha_k h) + \Omega \sin(r(t+\alpha_k h)), \quad 0 < \alpha_k \leq 1, \quad t \in (h\mathbb{N})_{k+1-(1-\alpha_k)}, \quad k = 0, 1, 2, 3. \quad (6.6)$$

where

$$\sin(r(t)) = (\sin(r_1(t)), \sin(r_2(t)))^T.$$

Suppose that

$$r(0) = (1, -1)^T, \quad (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0.5, 0.4, 0.3, 0.2), \quad h = 1.3, \quad m = 4.$$

The matrices are given by

$$A = \begin{bmatrix} 0.046 & 0 \\ 0 & 0.046 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0.0003 & 0 \\ -0.0004 & 0.0001 \end{bmatrix}.$$

We can investigate the stability analysis across various time domains, ensuring that the parameters satisfy all conditions stated in Theorems (3.6) and (3.7). Since

$$\mathcal{M} = 0.9858 < 1,$$

the required conditions are satisfied, and consequently the stability of the system can be examined as illustrated in Figure 3.7.

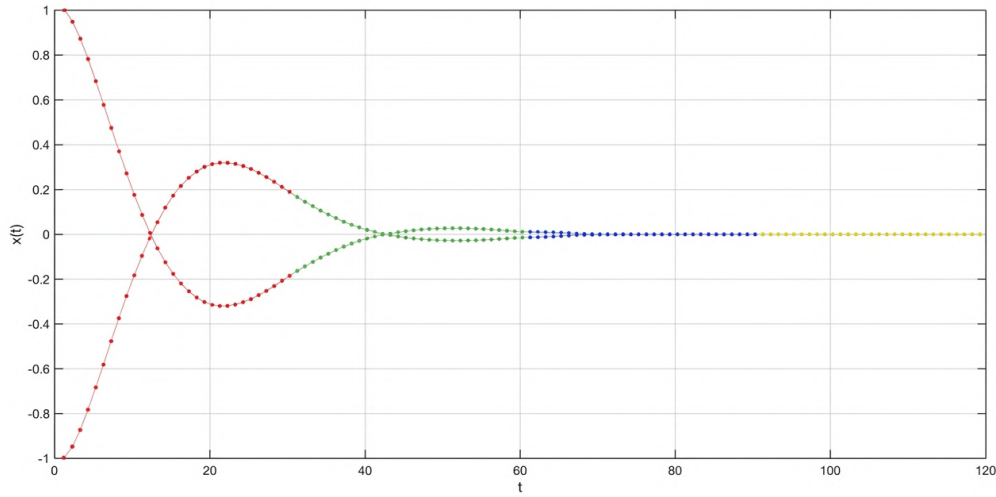


Figure 3.7: The numerical solution of neural network

3.5 Conclusion

In this chapter, we presented an overview of fractional-order neural networks and their importance in modeling complex dynamical systems with memory and hereditary properties. We first introduced the biological neural networks and the artificial neural networks, including neuron models, activation functions, and the general properties of neural systems.

After that, we focused on fractional-order neural networks and discussed their dynamic behavior, applications, and recent developments in different fractional neural network models. The use of fractional calculus provides more flexibility and accuracy in describing nonlinear phenomena compared to classical integer-order models.

Furthermore, we studied discrete fractional-order neural networks, emphasizing important concepts such as stability and chaotic behavior. In particular, the Ulam–Hyers stability of discrete fractional multiple-order neural networks was investigated through a detailed mathematical example.

Finally, numerical simulations were presented to illustrate the theoretical results and to confirm the effectiveness of the proposed analysis. The obtained results demonstrate that fractional discrete neural networks constitute a powerful framework for analyzing complex systems and provide promising perspectives for future research in mathematics, engineering, and applied sciences.

Chapter 4

Stabilization of Chaotic Fractional Discrete Neural Networks Based on the Caputo h -Difference Operator

4.1 Introduction

Discrete fractional calculus provides a nonlocal difference framework with memory. For neural network models, this permits the present state to depend on a weighted discrete history. The Caputo h -difference operator is suitable for such systems since it generates a discrete-time dynamics with hereditary effects and admits a Lyapunov analysis for stability.

In this chapter, a three-dimensional fractional discrete neural network is considered. The model is defined by means of the Caputo h -difference operator and a hyperbolic tangent activation law. For suitable fractional orders, the network exhibits chaotic dynamics. A sign-based feedback law is then constructed in order to stabilize the chaotic trajectories at the origin. Finally, a variable-order extension is introduced by replacing the constant fractional order with an interval-dependent order.

4.2 Fractional Discrete Neural Network Based on the h -Fractional Difference

Let[9] $h > 0$, $a \in \mathbb{R}$, and $0 < \nu \leq 1$. The forward h -difference operator is defined by

$$\Delta_h x(s) = \frac{x(s+h) - x(s)}{h}. \quad (4.1)$$

The h -falling factorial function is given by

$$s_h^{(\nu)} = h^\nu \frac{\Gamma(\frac{s}{h} + 1)}{\Gamma(\frac{s}{h} + 1 - \nu)}. \quad (4.2)$$

The h -fractional sum of order ν is

$${}_a \Delta_h^{-\nu} x(s) = \frac{h}{\Gamma(\nu)} \sum_{k=a/h}^{s/h-\nu} (s - \sigma(kh))_h^{(\nu-1)} x(kh), \quad \sigma(kh) = (k+1)h. \quad (4.3)$$

For $0 < \nu \leq 1$, the Caputo-like h -difference operator is defined by

$${}_a^C \Delta_h^\nu x(s) = \Delta_h^{-(1-\nu)} \Delta_h x(s), \quad s \in (h\mathbb{N})_{a+(1-\nu)h}. \quad (4.4)$$

4.2.1 Model

Consider the three-dimensional fractional discrete neural network[9]

$$\begin{cases} {}_a^C \Delta_h^\nu x_1(s) = -x_1(s + \nu h) + a_{11} \tanh(x_1(s + \nu h)) + a_{12} \tanh(x_2(s + \nu h)), \\ {}_a^C \Delta_h^\nu x_2(s) = -x_2(s + \nu h) + a_{21} \tanh(x_1(s + \nu h)) + a_{22} \tanh(x_2(s + \nu h)) \\ \quad + a_{23} \tanh(x_3(s + \nu h)), \\ {}_a^C \Delta_h^\nu x_3(s) = -x_3(s + \nu h) + a_{31} \tanh(x_1(s + \nu h)) + a_{33} \tanh(x_3(s + \nu h)) \end{cases} \quad (4.5)$$

for $s \in (h\mathbb{N})_{a+(1-\nu)h}$.

The parameter matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & -1.2 & 0 \\ 2 & 1.71 & 1.15 \\ -4.75 & 0 & 1.1 \end{pmatrix}. \quad (4.6)$$

The activation function is the hyperbolic tangent function $\tanh(\cdot)$. The numerical experiments are generated from the initial data

$$(x_1(0), x_2(0), x_3(0)) = (0.02, 0.5, 0.8), \quad (4.7)$$

$$(x_1(0), x_2(0), x_3(0)) = (-0.02, -0.5, -0.8), \quad (4.8)$$

and

$$(x_1(0), x_2(0), x_3(0)) = (0.02, -0.5, 0.8). \quad (4.9)$$

Applying the h -fractional sum operator to (4.5) yields a discrete Volterra-type representation, which is used in the predictor-corrector numerical procedure. For each component, one obtains a formula of the form

$$x_i(s) = x_i(a) + \frac{h}{\Gamma(\nu)} \sum (s - \sigma(\tau))_h^{(\nu-1)} F_i(\tau, x(\tau + \nu h)), \quad i = 1, 2, 3, \quad (4.10)$$

where F_i denotes the right-hand side of the i -th equation in (4.5).

4.2.2 Chaotic Behavior

For the parameter matrix (4.6), the network (4.5) exhibits chaotic behavior when the fractional order is sufficiently large. The reported numerical analysis shows that chaotic dynamics arise for[9]

$$\nu > 0.7. \quad (4.11)$$

In particular, chaotic attractors are observed for the integer-order case

$$\nu = 1 \quad (4.12)$$

and for the fractional-order case

$$\nu = 0.9. \quad (4.13)$$

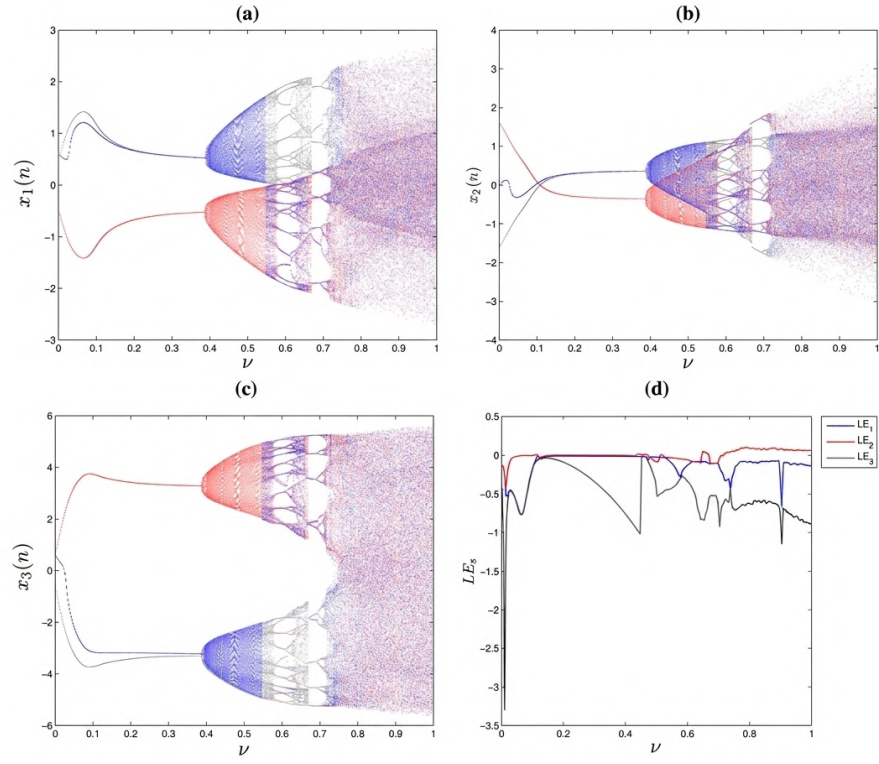


Figure 4.1: Bifurcation Diagram and Lyapunov Exponents

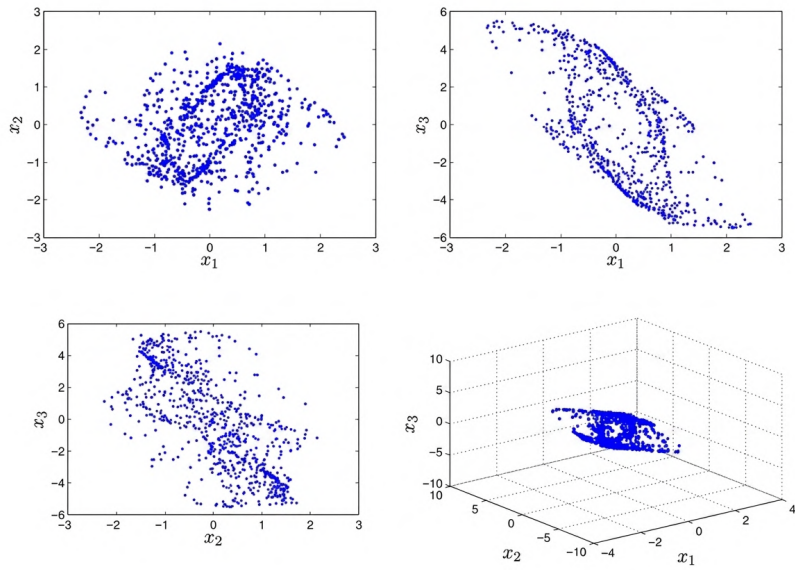


Figure 4.2: Chaotic Attractor at $\nu = 1$

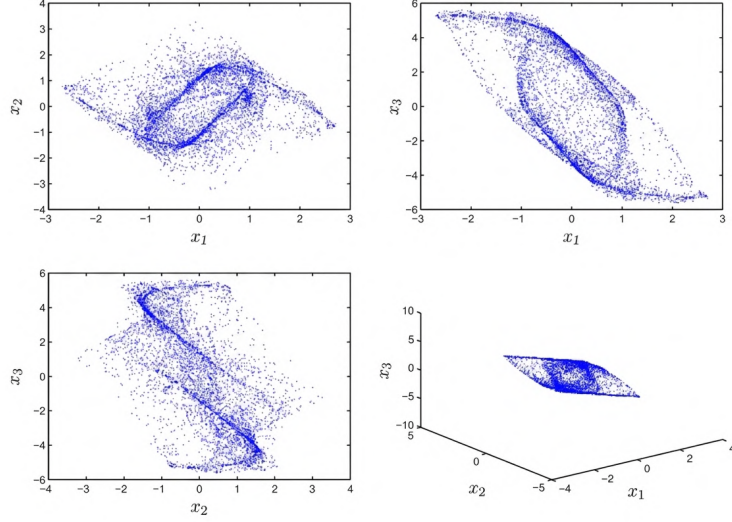


Figure 4.3: Chaotic Attractor at $v = 0.9$

The transition to chaos is detected numerically by means of bifurcation diagrams and the largest Lyapunov exponent. A positive largest Lyapunov exponent indicates chaotic evolution. Hence, the memory generated by the Caputo h -difference operator does not suppress chaos; rather, it modifies the geometry of the attractor and the discrete temporal evolution.

The estimate

$$|\tanh(z)| \leq |z|, \quad z \in \mathbb{R}, \quad (4.14)$$

will be used in the stabilization analysis.

4.3 Stabilization

The stabilization problem consists in constructing a feedback law such that the origin becomes an asymptotically stable equilibrium of the controlled system[11].

A basic inequality for the Caputo h -difference operator is

$${}_h^C \Delta_a^\nu x^2(s) \leq 2x(s + \nu h) {}_h^C \Delta_a^\nu x(s), \quad 0 < \nu \leq 1. \quad (4.15)$$

Consider now the controlled neural network

$$\begin{cases} {}_a^C \Delta_h^\nu x_1(s) = -x_1(s + \nu h) + a_{11} \tanh(x_1(s + \nu h)) + a_{12} \tanh(x_2(s + \nu h)) + C_1(s), \\ {}_a^C \Delta_h^\nu x_2(s) = -x_2(s + \nu h) + a_{21} \tanh(x_1(s + \nu h)) + a_{22} \tanh(x_2(s + \nu h)) \\ \quad + a_{23} \tanh(x_3(s + \nu h)) + C_2(s), \\ {}_a^C \Delta_h^\nu x_3(s) = -x_3(s + \nu h) + a_{31} \tanh(x_1(s + \nu h)) + a_{33} \tanh(x_3(s + \nu h)) + C_3(s) \end{cases} \quad (4.16)$$

The control law is chosen as

$$\begin{cases} C_1(s) = -\operatorname{sgn}(x_1(s)) (|a_{11}| + |a_{12}|), \\ C_2(s) = -\operatorname{sgn}(x_2(s)) (|a_{21}| + |a_{22}| + |a_{23}|), \\ C_3(s) = -\operatorname{sgn}(x_3(s)) (|a_{31}| + |a_{33}|), \end{cases} \quad (4.17)$$

where

$$\text{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases} \quad (4.18)$$

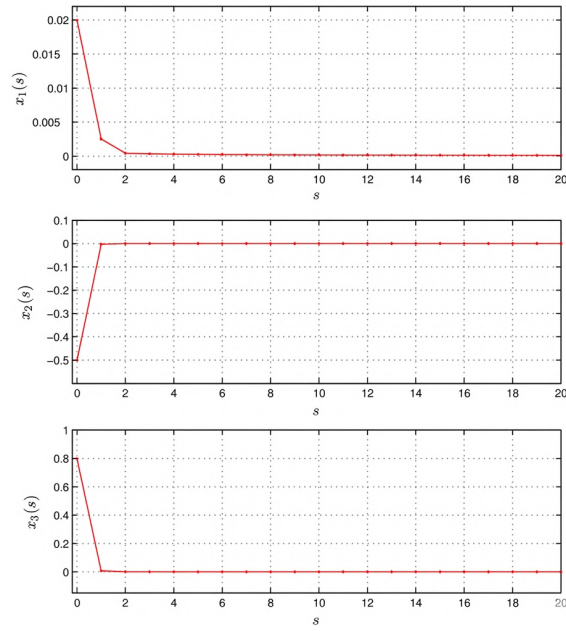


Figure 4.4: Stabilization of Chaotic States Using a Control Law

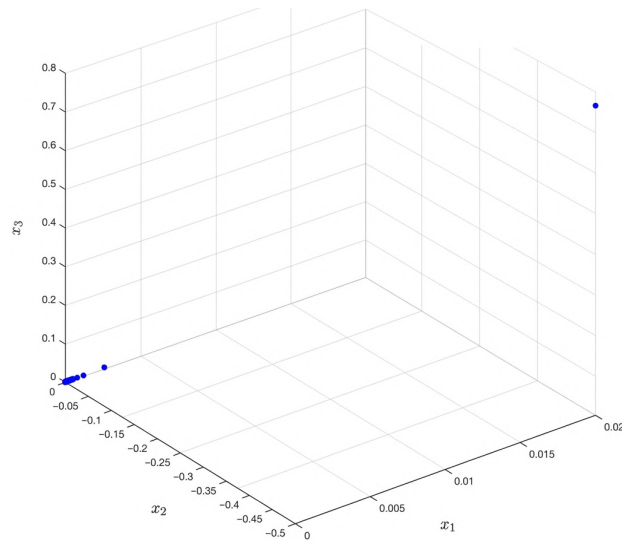


Figure 4.5: Phase Space After Control

Theorem 4.1. [22] *The h -fractional neural network (4.5) is asymptotically stabilized at the origin under the feedback law (4.17).*

Proof. [22] Consider the Lyapunov function

$$V(s) = \frac{1}{2} (x_1^2(s) + x_2^2(s) + x_3^2(s)). \quad (4.19)$$

The function V is positive definite and radially unbounded.

Using (4.15), one obtains

$${}_h^C \Delta_a^\nu V(s) \leq x_1(s + \nu h) {}_h^C \Delta_a^\nu x_1(s) + x_2(s + \nu h) {}_h^C \Delta_a^\nu x_2(s) + x_3(s + \nu h) {}_h^C \Delta_a^\nu x_3(s). \quad (4.20)$$

Substituting (4.16) into (4.20) gives

$$\begin{aligned} {}_h^C \Delta_a^\nu V(s) \leq & -x_1^2(s + \nu h) - x_2^2(s + \nu h) - x_3^2(s + \nu h) \\ & + a_{11}x_1(s + \nu h) \tanh(x_1(s + \nu h)) + a_{12}x_1(s + \nu h) \tanh(x_2(s + \nu h)) \\ & + a_{21}x_2(s + \nu h) \tanh(x_1(s + \nu h)) + a_{22}x_2(s + \nu h) \tanh(x_2(s + \nu h)) \\ & + a_{23}x_2(s + \nu h) \tanh(x_3(s + \nu h)) + a_{31}x_3(s + \nu h) \tanh(x_1(s + \nu h)) \\ & + a_{33}x_3(s + \nu h) \tanh(x_3(s + \nu h)) + x_1(s + \nu h)C_1(s) \\ & + x_2(s + \nu h)C_2(s) + x_3(s + \nu h)C_3(s). \end{aligned} \quad (4.21)$$

By (4.14),

$$|x_i \tanh(x_j)| \leq |x_i| |x_j|.$$

Hence the nonlinear coupling terms are bounded above by the corresponding parameter moduli. By the choice of the controllers in (4.17), these terms are dominated componentwise. Therefore,

$${}_h^C \Delta_a^\nu V(s) \leq -x_1^2(s + \nu h) - x_2^2(s + \nu h) - x_3^2(s + \nu h) < 0 \quad (4.22)$$

for every nonzero state. It follows that the zero equilibrium is asymptotically stable. $\square$

For the numerical verification of stabilization, the same parameter values as in (4.6) are used with

$$\nu = 0.8. \quad (4.23)$$

The controlled trajectories converge to the origin, and the associated phase portrait collapses to the equilibrium point.

4.4 Variable-Order Fractional Discrete Neural Network Based on the Caputo h -Difference Operator

The previous model is extended by replacing the constant order ν with an interval-dependent order ν_k . Let the time axis be partitioned into m intervals of equal length l , and define[25]

$$s_{kl} = kl, \quad k = 0, 1, \dots, m-1.$$

On each interval, the order is taken to be constant and equal to ν_k , with

$$0 < \nu_k \leq 1, \quad k = 0, 1, \dots, m-1. \quad (4.24)$$

The variable-order fractional discrete neural network is

$$\begin{cases} {}_h^C \Delta_{s_{kl}}^{\nu_k} x_1(s) = -x_1(s + \nu_k h) + a_{11} \tanh(x_1(s + \nu_k h)) + a_{12} \tanh(x_2(s + \nu_k h)), \\ {}_h^C \Delta_{s_{kl}}^{\nu_k} x_2(s) = -x_2(s + \nu_k h) + a_{21} \tanh(x_1(s + \nu_k h)) + a_{22} \tanh(x_2(s + \nu_k h)) \\ \quad + a_{23} \tanh(x_3(s + \nu_k h)), \\ {}_h^C \Delta_{s_{kl}}^{\nu_k} x_3(s) = -x_3(s + \nu_k h) + a_{31} \tanh(x_1(s + \nu_k h)) + a_{33} \tanh(x_3(s + \nu_k h)), \end{cases} \quad (4.25)$$

for

$$s \in (h\mathbb{N})_{s_{kl} + (1-\nu_k)h}. \quad (4.26)$$

The parameter values remain

$$a_{11} = 2, \quad a_{12} = -1.2, \quad a_{21} = 2, \quad a_{22} = 1.71, \quad a_{23} = 1.15, \quad a_{31} = -4.75, \quad a_{33} = 1.1. \quad (4.27)$$

The initial condition used in the numerical simulation is

$$x_1(0) = 0.02, \quad x_2(0) = -0.5, \quad x_3(0) = 0.8, \quad (4.28)$$

and the interval length is

$$l = 800. \quad (4.29)$$

A piecewise-constant order law may be written as

$$\nu(s) = \nu_k, \quad s \in [s_{kl}, s_{(k+1)l}), \quad (4.30)$$

where typical values reported in the numerical study are

$$\nu = 0.9, \quad \nu = 0.8, \quad \nu = 0.7, \quad \nu = 0.5. \quad (4.31)$$

Applying the h -fractional sum operator on each interval to (4.25) yields the corresponding numerical formula, which is the variable-order analogue of the constant-order Volterra-type representation. The memory kernel therefore changes from one interval to another.

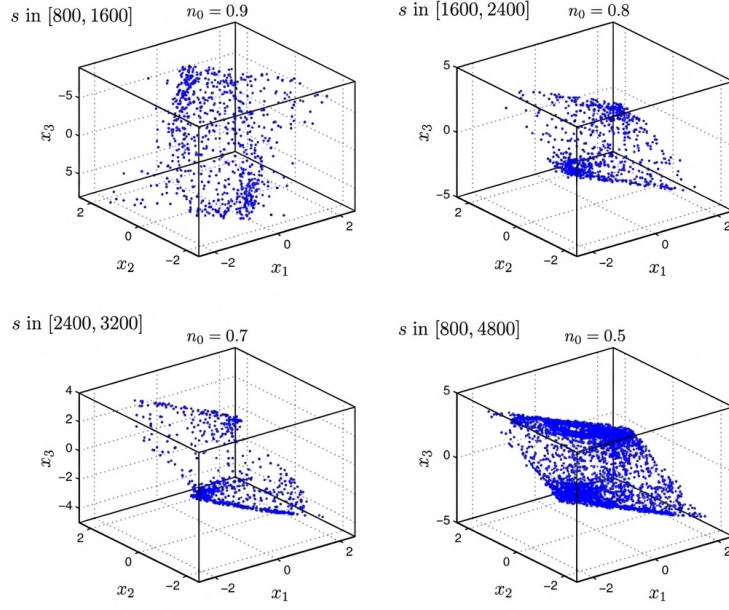


Figure 4.6: Chaotic Attractors of the Variable-Order Network

The numerical simulations show that different choices of the order generate different chaotic attractors. In particular, the system remains chaotic for

$$\nu = 0.5 \quad (4.32)$$

on the interval

$$[800, 4800]. \quad (4.33)$$

Thus, the variable-order network preserves a rich dynamical structure under order variation.

4.5 Conclusion

A three-dimensional fractional discrete neural network based on the Caputo h -difference operator has been studied. The model exhibits chaotic behavior for the parameter matrix (4.6), with chaotic regimes detected for sufficiently large fractional order, in particular for $\nu > 0.7$.

A stabilization law based on the sign function has been constructed. With the Lyapunov function (4.19), the controlled system satisfies

$${}^C_h \Delta_a^\nu V(s) \leq -\|x(s + \nu h)\|^2,$$

which implies asymptotic stability of the origin.

A variable-order extension has also been considered. The model (4.25) shows that non-trivial and chaotic dynamics persist under interval-dependent order variation. Therefore, variable-order fractional discrete neural networks defined through the Caputo h -difference operator constitute a mathematically relevant class of memory-dependent discrete dynamical systems.

General Conclusion

This dissertation has presented a comprehensive study of discrete fractional-order dynamical systems, starting from the foundational concepts of discrete fractional calculus and progressing toward advanced topics in stability theory, chaos analysis, and neural network modeling.

The results highlight that fractional-order systems exhibit richer and more complex dynamics compared to classical integer-order systems, mainly due to the presence of memory effects. These effects significantly influence stability properties and system evolution over time.

The study of stability revealed that classical notions must be extended in the fractional context, with Mittag–Leffler stability playing a central role in describing asymptotic behavior of solutions in discrete fractional systems.

Furthermore, chaotic behavior in fractional discrete systems was shown to depend strongly on system parameters and fractional order, with tools such as Lyapunov exponents and fractal dimensions providing a rigorous framework for analysis.

Finally, fractional-order discrete neural networks were shown to be effective models for nonlinear dynamical behavior, combining the adaptability of neural networks with the memory properties of fractional calculus. The presented example on Ulam–Hyers stability illustrates the theoretical results in a concrete setting.

Overall, discrete fractional-order systems constitute a powerful framework for modeling complex dynamical phenomena, with promising applications in science and engineering.

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