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Numerical Investigation of Iterative Method for Fractional Differential Equations

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ



الاهداء

" بسم الله خالقي و ميسر أموري و عصمة أمري، لك الحمد و الامتنان "

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List of Abbreviations

DEs	Differential Equations
FDEs	Fractional Differential Equations
Eq/Eqs	Equation/ Equations.
ABC	Atangana–Baleanu fractional derivative in Caputo sense
RKHS	Reproducing Kernel Hilbert Space
LRKHS	Laplace Reproducing Kernel Hilbert Space
IVP	Initial Value Problem
BVP	Boundary Value Problem
IC's	Initial Conditions
BC's	Boundary Conditions

List of Symbols

$\mathbb{F}$	Set of Real or Complex Numbers
$\mathbb{R}$	Set of Real Numbers
$\mathbb{C}$	Set of Complex Numbers
Γ	Gamma Function
$E_{(\alpha,\beta)}$	Mittag–Leffler Function
$\mathcal{L}$	Laplace Transform
$\mathcal{L}^{-1}$	Inverse Laplace Transform

Abstract

This thesis presents a hybrid numerical framework that combines the Laplace transform with the Reproducing Kernel Hilbert Space (RKHS) method within the Atangana–Baleanu fractional setting to solve nonlinear fractional differential systems with memory effects. The proposed approach reformulates the governing equations into an equivalent integral representation, enabling the construction of approximate solutions in the form of uniformly convergent series. This ensures both accuracy and numerical stability, while the Laplace transform facilitates the treatment of Atangana–Baleanu fractional derivatives and the RKHS structure provides a rigorous functional-analytic basis for the approximation process. A convergence analysis is established to validate the method, and several numerical experiments demonstrate its effectiveness in capturing the dynamical behavior of fractional systems and the impact of the fractional order. Overall, the method offers a robust and flexible tool that can be extended to more complex and higher-dimensional fractional models.

Keywords: Fractional Differential Equations; Laplace Transform; Reproducing Kernel Hilbert Space (RKHS); Atangana–Baleanu Fractional Operator.

Résumé

Cette thèse propose un cadre numérique hybride combinant la transformée de Laplace et la méthode des espaces de Hilbert à noyau reproduisant (RKHS) dans le cadre des dérivées fractionnaires d'Atangana–Baleanu, afin de résoudre des systèmes non linéaires d'équations différentielles fractionnaires présentant des effets de mémoire. L'approche proposée reformule les équations gouvernantes sous une représentation intégrale équivalente, ce qui permet de construire des solutions approchées sous forme de séries uniformément convergentes. Cette formulation garantit à la fois une grande précision et une bonne stabilité numérique. La transformée de Laplace facilite le traitement des dérivées fractionnaires d'Atangana–Baleanu, tandis que la structure RKHS fournit un cadre rigoureux d'analyse fonctionnelle pour le processus d'approximation. Une étude de convergence est établie afin de valider la méthode, et plusieurs expériences numériques mettent en évidence son efficacité pour décrire le comportement dynamique des systèmes fractionnaires ainsi que l'influence de l'ordre fractionnaire. Dans l'ensemble, la méthode constitue un outil robuste et flexible pouvant être étendu à des modèles fractionnaires plus complexes et de dimension supérieure.

Mots-clés : Équations différentielles fractionnaires ; Transformée de Laplace ; Espace de Hilbert à noyau reproduisant (RKHS) ; Opérateur fractionnaire d'Atangana–Baleanu.

ملخص

تتناول هذه الرسالة تطوير إطار عددي هجين يجمع بين تحويل لابلاس وطريقة فضاءات هيلبرت ذات النواة المولدة (RKHS) ضمن سياق مشتقة أتانجانا-باليانو الكسرية، وذلك بهدف حل الأنظمة التفاضلية الكسرية غير الخطية التي تتميز بوجود تأثيرات الذاكرة. تعتمد الطريقة المقترحة على إعادة صياغة المعادلات المدروسة في صورة تكاملية مكافئة، مما يسمح ببناء حلول تقريبية على شكل متسلسلات متقاربة بانتظام، وهو ما يضمن الدقة والاستقرار العددي في آنٍ واحد. كما يسهم تحويل لابلاس في تسهيل التعامل مع المشتقات الكسرية من نوع أتانجانا-باليانو، بينما يوفر إطار RKHS أساساً تحليلياً صارماً لعملية التقريب. وقد تم إثبات تقارب الطريقة للتحقق من صحتها الرياضية، إضافةً إلى إجراء عدة تجارب عددية أظهرت كفاءة المنهج المقترح في وصف السلوك الديناميكي للأنظمة الكسرية ودراسة تأثير الرتبة الكسرية على الحلول. و بصورة عامة، تمثل هذه الطريقة أداة عددية فعالة ومرنة يمكن توسيعها مستقبلاً لمعالجة نماذج كسرية أكثر تعقيداً وذات أبعاد أعلى.

الكلمات المفتاحية: المعادلات التفاضلية الكسرية، تحويل لابلاس، فضاءات هيلبرت ذات النواة المولدة (RKHS)، مؤثر أتانجانا-باليانو الكسري.

Introduction

Fractional differential equations (FDEs) have emerged as an important extension of classical differential equations, where derivatives of non-integer order are employed. This generalization provides a powerful and flexible framework for modeling complex phenomena characterized by memory and nonlocal effects. As a result, FDEs have found numerous applications in various fields, including physics, engineering, biology, and finance.

In contrast to classical models, fractional derivatives incorporate both memory and hereditary properties, allowing a system to depend not only on its current state but also on its entire history. This leads to a more accurate and realistic description of many physical processes. This distinctive feature has motivated extensive research in the development and application of fractional calculus. Among the various definitions of fractional derivatives, the Atangana–Baleanu derivative has attracted significant attention due to its nonlocal and nonsingular kernel, which provides an improved representation of memory effects.

Despite these advantages, most fractional differential equations, particularly nonlinear ones, do not admit exact analytical solutions because of the nonlocal nature of fractional operators. Consequently, a wide range of analytical and numerical methods has been developed to obtain approximate solutions, including Laplace transform techniques, decomposition methods, iterative procedures, and spectral approaches.

In this context, the reproducing kernel Hilbert space (RKHS) method has proven to be an effective and powerful tool for solving differential equations, including fractional

models. The RKHS framework is based on constructing a Hilbert space equipped with a reproducing kernel, which enables functions to be represented in terms of inner products. A key advantage of this approach is its ability to transform complex problems into simpler functional forms, leading to accurate and efficient approximations. Moreover, it avoids discretization and significantly reduces computational effort. Owing to these properties, the RKHS method has been successfully applied to a wide range of problems in mathematical analysis and numerical computation, particularly in the study of fractional differential equations.

Motivated by these advantages, this thesis develops a Laplace–RKHS method for solving systems of fractional differential equations involving the Atangana–Baleanu derivative. The proposed approach is based on constructing an orthonormal system derived from reproducing kernel functions, allowing the solution to be expressed as a convergent series in a suitable Hilbert space.

Within this framework, both the analytical formulation and the approximate solutions are obtained in the form of uniformly convergent series, ensuring high accuracy and computational efficiency. Furthermore, a convergence analysis is carried out to validate the reliability of the method. Several numerical examples are also presented to illustrate its effectiveness and applicability.

The remainder of this thesis is organized as follows:

- **Chapter 1:** introduces the fundamental concepts of functional analysis and fractional calculus, which are essential for the development of the subsequent chapters.
- **Chapter 2:** presents the theoretical foundations of reproducing kernel Hilbert spaces (RKHS), including the main definitions, properties, and essential tools of

the method.

- **Chapter 3:** is devoted to the application of the Laplace–RKHS method to non-linear fractional differential equations involving the Atangana–Baleanu derivative. Numerical examples are provided to illustrate the effectiveness of the proposed approach.
- **Chapter 4:** concludes the thesis by summarizing the main results and outlining possible directions for future research.

CHAPTER 1

Preliminaries.

This chapter provides a systematic presentation of the fundamental definitions and central results of Functional Analysis, together with a brief account of key properties of Fractional Calculus.

1.1 Functional Analysis Framework

Definition 1.1.1. *Let X be a vector space over the field $\mathbb{F}$ (real or complex). A norm on X is a mapping*

$$\|\cdot\| : X \rightarrow [0, \infty)$$

satisfying the following properties for all $x, y \in X$ and $\lambda \in \mathbb{F}$:

(i) $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda| \|x\|$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

*The pair $(X, \|\cdot\|)$ is called a **normed space**.*

Definition 1.1.2. *Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{F}$, and let $(x_n)_{n \in \mathbb{N}} \subset X$ be a sequence. The sequence (x_n) is said to be a **Cauchy sequence** if,*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \|x_m - x_n\| < \varepsilon, \quad \forall m, n \geq N.$$

Equivalently,

$$\lim_{m, n \rightarrow \infty} \|x_m - x_n\| = 0..$$

*The space X is said to be **complete** if every Cauchy sequence $(x_n) \subset X$ converges in X ,*

$\exists x \in X$ such that

$$\lim_{n \rightarrow \infty} \|x_n - x\| = 0.$$

In this case, the normed space $(X, \|\cdot\|)$ is called a **Banach space**, and the convergence $x_n \rightarrow x$ is understood in the norm (or strong) sense.

Definition 1.1.3. Let X be a vector space over the field $\mathbb{F}$. An **inner product** on X is a mapping $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{F}$ satisfying the following properties for all $x, y, z \in X$ and $\alpha \in \mathbb{F}$:

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$. (**Additivity**)
2. $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$. (**Homogeneity**)
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$. (**Conjugate symmetry**)
4. $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0 \iff x = 0$. (**Positive-definiteness**)

The pair $(X, \langle \cdot, \cdot \rangle)$ is called an **inner product space**.

An inner product on X induces a norm on X defined by $|x| = \sqrt{\langle x, x \rangle}$.

An inner product space that is complete with respect to the norm induced by its inner product is called a **Hilbert space**.

Theorem 1.1.4 (Cauchy–Schwarz Inequality). Let H be an inner product space.

Then, for all $x, y \in H$,

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Equality holds if and only if x and y are linearly dependent.

Definition 1.1.5. A **Hilbert space** is an inner product space H over $\mathbb{F}$ that is complete with respect to the norm induced by its inner product, that is,

$$|x| = \sqrt{\langle x, x \rangle}$$

and for every Cauchy sequence $(x_n) \subset H$, there exists an element $x \in H$ such that

$$\lim_{n \rightarrow \infty} |x_n - x| = 0.$$

Definition 1.1.6. Let $\Omega \subset \mathbb{R}$ be an interval and let $m \in \mathbb{N}_0$. The Sobolev space $W_2^m(\Omega)$ is defined by

$$W_2^m(\Omega) = \{u \in L^2(\Omega) : D^k u \in L^2(\Omega), \quad k = 0, 1, \dots, m\},$$

where $D^k u$ denotes the weak derivative of order k .

The inner product on $W_2^m(\Omega)$ is given by

$$\langle u, v \rangle_{W_2^m} = \sum_{k=0}^m \int_{\Omega} D^k u(x) D^k v(x) dx,$$

and the associated norm is

$$\|u\|_{W_2^m} = \left(\sum_{k=0}^m \|D^k u\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Equipped with this inner product, $W_2^m(\Omega)$ is a Hilbert space.

Definition 1.1.7. Let X and Y be vector spaces over the same field $\mathbb{F}$. A mapping $T : X \rightarrow Y$ is called a **linear operator** if, for all $x, y \in X$ and $\alpha \in \mathbb{F}$, the following properties hold:

(i) $T(x + y) = T(x) + T(y),$

(ii) $T(\alpha x) = \alpha T(x).$

Definition 1.1.8. Let X and Y be normed spaces and let $T : X \rightarrow Y$ be a linear operator, The operator T is said to be **bounded** if there exists a constant $c > 0$ such that

$$\|Tx\| \leq c\|x\|, \quad \forall x \in X.$$

Definition 1.1.9. Let X and Y be normed spaces, and let $T : X \rightarrow Y$ be a linear operator. The operator T is said to be **continuous** at a point $x_0 \in X$ if for every $\varepsilon > 0$,

there exists $\delta > 0$ such that

$$\|Tx - Tx_0\| < \varepsilon \quad \text{whenever} \quad \|x - x_0\| < \delta, \quad \text{for all } x \in X.$$

The operator T is said to be **continuous** if it is continuous at every point $x \in X$.

Theorem 1.1.10. *Let X and Y be normed spaces, and let $T : X \rightarrow Y$ be a linear operator. Then the operator T is bounded if and only if it is continuous on X .*

Definition 1.1.11. *Let Y be a vector space over the field $\mathbb{F}$. A mapping $f : Y \rightarrow \mathbb{F}$ is called a **linear functional** if it is linear. If, in addition, Y is a normed space, then f is said to be **bounded** if there exists a constant $c > 0$ such that*

$$|f(y)| \leq c\|y\|, \quad \forall y \in Y.$$

Theorem 1.1.12 (Riesz Representation Theorem). *Let H be a Hilbert space over $\mathbb{F}$. For every bounded linear functional f on H , there exists a unique element $y \in H$ such that*

$$f(x) = \langle x, y \rangle, \quad \forall x \in H.$$

Moreover, we have $\|f\| = \|y\|$.

Definition 1.1.13. *Let H_1 and H_2 be Hilbert spaces, and let $T : H_1 \rightarrow H_2$ be a bounded linear operator. The **adjoint operator** of T is the operator*

$$T^* : H_2 \rightarrow H_1$$

defined by the relation

$$\langle Tx, y \rangle = \langle x, T^*y \rangle, \quad \forall x \in H_1, \quad \forall y \in H_2.$$

Definition 1.1.14. *Let H be a Hilbert space and let $T : H \rightarrow H$ be a bounded linear operator. The operator T is said to be **self-adjoint** if $T = T^*$.*

1.2 Introduction of Fractional Calculus

Fractional calculus extends the classical concepts of differentiation and integration to non-integer orders. It provides a powerful mathematical framework for modeling memory and hereditary effects in many physical, biological, and engineering systems. This section introduces the fundamental concepts and operators of fractional calculus used throughout this thesis.

1.2.1 Special Functions in Fractional Calculus

1.2.1.1 Gamma Function

The Gamma function, introduced by Euler, is a fundamental special function widely used in fractional-order differential equations. It is defined as follows:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x \in \mathbb{R}^+.$$

The Gamma function satisfies the following properties:

- $\Gamma(x+1) = x\Gamma(x), \quad x \in \mathbb{R}^+.$
- $\Gamma(1) = 1.$
- $\Gamma(n+1) = n!, \quad n \in \mathbb{N}.$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$
- $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$

The Gamma function can also be represented in the following limit form:

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n! n^x}{x(x+1)(x+2) \cdots (x+n)}.$$

1.2.1.2 Mittag-Leffler Function

The Mittag-Leffler function is a fundamental special function in fractional calculus that generalizes the classical exponential function and plays an important role in fractional differential equations. It is defined as follows:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, z \in \mathbb{C}.$$

A more general form is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, z \in \mathbb{C}.$$

The Mittag-Leffler function satisfies the following properties:

- $E_1(z) = e^z$.
- $E_\alpha(0) = 1$.
- $E_{\alpha,\beta}(0) = \frac{1}{\Gamma(\beta)}$.
- The series defining $E_\alpha(z)$ converges for all $z \in \mathbb{C}$.
- The function $E_{\alpha,\beta}(z)$ generalizes $E_\alpha(z)$.

1.2.2 Fractional Calculus Operators

Fractional calculus has its origins in the 17th century, when the concept of derivatives of non-integer order was first introduced in correspondence between mathematicians such as Leibniz and L'Hôpital. Over time, this idea was further developed by several researchers, including Euler, Riemann, and Liouville, leading to the formulation of various definitions of fractional integrals and derivatives.

In the 20th century, new definitions were introduced to better model physical phenomena, among which the Caputo fractional derivative (1967) and the more recent Atangana–Baleanu derivative stand out due to their applicability in real-world problems involving memory and hereditary properties.

1.2.2.1 Fractional Integral

The fractional integral is a generalization of the classical integral to non-integer orders and plays a fundamental role in fractional calculus. It is defined as follows:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0.$$

The fractional integral satisfies the following properties:

- $I_t^0 f(t) = f(t)$.
- $I_t^\alpha I_t^\beta f(t) = I_t^{\alpha+\beta} f(t), \quad \alpha, \beta > 0.$
- The operator I_t^α is linear.

- For a power function, we have

$$I_t^\alpha t^m = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} t^{m+\alpha}, \quad m > -1.$$

1.2.2.2 Caputo Fractional Derivative

In 1967, M. Caputo introduced a new definition of the fractional derivative, where he showed its relation to the Riemann–Liouville fractional derivative. This definition is widely used in fractional calculus due to its suitability for initial value problems. It is defined as follows:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad n-1 < \alpha < n, \quad n \in \mathbb{N}.$$

The Caputo fractional derivative satisfies the following properties:

- $D_t^\alpha c = 0$, for any constant c .
- $\lim_{\alpha \rightarrow n} D_t^\alpha f(t) = f^{(n)}(t)$.
- It allows the use of classical initial conditions:

$$f(0), f'(0), \dots, f^{(n-1)}(0);$$

- For a power function, we have

$$D_t^\alpha t^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} t^{m-\alpha}, \quad m > \alpha - 1.$$

Despite its wide applicability, the Caputo fractional derivative involves a singular

kernel, which may limit its effectiveness in modeling certain physical phenomena. To overcome this limitation, more recent definitions have been introduced. In particular, the Atangana–Baleanu fractional derivative can be viewed as a modern extension of the Caputo operator, characterized by a non-singular and non-local kernel.

1.2.2.3 Atangana-Baleanu Fractional Derivative

In 2016, Atangana and Baleanu introduced a new class of fractional differential operators based on the generalized Mittag–Leffler function, characterized by a non-singular and non-local kernel. Among the different formulations of this operator, we consider the Atangana–Baleanu fractional derivative in Caputo sense, which is defined as follows:

$${}^{ABC}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t f'(s) E_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds, \quad 0 < \alpha < 1,$$

where $B(\alpha)$ is a normalization function such that $B(0) = B(1) = 1$.

The Atangana–Baleanu fractional derivative satisfies the following properties:

- ${}^{ABC}D_t^\alpha c = 0$, for any constant c ;
- The operator is linear;
- The kernel is non-singular and non-local;
- It involves the Mittag–Leffler function.

Example 1.2.1. *Let $f(t) = t^m$, where $m > 0$. We compute the Atangana–Baleanu fractional derivative in the Caputo sense. Starting from the definition:*

$${}^{ABC}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t f'(s) E_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds.$$

Since $f(t) = t^m$, we have

$$f'(s) = ms^{m-1}$$

. Substituting into the definition gives

$${}^{ABC}D_t^\alpha t^m = \frac{B(\alpha)}{1-\alpha} \int_0^t ms^{m-1} E_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds.$$

Factoring out the constant m , we obtain

$${}^{ABC}D_t^\alpha t^m = \frac{mB(\alpha)}{1-\alpha} \int_0^t s^{m-1} E_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds.$$

Therefore,

$${}^{ABC}D_t^\alpha t^m = \frac{mB(\alpha)}{1-\alpha} \int_0^t s^{m-1} E_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds$$

1.2.3 Laplace Transform in Fractional Calculus

The Laplace transform is an essential tool in fractional calculus, as it transforms differential equations into algebraic equations in the frequency domain, thereby simplifying the solution process, especially for problems involving fractional operators.

Definition 1.2.2. *The Laplace transform of a function $f(t)$ is defined by*

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) dt, \quad s > 0.$$

The Laplace transform satisfies the following properties:

- $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$ (Linearity);
- $\mathcal{L}\{1\} = \frac{1}{s}$;

- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n \in \mathbb{N};$
- $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$

The inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Another important concept is the convolution of two functions f and g , defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s) ds.$$

Its Laplace transform satisfies

$$\mathcal{L}\{(f * g)(t)\}(s) = \mathcal{L}\{f(t)\}(s) \cdot \mathcal{L}\{g(t)\}(s).$$

Moreover, the Laplace transform of an integer-order derivative is given by

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \mathcal{L}\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0).$$

1.2.3.1 Application to Fractional Derivatives

The Laplace transforms of fractional derivatives are given as follows:

- **Caputo fractional derivative:**

$$\mathcal{L}\{{}^C D_t^\alpha f(t)\}(s) = s^\alpha \mathcal{L}\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0), \quad n-1 < \alpha < n.$$

- **Atangana–Baleanu fractional derivative (in Caputo sense):**

$$\mathcal{L}\{^{ABC}D_t^\alpha f(t)\}(s) = \frac{B(\alpha)}{1-\alpha} \frac{s^\alpha \mathcal{L}\{f(t)\}(s) - s^{\alpha-1} f(0)}{s^\alpha + \frac{\alpha}{1-\alpha}}.$$

The Laplace transform of the Atangana–Baleanu fractional derivative satisfies the following properties:

- **Linearity:**

$$\mathcal{L}\{^{ABC}D_t^\alpha (au + bv)\} = a \mathcal{L}\{^{ABC}D_t^\alpha u\} + b \mathcal{L}\{^{ABC}D_t^\alpha v\}.$$

- **Zero initial condition:** If $u(0) = 0$, then

$$\mathcal{L}\{^{ABC}D_t^\alpha u(t)\} = \frac{B(\alpha)}{1-\alpha} \frac{s^\alpha}{s^\alpha + \frac{\alpha}{1-\alpha}} U(s).$$

- **Classical limit:** As $\alpha \rightarrow 1$, we recover

$$\mathcal{L}\{u'(t)\} = sU(s) - u(0).$$

CHAPTER 2

Reproducing Kernel Hilbert Space

This chapter presents a systematic exposition of the fundamental concepts and main results of Reproducing Kernel Hilbert Spaces (RKHS), together with the essential properties and theoretical tools required for the development of the proposed method.

2.1 Fundamentals of Reproducing Kernel Hilbert Spaces

Definition 2.1.1. *Let E be a nonempty set. A function $K : E \times E \rightarrow \mathbb{C}$ is called a **kernel** on E . The kernel K is said to be **Hermitian** if*

$$K(x, y) = \overline{K(y, x)}, \quad \forall x, y \in E.$$

*Moreover, K is called **positive definite** if for any finite set of points $\{x_1, x_2, \dots, x_n\} \subseteq E$ and any complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$, one has*

$$\sum_{i,j=1}^n \overline{\alpha_i} \alpha_j K(x_i, x_j) \geq 0.$$

Definition 2.1.2 (Aronszajn, 1950). *Let H be a Hilbert space of functions $f : E \rightarrow \mathbb{F}$ defined on a nonempty set E . A function $K : E \times E \rightarrow \mathbb{C}$ is called a **reproducing kernel** of H if the following conditions are satisfied:*

1. $K(\cdot, x) \in H$ for all $x \in E$;
2. $\langle f, K(\cdot, x) \rangle = f(x)$ for all $f \in H$ and all $x \in E$.

*The second condition is called the **reproducing property**, which means that the value of the function f at the point x can be obtained through the inner product of f with $K(\cdot, x)$.*

For each $x \in E$, define $K_x(y) = K(y, x)$. Then $K_x \in H$, and by the reproducing property we have

$$K(x, y) = \langle K(\cdot, y), K(\cdot, x) \rangle, \quad \forall x, y \in E.$$

In particular,

$$\|K(\cdot, x)\|^2 = \langle K(\cdot, x), K(\cdot, x) \rangle = K(x, x).$$

Definition 2.1.3. Let H be a Hilbert space of functions defined on a set E . Then H is called a **Reproducing Kernel Hilbert Space (RKHS)** if there exists a reproducing kernel k associated with it.

The space is commonly denoted by H_k , and its inner product and norm are written as $\langle \cdot, \cdot \rangle_{H_k}$ and $\|\cdot\|_{H_k}$, respectively.

Theorem 2.1.4 (Aronszajn, 1950). Let H be a Hilbert space of functions defined on a set E . If H admits a reproducing kernel, then this reproducing kernel is unique.

Proof : Let K and R be two reproducing kernels of H . Then for all $f \in H$ and all $x \in E$, we have

$$f(x) = \langle f, K(\cdot, x) \rangle = \langle f, R(\cdot, x) \rangle.$$

Taking $f = K(\cdot, y)$, we obtain

$$K(y, x) = \langle K(\cdot, y), R(\cdot, x) \rangle.$$

Similarly, taking $f = R(\cdot, y)$, we get

$$R(y, x) = \langle R(\cdot, y), K(\cdot, x) \rangle.$$

Hence, $K(x, y) = R(x, y)$ for all $x, y \in E$, and therefore $K = R$. □

Theorem 2.1.5. Let H be a Hilbert space of functions defined on a set E . Then H admits

a reproducing kernel if and only if, for every $x \in E$, the evaluation (Dirac) functional

$$\delta_x : H \rightarrow \mathbb{F}, \quad \delta_x(f) = f(x),$$

is a bounded linear functional.

Theorem 2.1.6. *Let H be a reproducing kernel Hilbert space with reproducing kernel k .*

Then k is a positive definite kernel.

Proof : Let $\{x_1, \dots, x_n\} \subset E$ and let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Consider the element

$$f = \sum_{i=1}^n \alpha_i k_{x_i} \in H.$$

Then, using the properties of the inner product, we obtain

$$\|f\|^2 = \left\langle \sum_{i=1}^n \alpha_i k_{x_i}, \sum_{j=1}^n \alpha_j k_{x_j} \right\rangle = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} \langle k_{x_i}, k_{x_j} \rangle.$$

By the reproducing property, we have

$$\langle k_{x_i}, k_{x_j} \rangle = k(x_i, x_j).$$

Hence,

$$\|f\|^2 = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} k(x_i, x_j) \geq 0.$$

Therefore, k is a positive definite kernel. □

Remark 2.1.7. *Let H be a reproducing kernel Hilbert space with kernel k on E . Then for all $x, y \in E$, the following properties hold:*

(i) $|k(x, y)|^2 \leq k(x, x) k(y, y);$

(ii) *For a fixed point $x_0 \in E$, the following statements are equivalent:*

- $k(x_0, x_0) = 0;$
- $k(x_0, y) = 0$ for all $y \in E;$
- $f(x_0) = 0$ for all $f \in H.$

Proof : Let H be a RKHS with reproducing kernel K .

(i) For all $x, y \in E$, we prove that

$$|K(x, y)|^2 \leq K(x, x) K(y, y).$$

Using the reproducing property, we have

$$K(x, y) = \langle K(\cdot, y), K(\cdot, x) \rangle.$$

Applying the Cauchy–Schwarz inequality in H , we obtain

$$|K(x, y)|^2 = |\langle K(\cdot, y), K(\cdot, x) \rangle|^2 \leq \|K(\cdot, y)\|^2 \|K(\cdot, x)\|^2.$$

Using again the reproducing property,

$$\|K(\cdot, x)\|^2 = K(x, x), \quad \|K(\cdot, y)\|^2 = K(y, y).$$

Thus,

$$|K(x, y)|^2 \leq K(x, x) K(y, y).$$

(ii) Let $x_0 \in E$. We prove the equivalence.

(a) $\Rightarrow$ (b): Suppose that $K(x_0, x_0) = 0$. Then, using part (i),

$$|K(x_0, y)|^2 \leq K(x_0, x_0) K(y, y) = 0,$$

which implies

$$K(x_0, y) = 0, \quad \forall y \in E.$$

(b) $\Rightarrow$ (c): Assume that, $K(x_0, y) = 0$ for all $y \in E$. Then for any $f \in H$,

$$f(x_0) = \langle f, K(\cdot, x_0) \rangle = 0,$$

since $K(\cdot, x_0) = 0$.

(c) $\Rightarrow$ (a): Suppose that $f(x_0) = 0$ for all $f \in H$. In particular, taking $f = K(\cdot, x_0)$,

we obtain

$$K(x_0, x_0) = \langle K(\cdot, x_0), K(\cdot, x_0) \rangle = 0.$$

Hence, (a), (b) and (c) are equivalent. $\square$

Theorem 2.1.8. *Let X be a nonempty set and let $K : X \times X \rightarrow \mathbb{C}$ be a positive definite kernel. Then there exists a unique Hilbert space H_k of functions on X such that:*

1. $k(\cdot, x) \in H_k$ for all $x \in X$;
2. k is the reproducing kernel of H_k , i.e.,

$$f(x) = \langle f, k(\cdot, x) \rangle_{H_k}, \quad \forall f \in H_k, \forall x \in X.$$

Proof : The proof can be found in Aronszajn (1950). □

Theorem 2.1.9. *Let $H_k(X)$ be a reproducing kernel Hilbert space with kernel k , and let (f_n) be a sequence in $H_k(X)$ such that*

$$\|f_n - f\|_{H_k} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then (f_n) converges pointwise to f , that is,

$$f_n(x) \rightarrow f(x), \quad \forall x \in E.$$

Moreover, the convergence is uniform on every subset $A \subset E$ for which there exists a constant $M > 0$ such that

$$k(x, x) \leq M, \quad \forall x \in A.$$

Proof : Let (f_n) be a sequence in $H_k(X)$ such that

$$\|f_n - f\|_{H_k} \rightarrow 0.$$

For any $x \in E$, using the reproducing property, we have

$$f_n(x) - f(x) = \langle f_n - f, k(\cdot, x) \rangle_{H_k}.$$

By the Cauchy–Schwarz inequality,

$$|f_n(x) - f(x)| \leq \|f_n - f\|_{H_k} \|k(\cdot, x)\|_{H_k}.$$

Since $\|k(\cdot, x)\|_{H_k}^2 = k(x, x)$, it follows that

$$|f_n(x) - f(x)| \leq \|f_n - f\|_{H_k} \sqrt{k(x, x)}.$$

Thus, for each $x \in E$, we have $f_n(x) \rightarrow f(x)$.

Now, let $A \subset E$ be such that there exists $M > 0$ with $k(x, x) \leq M$ for all $x \in A$.

Then

$$|f_n(x) - f(x)| \leq \sqrt{M} \|f_n - f\|_{H_k}, \quad \forall x \in A.$$

Taking the supremum over A , we obtain

$$\sup_{x \in A} |f_n(x) - f(x)| \leq \sqrt{M} \|f_n - f\|_{H_k}.$$

Since $\|f_n - f\|_{H_k} \rightarrow 0$, it follows that $f_n \rightarrow f$ uniformly on A . □

Definition 2.1.10. Let $[a, b] \subset \mathbb{R}$ and $m \in \mathbb{N}$. The Sobolev space $W_2^m[a, b]$ is defined by:

$$W_2^m[a, b] = \{u : u^{(i)} \in AC[a, b], i = 0, 1, \dots, m-1, \text{ and } u^{(m)} \in L^2[a, b]\}. \quad (2.1.1)$$

The inner product in $W_2^m[a, b]$ is defined, for all $u, v \in W_2^m[a, b]$, by

$$\langle u, v \rangle_{W_2^m[a, b]} = \sum_{i=0}^{m-1} u^{(i)}(a) v^{(i)}(a) + \int_a^b u^{(m)}(x) v^{(m)}(x) dx. \quad (2.1.2)$$

The associated norm is given by

$$\|u\|_{W_2^m[a, b]} = \sqrt{\langle u, u \rangle_{W_2^m[a, b]}}. \quad (2.1.3)$$

Theorem 2.1.11. *The space $W_2^m[a, b]$ is a Hilbert space.*

Theorem 2.1.12. *The Hilbert space $W_2^m[a, b]$ is a reproducing kernel Hilbert space. That is, for every fixed $x \in [a, b]$, there exists a function $k(\cdot, x) \in W_2^m[a, b]$ such that*

$$u(x) = \langle u(y), k(y, x) \rangle_{W_2^m[a, b]}, \quad \forall u \in W_2^m[a, b].$$

The function $k : [a, b] \times [a, b] \rightarrow \mathbb{C}$ is called the reproducing kernel of $W_2^m[a, b]$.

2.2 Reproducing Kernel Function

In this section, we study the reproducing kernel function and derive its explicit algebraic form in the space $W_2^m[a, b]$. It will be shown that the reproducing kernel can be represented as a piecewise polynomial of degree $2m - 1$. Moreover, the kernel admits a unique representation. Some related properties and remarks will also be presented, followed by illustrative examples in the case $W_2^1[a, b]$.

Let $k_x(y)$ denote the reproducing kernel of the space $W_2^m[a, b]$. Then, for every fixed $x \in [a, b]$ and for any $u \in W_2^m[a, b]$, we have

$$\langle u(\cdot), k_x(\cdot) \rangle_{W_2^m[a, b]} = u(x).$$

Using the definition of the inner product, we obtain

$$\langle u(y), k_x(y) \rangle_{W_2^m[a, b]} = \sum_{i=0}^{m-1} u^{(i)}(a) k_x^{(i)}(a) + \int_a^b u^{(m)}(y) k_x^{(m)}(y) dy. \quad (2.2.1)$$

Applying integration by parts repeatedly to the integral term, we get

$$\int_a^b u^{(m)}(y) k_x^{(m)}(y) dy = \sum_{i=0}^{m-1} (-1)^i u^{(m-i-1)}(y) k_x^{(m+i)}(y) \Big|_a^b + \int_a^b (-1)^m u(y) k_x^{(2m)}(y) dy.$$

Substituting into (2.2.1), we obtain

$$\begin{aligned} \langle u(y), k_x(y) \rangle_{W_2^m[a,b]} &= \sum_{i=0}^{m-1} u^{(i)}(a) \left(k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a) \right) \\ &\quad + \sum_{i=0}^{m-1} (-1)^{m-i-1} u^{(i)}(b) k_x^{(2m-i-1)}(b) \\ &\quad + \int_a^b (-1)^m u(y) k_x^{(2m)}(y) dy. \end{aligned}$$

Since $k_x(y) \in W_2^m[a, b]$, we impose the boundary conditions

$$k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a) = 0, \quad k_x^{(2m-i-1)}(b) = 0, \quad i = 0, 1, \dots, m-1.$$

Thus, the reproducing property reduces to

$$\langle u(y), k_x(y) \rangle_{W_2^m[a,b]} = \int_a^b u(y) (-1)^m k_x^{(2m)}(y) dy.$$

To ensure that $\langle u, k_x \rangle = u(x)$ for all $u \in W_2^m[a, b]$, it is necessary that

$$(-1)^m k_x^{(2m)}(y) = \delta(x - y),$$

where $\delta(x)$ denotes the Dirac delta function. Hence, $k_x(y)$ is the reproducing kernel of $W_2^m[a, b]$.

Therefore, $k_x(y)$ satisfies the following boundary value problem:

$$\begin{cases} (-1)^m k_x^{(2m)}(y) = \delta(x - y), \\ k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a) = 0, \quad i = 0, \dots, m-1, \\ k_x^{(2m-i-1)}(b) = 0, \quad i = 0, \dots, m-1. \end{cases} \quad (2.2.2)$$

For $y \neq x$, we have

$$k_x^{(2m)}(y) = 0,$$

whose characteristic equation is $\lambda^{2m} = 0$. Hence, the general solution is a polynomial of degree $2m - 1$.

Thus, the kernel admits the piecewise representation

$$k_x(y) = \begin{cases} \sum_{i=0}^{2m-1} p_i(x) y^i, & y \leq x, \\ \sum_{i=0}^{2m-1} q_i(x) y^i, & y > x. \end{cases} \quad (2.2.3)$$

Moreover, continuity conditions at $y = x$ yield

$$k_x^{(i)}(x^-) = k_x^{(i)}(x^+), \quad i = 0, \dots, 2m-2,$$

and the jump condition

$$(-1)^m (k_x^{(2m-1)}(x^+) - k_x^{(2m-1)}(x^-)) = 1.$$

These conditions, together with the boundary conditions, form a linear system that

uniquely determines the coefficients $p_i(x)$ and $q_i(x)$.

Therefore, the reproducing kernel $k_x(y)$ is uniquely determined and can be expressed as a piecewise polynomial of degree $2m - 1$.

Corollary 2.2.1. *The reproducing kernel $k(x, y)$ of the space $W_2^m[a, b]$ is symmetric, unique, and satisfies $k(x, x) \geq 0$, $\forall x \in [a, b]$.*

Proof : Symmetry. For all $x, y \in [a, b]$, by the reproducing property we have

$$k(x, y) = \langle k(\cdot, y), k(\cdot, x) \rangle.$$

Using the conjugate symmetry of the inner product,

$$k(x, y) = \overline{\langle k(\cdot, x), k(\cdot, y) \rangle} = \overline{k(y, x)}.$$

In the real case, this reduces to $k(x, y) = k(y, x)$.

Uniqueness. Assume that k and R are two reproducing kernels of $W_2^m[a, b]$. Then, for all $x, y \in [a, b]$,

$$k(x, y) = \langle k(\cdot, y), R(\cdot, x) \rangle = \langle R(\cdot, x), k(\cdot, y) \rangle = R(x, y).$$

Hence, $k = R$, and the reproducing kernel is unique.

Positivity. For any $x \in [a, b]$, we have

$$k(x, x) = \langle k(\cdot, x), k(\cdot, x) \rangle = \|k(\cdot, x)\|^2 \geq 0.$$

Thus, $k(x, x) \geq 0$ for all $x \in [a, b]$.

□

Now, we present some explicit expressions of the reproducing kernel function in the space $W_2^1[a, b]$ under different inner products, using the approach developed in this section.

Example 2.2.2. *Consider the space*

$$W_2^1[a, b] = \{u : [a, b] \rightarrow \mathbb{R} ; u \text{ is absolutely continuous and } u' \in L^2[a, b]\}.$$

The inner product and the associated norm are defined by

$$\langle u_1, u_2 \rangle_{W_2^1} = u_1(a)u_2(a) + \int_a^b u_1'(y)u_2'(y) dy, \quad \|u\|_{W_2^1} = \sqrt{\langle u, u \rangle_{W_2^1}}.$$

To determine the reproducing kernel function $k_x(y)$, we use the reproducing property

$$\langle u(y), k_x(y) \rangle_{W_2^1} = u(x), \quad \forall u \in W_2^1[a, b].$$

Applying integration by parts, we obtain

$$\langle u(y), k_x(y) \rangle = u(a)k_x(a) + u(y)k_x'(y) \Big|_a^b - \int_a^b u(y)k_x''(y) dy.$$

Since $u(y), k_x(y) \in W_2^1[a, b]$, we impose the boundary conditions

$$k_x(a) - k_x'(a) = 0, \quad k_x'(b) = 0.$$

Thus, $k_x(y)$ satisfies the boundary value problem

$$-k_x''(y) = \delta(x - y).$$

For $y \neq x$, we have $k_x''(y) = 0$, hence

$$k_x(y) = \begin{cases} p_1(x) + p_2(x)y, & y \leq x, \\ q_1(x) + q_2(x)y, & y > x. \end{cases}$$

The coefficients are determined using the following conditions:

$$k_x(a) - k'_x(a) = 0,$$

$$k'_x(b) = 0,$$

$$k_x(x^-) = k_x(x^+),$$

$$k'_x(x^+) - k'_x(x^-) = -1.$$

We now compute the coefficients. From the first condition,

$$k_x(a) = p_1 + p_2a, \quad k'_x(a) = p_2,$$

hence

$$p_1 = p_2(1 - a).$$

From $k'_x(b) = 0$, we obtain

$$q_2 = 0.$$

Continuity at $y = x$ gives

$$q_1 = p_1 + p_2 x.$$

Finally, the jump condition yields

$$q_2 - p_2 = -1 \Rightarrow p_2 = 1.$$

Thus,

$$p_1 = 1 - a, \quad q_1 = 1 - a + x, \quad q_2 = 0.$$

Therefore, the reproducing kernel function is

$$k_x(y) = \begin{cases} 1 - a + y, & y \leq x, \\ 1 - a + x, & y > x. \end{cases}$$

Remark 2.2.3. It was shown by Yao (2008) that the space $W_2^1[a, b]$ is a complete reproducing kernel Hilbert space. Moreover, its reproducing kernel function is explicitly given by

$$k_x(y) = \begin{cases} 1 + y, & y \leq x, \\ 1 + x, & y > x. \end{cases}$$

Example 2.2.4. Consider the space

$$W_2^1[a, b] = \{u : u \text{ is absolutely continuous on } [a, b], u' \in L^2[a, b], u(a) = u(b) = 0\}.$$

The inner product is defined by $\langle u_1, u_2 \rangle_{W_2^1} = \int_a^b u_1'(y) u_2'(y) dy$.

We seek the reproducing kernel $R_x(y)$. As in the previous example, it satisfies

$$-R_x''(y) = \delta(x - y),$$

with boundary conditions

$$R_x(a) = 0, \quad R_x(b) = 0.$$

For $y \neq x$, we have

$$R_x''(y) = 0,$$

hence $R_x(y)$ is linear. Thus,

$$R_x(y) = \begin{cases} p_1(x) + p_2(x)y, & y \leq x, \\ q_1(x) + q_2(x)y, & y > x. \end{cases}$$

Step 1: Boundary conditions

From $R_x(a) = 0$ (for $y \leq x$),

$$p_1 + p_2a = 0 \quad \Rightarrow \quad p_1 = -p_2a.$$

From $R_x(b) = 0$ (for $y > x$),

$$q_1 + q_2b = 0 \quad \Rightarrow \quad q_1 = -q_2b.$$

Step 2: Continuity at $y = x$

$$p_1 + p_2x = q_1 + q_2x.$$

Substitute $p_1 = -p_2a$ and $q_1 = -q_2b$:

$$-p_2a + p_2x = -q_2b + q_2x.$$

Step 3: Jump condition

$$R'_x(x^+) - R'_x(x^-) = -1 \quad \Rightarrow \quad q_2 - p_2 = -1.$$

Step 4: Solve the system

From the continuity equation:

$$p_2(x - a) = q_2(x - b).$$

Using $q_2 = p_2 - 1$:

$$p_2(x - a) = (p_2 - 1)(x - b).$$

Expanding:

$$p_2(x - a) = p_2(x - b) - (x - b).$$

Thus,

$$p_2[(x - a) - (x - b)] = -(x - b)$$

$$p_2(b - a) = -(x - b) \quad \Rightarrow \quad p_2 = \frac{b - x}{b - a}.$$

Then,

$$q_2 = p_2 - 1 = \frac{b - x}{b - a} - 1 = \frac{a - x}{b - a}.$$

Now,

$$p_1 = -p_2a = -a \frac{b-x}{b-a}, \quad q_1 = -q_2b = -b \frac{a-x}{b-a}.$$

Final expression

$$R_x(y) = \begin{cases} \frac{(b-x)(y-a)}{b-a}, & y \leq x, \\ \frac{(a-x)(y-b)}{b-a}, & y > x. \end{cases}$$

Remark 2.2.5. *Paulsen (2009) showed that the space $W_2^1[a, b]$ in Example (2.2.4) is a complete reproducing kernel Hilbert space.*

In the special case $[a, b] = [0, 1]$, the reproducing kernel reduces to

$$R_x(y) = \begin{cases} (1-x)y, & y \leq x, \\ (1-y)x, & y > x. \end{cases}$$

Example 2.2.6. *Consider the space*

$$W_2^1[a, b] = \{u : u(x) \text{ is absolutely continuous on } [a, b], \ u'(x) \in L^2[a, b]\},$$

and define a new inner product in this space by $\langle u, v \rangle_{W_2^1} = \int_a^b (u(y)v(y) + u'(y)v'(y)) dy$,

with the corresponding norm $\|u\|_{W_2^1} = \sqrt{\langle u, u \rangle_{W_2^1}}$.

To find the reproducing kernel function $T_x(y)$, we use integration by parts:

$$\langle u, T_x \rangle_{W_2^1} = \int_a^b (u(y)T_x(y) + u'(y)T_x'(y)) dy.$$

Applying integration by parts to the second term, we get

$$\langle u, T_x \rangle_{W_2^1} = u(y)T'_x(y) \Big|_a^b + \int_a^b u(y) (T_x(y) - T''_x(y)) dy.$$

Since there are no boundary conditions imposed and $u, T_x \in W_2^1[a, b]$, we take

$$T'_x(a) = 0, \quad T'_x(b) = 0.$$

Thus, the reproducing property implies

$$T_x(y) - T''_x(y) = \delta(x - y).$$

The associated homogeneous equation is

$$T''_x(y) - T_x(y) = 0,$$

whose characteristic equation is

$$\lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda = \pm 1.$$

Hence, the general solution is

$$T_x(y) = \begin{cases} p_1(x)e^{-y} + p_2(x)e^y, & y \leq x, \\ q_1(x)e^{-y} + q_2(x)e^y, & y > x. \end{cases}$$

The unknown coefficients are determined from:

$$1. \quad T'_x(a) = 0$$

$$2. \quad T'_x(b) = 0$$

$$3. \quad T_x(x^+) = T_x(x^-)$$

$$4. \quad T'_x(x^+) - T'_x(x^-) = -1$$

Solving this system, we obtain

$$T_x(y) = \begin{cases} -\frac{e^{-x-y}(e^{2b} + e^{2x})(e^{2a} + e^{2y})}{2(e^{2a} - e^{2b})}, & y \leq x, \\ -\frac{e^{-x-y}(e^{2a} + e^{2x})(e^{2b} + e^{2y})}{2(e^{2a} - e^{2b})}, & y > x. \end{cases}$$

Moreover, it can be written in a simpler symmetric form as

$$T_x(y) = \frac{1}{2 \sinh(b-a)} [\cosh(x+y-b-a) + \cosh(|x-y|-b+a)].$$

Remark 2.2.7. *Cui and Li (2003) proved that the space $W_2^1[a, b]$ in this example is a complete reproducing kernel space and its reproducing kernel function can be written as*

$$T_x(y) = \frac{1}{2 \sinh(b-a)} [\cosh(x+y-b-a) + \cosh(|x-y|-b+a)].$$

2.3 Description of Reproducing Kernel Method

In this section, we establish an iterative framework to construct and approximate the solution of a general m^{th} -order boundary value problem (BVP). The exact solution is represented in the reproducing kernel Hilbert space (RKHS), while an approximate solution is obtained by truncating the resulting series expansion.

Consider the following general m^{th} -order BVP:

$$u^{(m)}(x) + a_1(x)u^{(m-1)}(x) + \cdots + a_{m-1}(x)u'(x) = F(x, u(x)), \quad a \leq x \leq b, \quad (2.3.1)$$

subject to the boundary conditions

$$u^{(i)}(a) = c_i, \quad i = 0, 1, \dots, r-1, \quad (2.3.2)$$

$$u^{(i)}(b) = d_i, \quad i = r, r+1, \dots, m-1.$$

where $a_i(x)$, $i = 1, 2, \dots, m-1$, are continuous real-valued functions, c_i and d_i are real constants, and $F(x, u(x))$ may be linear or nonlinear.

To solve the BVP (2.3.1)–(2.3.2) using the RKHS method, we first construct the reproducing kernel space $W_2^{m+1}[a, b]$ such that every function satisfies the homogeneous boundary conditions. Then, we utilize the space $W_2^1[a, b]$.

The inner product and the norm in $W_2^{m+1}[a, b]$ are defined as in Equations (2.1.2) and (2.1.3).

Let $k_x(y)$ and $R_x(y)$ be the reproducing kernel functions of the spaces $W_2^{m+1}[a, b]$ and $W_2^1[a, b]$, respectively. Define the differential operator $L : W_2^{m+1}[a, b] \longrightarrow W_2^1[a, b]$ by

$$Lu(x) = u^{(m)}(x) + a_1(x)u^{(m-1)}(x) + \cdots + a_{m-1}(x)u'(x).$$

Note that L is a bounded linear operator, as shown in the following lemma.

Lemma 2.3.1. *Let $u \in W_2^{m+1}[a, b]$. Then there exist positive constants $M_i > 0$,*

$i = 0, 1, \dots, m$, such that $|u(x)| \leq M_0 \|u\|_{W_2^{m+1}}, \quad \forall x \in [a, b]$, and

$$|u^{(i)}(x)| \leq M_i \|u\|_{W_2^{m+1}}, \quad \forall x \in [a, b], \quad i = 1, 2, \dots, m.$$

Proof : Using the reproducing property of the kernel $k_x(y)$, we have

$$u(x) = \langle u(y), k_x(y) \rangle_{W_2^{m+1}}.$$

Applying the Cauchy–Schwarz inequality, we obtain

$$|u(x)| \leq \|u\|_{W_2^{m+1}} \|k_x(y)\|_{W_2^{m+1}}.$$

Since $\|k_x(y)\|_{W_2^{m+1}}$ is uniformly bounded with respect to x and y , there exists a constant

$M_0 > 0$ such that

$$|u(x)| \leq M_0 \|u\|_{W_2^{m+1}}.$$

Similarly, for the derivatives, we use the representation

$$u^{(i)}(x) = \langle u(y), k_x^{(i)}(y) \rangle_{W_2^{m+1}}, \quad i = 1, 2, \dots, m.$$

Again, by the Cauchy–Schwarz inequality, we have

$$|u^{(i)}(x)| \leq \|u\|_{W_2^{m+1}} \|k_x^{(i)}(y)\|_{W_2^{m+1}}.$$

Since $k_x^{(i)}(y)$ is uniformly bounded, there exist constants $M_i > 0$ such that

$$|u^{(i)}(x)| \leq M_i \|u\|_{W_2^{m+1}}, \quad i = 1, 2, \dots, m.$$

This completes the proof. □

Hence, after homogenizing the boundary conditions (2.3.2), the BVP (2.3.1)–(2.3.2) can be rewritten in the following equivalent form:

$$Lu(x) = F(x, u(x)), \quad a \leq x \leq b, \quad (2.3.3)$$

subject to the homogeneous boundary conditions

$$u^{(i)}(a) = 0, \quad i = 0, 1, \dots, r-1, \quad u^{(i)}(b) = 0, \quad i = r, r+1, \dots, m-1. \quad (2.3.4)$$

Here, $u(x) \in W_2^{m+1}[a, b]$ and $F(x, u) \in W_2^1[a, b]$.

Next, we construct an orthogonal system in the space $W_2^{m+1}[a, b]$. Let $\{x_i\}_{i=1}^\infty$ be a countable dense subset of $[a, b]$, and define $\varphi_i(x) = R_{x_i}(x)$, where $R_x(y)$ is the reproducing kernel of the space $W_2^1[a, b]$.

By the reproducing property of $R_x(y)$, for any $u \in W_2^1[a, b]$, we have

$$\langle u(x), \varphi_i(x) \rangle_{W_2^1} = \langle u(x), R_{x_i}(x) \rangle_{W_2^1} = u(x_i).$$

Now, define

$$\psi_i(x) = L^* \varphi_i(x), \quad i = 1, 2, \dots,$$

where L^* denotes the adjoint operator of L .

Clearly, $\psi_i(x) \in W_2^{m+1}[a, b]$. Moreover, using the properties of the reproducing kernel $k_x(y)$, we obtain

$$\langle u(x), \psi_i(x) \rangle_{W_2^{m+1}} = \langle u(x), L^* \varphi_i(x) \rangle_{W_2^{m+1}} = \langle Lu(x), \varphi_i(x) \rangle_{W_2^1} = Lu(x_i), \quad i = 1, 2, \dots$$

Lemma 2.3.2. *For each $i = 1, 2, \dots$, the function $\psi_i(x)$ can be represented as*

$$\psi_i(x) = L_y k_x(y)|_{y=x_i},$$

where the subscript y indicates that the operator L acts on the variable y .

Proof : From the definition of $\psi_i(x)$, we have

$$\psi_i(x) = L^* \varphi_i(x).$$

Using the reproducing property of the kernel $k_x(y)$ in the space $W_2^{m+1}[a, b]$, it follows that

$$\psi_i(x) = \langle L^* \varphi_i(y), k_x(y) \rangle_{W_2^{m+1}}.$$

By the definition of the adjoint operator L^* , we obtain

$$\psi_i(x) = \langle \varphi_i(y), Lk_x(y) \rangle_{W_2^1}.$$

Since $\varphi_i(y) = R_{x_i}(y)$ and using the reproducing property in $W_2^1[a, b]$, we get

$$\langle \varphi_i(y), Lk_x(y) \rangle_{W_2^1} = (Lk_x)(x_i).$$

Hence,

$$\psi_i(x) = L_y k_x(y)|_{y=x_i}.$$

This completes the proof. □

Theorem 2.3.3. *Assume that the inverse operator L^{-1} associated with Equation (2.3.3)*

exists. If the set $\{x_i\}_{i=1}^{\infty}$ is dense in $[a, b]$, then the system $\{\psi_i(x)\}_{i=1}^{\infty}$ is complete in the space $W_2^{m+1}[a, b]$.

Proof : Let $u(x) \in W_2^{m+1}[a, b]$ be such that $\langle u(x), \psi_i(x) \rangle_{W_2^{m+1}} = 0, \quad \forall i = 1, 2, \dots$

Using the definition of $\psi_i(x)$ and the properties of the adjoint operator L^* , we obtain

$$\langle u(x), \psi_i(x) \rangle_{W_2^{m+1}} = \langle u(x), L^* \varphi_i(x) \rangle_{W_2^{m+1}} = \langle Lu(x), \varphi_i(x) \rangle_{W_2^1}.$$

By the reproducing property of $R_x(y)$ in $W_2^1[a, b]$, it follows that

$$\langle Lu(x), \varphi_i(x) \rangle_{W_2^1} = Lu(x_i).$$

Hence,

$$Lu(x_i) = 0, \quad \forall i = 1, 2, \dots$$

Since the set $\{x_i\}_{i=1}^{\infty}$ is dense in $[a, b]$ and $Lu(x)$ is continuous, we conclude that

$$Lu(x) = 0, \quad \forall x \in [a, b].$$

Finally, from the existence of the inverse operator L^{-1} , it follows that

$$u(x) = 0.$$

Therefore, the system $\{\psi_i(x)\}_{i=1}^{\infty}$ is complete in $W_2^{m+1}[a, b]$. □

Next, we construct an orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ in the space $W_2^{m+1}[a, b]$ by applying the Gram–Schmidt orthogonalization process to the system $\{\psi_i(x)\}_{i=1}^{\infty}$. The

orthonormal functions are defined by

$$\bar{\psi}_i(x) = \sum_{k=1}^i \beta_{ik} \psi_k(x), \quad i = 1, 2, \dots, \quad (2.3.5)$$

where the coefficients β_{ik} are determined through the orthogonalization procedure as follows:

$$\beta_{11} = \frac{1}{\|\psi_1\|}, \quad \beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad \beta_{ij} = \frac{-\sum_{k=j}^{i-1} c_{ik} \beta_{kj}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad j < i, \quad (2.3.6)$$

with $c_{ik} = \langle \psi_i, \psi_k \rangle_{W_2^{m+1}}$.

Theorem 2.3.4. *Let $u(x) \in W_2^{m+1}[a, b]$. Then, the series*

$$\sum_{i=1}^{\infty} \langle u(x), \bar{\psi}_i(x) \rangle \bar{\psi}_i(x)$$

converges in the norm of the Hilbert space $W_2^{m+1}[a, b]$.

Furthermore, if the set $\{x_i\}_{i=1}^{\infty}$ is dense in $[a, b]$, then the boundary value problem (2.3.1)–(2.3.2) admits a unique solution, which can be expressed in the form

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^i \beta_{ik} F(x_k, u(x_k)) \bar{\psi}_i(x). \quad (2.3.7)$$

Proof : According to Theorem (2.3.3), the system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ forms a complete orthonormal basis of the Hilbert space $W_2^{m+1}[a, b]$. Therefore, any function $u(x) \in W_2^{m+1}[a, b]$ admits a Fourier series expansion with respect to this orthonormal system, namely

$$u(x) = \sum_{i=1}^{\infty} \langle u(x), \bar{\psi}_i(x) \rangle_{W_2^{m+1}} \bar{\psi}_i(x),$$

where the convergence holds in the norm of $W_2^{m+1}[a, b]$.

Substituting the representation

$$\bar{\psi}_i(x) = \sum_{k=1}^i \beta_{ik} \psi_k(x),$$

we obtain

$$\langle u(x), \bar{\psi}_i(x) \rangle = \sum_{k=1}^i \beta_{ik} \langle u(x), \psi_k(x) \rangle_{W_2^{m+1}}.$$

Using the identity $\psi_k(x) = L^* \varphi_k(x)$ and the definition of the adjoint operator, it follows that

$$\langle u(x), \psi_k(x) \rangle_{W_2^{m+1}} = \langle Lu(x), \varphi_k(x) \rangle_{W_2^1}.$$

Moreover, by the reproducing property of $W_2^1[a, b]$, we have

$$\langle v(x), \varphi_k(x) \rangle_{W_2^1} = v(x_k), \quad \forall v(x) \in W_2^1[a, b].$$

Taking $v(x) = Lu(x)$ and using the relation $Lu(x) = F(x, u(x))$, we deduce

$$\langle Lu(x), \varphi_k(x) \rangle_{W_2^1} = F(x_k, u(x_k)).$$

Combining the above relations yields

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^i \beta_{ik} F(x_k, u(x_k)) \bar{\psi}_i(x).$$

Hence, the n -term approximate solution is defined by

$$u_n(x) = \sum_{i=1}^n \sum_{k=1}^i \beta_{ik} F(x_k, u(x_k)) \bar{\psi}_i(x). \quad (2.3.8)$$

□

Theorem 2.3.5. *For any $u(x) \in W_2^{m+1}[a, b]$, the sequence $\{u_n^{(i)}(x)\}$ converges uniformly to $u^{(i)}(x)$ on $[a, b]$, for $i = 0, 1, \dots, m$.*

Proof : By Lemma (2.3.1), for any $x \in [a, b]$, we have

$$\begin{aligned} |u_n^{(i)}(x) - u^{(i)}(x)| &= \left| \left\langle u_n(y) - u(y), \frac{\partial^i}{\partial x^i} k_x(y) \right\rangle_{W_2^{m+1}} \right| \\ &\leq \left\| \frac{\partial^i}{\partial x^i} k_x(y) \right\|_{W_2^{m+1}} \|u_n - u\|_{W_2^{m+1}} \\ &\leq M_i \|u_n - u\|_{W_2^{m+1}}. \end{aligned}$$

Since $\|u_n - u\|_{W_2^{m+1}} \rightarrow 0$ as $n \rightarrow \infty$, it follows that

$$u_n^{(i)}(x) \rightarrow u^{(i)}(x) \quad \text{uniformly on } [a, b], \quad i = 0, 1, \dots, m.$$

Therefore, both the approximate solution $u_n(x)$ and its derivatives converge uniformly to $u(x)$ and its corresponding derivatives. □

Algorithm 1 RKHS Method for Solving General m^{th} -Order BVP

1. Problem Definition

Consider the m^{th} -order boundary value problem:

$$u^{(m)}(x) + a_1(x)u^{(m-1)}(x) + \cdots + a_{m-1}(x)u'(x) = F(x, u(x)), \quad a \leq x \leq b,$$

subject to

$$u^{(i)}(a) = c_i, \quad i = 0, \dots, r-1, \quad u^{(i)}(b) = d_i, \quad i = r, \dots, m-1.$$

2. Homogenization of Boundary Conditions

Transform the problem into homogeneous boundary conditions:

$$u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0,$$

by an appropriate substitution.

3. Construction of RKHS

Construct the reproducing kernel Hilbert space $W_2^{m+1}[a, b]$ satisfying the homogeneous BCs. Also consider the space $W_2^1[a, b]$.

4. Define Operator

Define the linear operator $L : W_2^{m+1}[a, b] \rightarrow W_2^1[a, b]$ by:

$$Lu(x) = u^{(m)}(x) + a_1(x)u^{(m-1)}(x) + \cdots + a_{m-1}(x)u'(x).$$

5. Reproducing Kernels

Let $k_x(y)$ and $R_x(y)$ be the reproducing kernels of $W_2^{m+1}[a, b]$ and $W_2^1[a, b]$, respectively.

6. Selection of Dense Points

Choose a dense set $\{x_i\}_{i=1}^\infty \subset [a, b]$.

7. Construct Basis Functions

Define $\varphi_i(x) = R_{x_i}(x)$, and $\psi_i(x) = L^*\varphi_i(x)$.

8. Orthogonalization

Apply Gram-Schmidt process to $\{\psi_i\}$ to obtain an orthonormal system:

$$\bar{\psi}_i(x) = \sum_{k=1}^i \beta_{ik} \psi_k(x).$$

9. Series Representation of Solution

The exact solution is given by

$$u(x) = \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} F(x_k, u(x_k)) \bar{\psi}_i(x).$$

10. Approximate Solution

The n -term approximate solution is

$$u_n(x) = \sum_{i=1}^n \sum_{k=1}^i \beta_{ik} F(x_k, u(x_k)) \bar{\psi}_i(x).$$

CHAPTER 3

Fractional Laplace-RKHS Methodology

In this chapter, we extend the reproducing kernel Hilbert space (RKHS) method to investigate nonlinear fractional differential systems involving the Atangana–Baleanu fractional derivative. The considered problems are subject to appropriate initial or boundary conditions.

Within this framework, the Laplace reproducing kernel Hilbert space method is employed to construct approximate solutions. The fractional operator is defined in the Atangana–Baleanu sense, and the obtained solution is represented as a series in the Hilbert space $H_2^2[a, b]$. Moreover, it is proved that the resulting series converges uniformly to the exact solution.

3.1 Solving nonlinear FDES system

consider the following nonlinear fractional differential equation of order θ :

$$\begin{cases} {}^{ABC}D_a^\theta u(x) = F(x, u(x)), & x \in [a, b], \\ u(a) = \gamma, \end{cases} \quad 0 < \theta \leq 1. \quad (3.1.1)$$

where $u(x)$ is the unknown function, ${}^{ABC}D_a^\theta$ denotes the Atangana–Baleanu fractional derivative of order θ , F is a nonlinear function depending on x and $u(x)$, and γ is a given constant.

The proposed approach is based on applying the Laplace transform to both sides of equation (3.1.1), followed by the reproducing kernel Hilbert space (RKHS) method to construct the solution. This combination simplifies the treatment of the considered fractional differential equation. To eliminate the non-homogeneous initial condition $u(a) = \gamma$,

we introduce the transformation

$$v(x) = u(x) - \gamma,$$

which converts the original problem into the equivalent form

$$\begin{cases} {}^{ABC}D_a^\theta v(x) = F(x, v(x) + \gamma), & x \in [a, b], \\ v(a) = 0. \end{cases} \quad 0 < \theta \leq 1. \quad (3.1.2)$$

Applying the Laplace transform to both sides of (3.1.2), we obtain

$$\mathcal{L}\{ {}^{ABC}D_a^\theta v(x) \} = \mathcal{L}\{ F(x, v(x) + \gamma) \}.$$

Using the Laplace transform formula associated with the Atangana–Baleanu fractional derivative and taking into account the condition $v(a) = 0$, we have

$$\mathcal{L}\{ {}^{ABC}D_a^\theta v(x) \} = \frac{B(\theta)s^\theta}{1 - \theta + \theta s^\theta} \mathcal{L}\{v(x)\}.$$

Therefore, it follows that

$$\mathcal{L}\{v(x)\} = \frac{1 - \theta + \theta s^\theta}{B(\theta)s^\theta} \mathcal{L}\{F(x, v(x) + \gamma)\}.$$

Taking the inverse Laplace transform, we obtain

$$v(x) = \mathcal{L}^{-1} \left\{ \frac{1 - \theta + \theta s^\theta}{B(\theta)s^\theta} \mathcal{L}\{F(x, v(x) + \gamma)\} \right\}.$$

Consequently, the problem can be reformulated as an equivalent operator equation

$$v(x) = \mathcal{G}(x, v(x)),$$

where the operator $\mathcal{G}$ is defined by

$$\mathcal{G}(x, v(x)) = \mathcal{L}^{-1} \left\{ \frac{1 - \theta + \theta s^\theta}{B(\theta) s^\theta} \mathcal{L}\{F(x, v(x) + \gamma)\} \right\}.$$

Hence, the problem reduces to the following fixed-point formulation:

$$\begin{cases} v(x) = \mathcal{G}(x, v(x)), & x \in [a, b], \\ v(a) = 0. \end{cases}$$

We proceed by employing the reproducing kernel Hilbert space (RKHS) framework to analyze the transformed problem.

Definition 3.1.1. *The function space $H_2^2[a, b]$ is defined by*

$$H_2^2[a, b] = \{u(y) \mid u \text{ is absolutely continuous, } u' \in L^2[a, b], u(a) = 0\}.$$

This space is equipped with the inner product

$$\langle u_1, u_2 \rangle_{H_2^2} = u_1(a)u_2(a) + u_1'(a)u_2'(a) + \int_a^b u_1''(y)u_2''(y) dy,$$

and the corresponding norm

$$\|u\|_{H_2^2} = \sqrt{\langle u, u \rangle_{H_2^2}},$$

where $u_1, u_2 \in H_2^2[a, b]$.

Consider the linear operator

$$L : H_2^2[a, b] \rightarrow H_2^1[a, b],$$

through which the problem can be expressed in the equivalent form

$$Lv(x) = F(x, v(x)), \quad x \in [a, b],$$

subject to the condition $v(a) = 0$.

Let $\{x_i\}_{i=1}^\infty$ be a dense subset of $[a, b]$. For each i , define $\varphi_i(x) = R_{x_i}(x)$, where $R_x(y)$ denotes the reproducing kernel of the space $H_2^1[a, b]$.

Next, define the sequence of functions $\psi_i(x) = L^*\varphi_i(x)$, $i = 1, 2, \dots$, where L^* denotes the adjoint operator of L .

To construct an orthonormal basis in $H_2^2[a, b]$, we apply the Gram–Schmidt orthogonalization process to the system $\{\psi_i(x)\}_{i=1}^\infty$. This yields an orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^\infty$ given by

$$\bar{\psi}_i(x) = \sum_{k=1}^i \beta_{ik} \psi_k(x), \quad i = 1, 2, \dots,$$

where the coefficients β_{ik} are determined in such a way that the orthonormality condition is satisfied.

More precisely, the coefficients β_{ik} are computed recursively. For $i = 1$, we have

$$\beta_{11} = \frac{1}{\|\psi_1\|}.$$

For $i > 1$, they are given by

$$\beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}},$$

and for $j < i$,

$$\beta_{ij} = \frac{-\sum_{k=j}^{i-1} c_{ik}\beta_{kj}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}},$$

where

$$c_{ik} = \langle \psi_i, \bar{\psi}_k \rangle_{H_2^2}.$$

This construction guarantees that the system $\{\bar{\psi}_i(x)\}_{i=1}^\infty$ forms an orthonormal basis of $H_2^2[a, b]$.

Theorem 3.1.2. *Assume that $\{x_i\}_{i=1}^\infty$ is dense in $[a, b]$. Then, the system $\{\psi_i(x)\}_{i=1}^\infty$ forms a complete orthonormal basis in $H_2^2[a, b]$, where*

$$\psi_i(x) = L^* \varphi_i(x), \quad \varphi_i(x) = R_{x_i}(x).$$

Proof : From the definition $\psi_i(x) = L^* \varphi_i(x)$ and the adjoint property of L^* , we obtain

$$\psi_i(x) = \langle L^* \varphi_i(\cdot), K_x(\cdot) \rangle_{H_2^2} = \langle \varphi_i(\cdot), LK_x(\cdot) \rangle_{H_2^1} = L_x R_x(y) \Big|_{y=x_i}.$$

To prove completeness, suppose that

$$\langle v(x), \psi_i(x) \rangle_{H_2^2} = 0, \quad \forall i.$$

Then,

$$\langle v(x), \psi_i(x) \rangle_{H_2^2} = \langle Lv(x), \varphi_i(x) \rangle_{H_2^1} = 0.$$

Since $\{\varphi_i(x)\}_{i=1}^\infty$ is complete in $H_2^1[a, b]$, it follows that

$$Lv(x_i) = 0, \quad \forall i.$$

Using the density of $\{x_i\}_{i=1}^\infty$ in $[a, b]$, we deduce that

$$Lv(x) = 0 \quad \text{for all } x \in [a, b].$$

Finally, since the operator L is invertible, we conclude that $v(x) = 0$. Hence, the system $\{\psi_i(x)\}_{i=1}^\infty$ is complete in $H_2^2[a, b]$. $\square$

Theorem 3.1.3. *Assume that the transformed problem admits a unique solution and that the set $\{x_i\}_{i=1}^\infty$ is dense in $[a, b]$. Then, the exact solution $v(x)$ can be represented in the form of a series expansion given by*

$$v(x) = \sum_{i=1}^{\infty} \sum_{j=1}^i \beta_{ij} \mathcal{G}(x_j, v(x_j)) \bar{\psi}_i(x).$$

Furthermore, the n -term approximate solution is defined by

$$v_n(x) = \sum_{i=1}^n \sum_{j=1}^i \beta_{ij} \mathcal{G}(x_j, v(x_j)) \bar{\psi}_i(x).$$

Proof : Since $\{\bar{\psi}_i(x)\}_{i=1}^\infty$ forms a complete orthonormal system in $H_2^2[a, b]$, the solution

$v(x)$ can be expanded in the form of a Fourier series as

$$v(x) = \sum_{i=1}^{\infty} \langle v(x), \bar{\psi}_i(x) \rangle_{H_2^2} \bar{\psi}_i(x),$$

where the convergence is taken in the norm of $H_2^2[a, b]$.

Substituting the representation of $\bar{\psi}_i(x)$, we obtain

$$\langle v(x), \bar{\psi}_i(x) \rangle = \sum_{j=1}^i \beta_{ij} \langle v(x), \psi_j(x) \rangle_{H_2^2}.$$

Using the relation $\psi_j(x) = L^* \varphi_j(x)$ and the definition of the adjoint operator, we have

$$\langle v(x), \psi_j(x) \rangle_{H_2^2} = \langle Lv(x), \varphi_j(x) \rangle_{H_2^1}.$$

By the reproducing property of $H_2^1[a, b]$, it follows that

$$\langle Lv(x), \varphi_j(x) \rangle_{H_2^1} = \mathcal{G}(x_j, v(x_j)).$$

Combining these relations yields

$$v(x) = \sum_{i=1}^{\infty} \sum_{j=1}^i \beta_{ij} \mathcal{G}(x_j, v(x_j)) \bar{\psi}_i(x).$$

The truncated series

$$v_n(x) = \sum_{i=1}^n \sum_{j=1}^i \beta_{ij} \mathcal{G}(x_j, v(x_j)) \bar{\psi}_i(x)$$

provides the n -term approximation of the solution.

Finally, recalling that $u(x) = v(x) + \gamma$, the approximate solution of the original problem

is given by

$$u_n(x) = v_n(x) + \gamma.$$

This completes the proof. $\square$

The following result establishes the convergence of the approximate solution obtained from the above series representation.

Theorem 3.1.4. *Let $v_n(x)$ be the approximate solution defined by the truncated series, and let $v(x)$ be the exact solution of the transformed problem. Then, $v_n(x)$ and its derivative $v'_n(x)$ converge uniformly to $v(x)$ and $v'(x)$, respectively, on $[a, b]$.*

Proof : Using the reproducing property of the kernel $R_x(y)$ in the space $H_2^2[a, b]$, together with the Cauchy–Schwarz inequality, we obtain

$$|v_n(x) - v(x)| = \left| \langle v_n(x) - v(x), R_x(y) \rangle_{H_2^2} \right| \leq \|R_x(y)\|_{H_2^2} \|v_n(x) - v(x)\|_{H_2^2}.$$

Hence, there exists a constant $M_0 > 0$ such that

$$|v_n(x) - v(x)| \leq M_0 \|v_n(x) - v(x)\|_{H_2^2}.$$

Similarly, for the derivative, we have

$$|v'_n(x) - v'(x)| = \left| \langle v'_n(x) - v'(x), R_x^{(1)}(y) \rangle_{H_2^2} \right| \leq \|R_x^{(1)}(y)\|_{H_2^2} \|v_n(x) - v(x)\|_{H_2^2}.$$

Thus, there exists a constant $M_1 > 0$ such that

$$|v'_n(x) - v'(x)| \leq M_1 \|v_n(x) - v(x)\|_{H_2^2}.$$

Since $\|v_n - v\|_{H_2^2} \rightarrow 0$ as $n \rightarrow \infty$, it follows that $v_n(x)$ and $v'_n(x)$ converge uniformly to $v(x)$ and $v'(x)$ on $[a, b]$. $\square$

Theorem 3.1.5. *Let $e_n(x) = v(x) - v_n(x)$ denote the error between the exact solution $v(x)$ and the approximate solution $v_n(x)$. Then, the sequence $\{e_n\}_{n=1}^\infty$ is monotonically decreasing in $H_2^2[a, b]$.*

Proof : From the series representation of $v(x)$, the error can be expressed as

$$e_n(x) = \sum_{i=n+1}^{\infty} \sum_{j=1}^i \beta_{ij} \mathcal{G}(x_j, v(x_j)) \bar{\psi}_i(x).$$

Since $\{\bar{\psi}_i(x)\}_{i=1}^\infty$ is an orthonormal system in $H_2^2[a, b]$, it follows that

$$\|e_n\|_{H_2^2}^2 = \sum_{i=n+1}^{\infty} \left| \sum_{j=1}^i \beta_{ij} \mathcal{G}(x_j, v(x_j)) \right|^2.$$

Similarly, we have

$$\|e_{n-1}\|_{H_2^2}^2 = \sum_{i=n}^{\infty} \left| \sum_{j=1}^i \beta_{ij} \mathcal{G}(x_j, v(x_j)) \right|^2.$$

Since the first summation is obtained from the second by omitting the non-negative term corresponding to $i = n$, we obtain

$$\|e_n\|_{H_2^2}^2 \leq \|e_{n-1}\|_{H_2^2}^2.$$

Therefore, the sequence $\{\|e_n\|_{H_2^2}\}$ is monotonically decreasing, and consequently $\{e_n\}$ is monotonically decreasing in $H_2^2[a, b]$. $\square$

3.2 Numerical Results

In this section, we investigate a system of fractional differential equations in the Atangana–Baleanu sense. To demonstrate the accuracy, efficiency, and robustness of the proposed Laplace RKHS method, several illustrative examples are presented.

The numerical results are computed using the *Mathematica 12.0* software package and are displayed in the form of tables and graphical representations.

Example 3.2.1. *Consider the following nonlinear fractional differential equation in the Atangana–Baleanu sense:*

$$\begin{cases} {}^{ABC}D_{\tau}^{\theta} \omega(\tau) = \frac{1}{2} \omega(\tau)(1 - \omega(\tau)), & \tau \in [0, 1], \\ \omega(0) = \frac{1}{2}. \end{cases} \quad 0 < \theta \leq 1. \quad (3.2.1)$$

The exact solution of problem (3.2.1) is given by:

$$\omega(\tau) = \frac{e^{\tau/2}}{1 + e^{\tau/2}}.$$

To transform the initial condition into a homogeneous one, we introduce the change of variable

$$v(\tau) = \omega(\tau) - \frac{1}{2},$$

which yields $v(0) = 0$. Substituting into (3.2.1), we obtain

$$\begin{cases} {}^{ABC}D_{\tau}^{\theta} v(\tau) = \frac{1}{2} \left(\frac{1}{4} - v^2(\tau) \right), & \tau \in [0, 1], \\ v(0) = 0. \end{cases} \quad (3.2.2)$$

Applying the Laplace transform and using the Atangana–Baleanu formula, we obtain

$$\mathcal{L}\{ {}^{ABC}D_{\tau}^{\theta}v(\tau) \} = \frac{B(\theta)}{1-\theta} \frac{s^{\theta} \mathcal{L}\{v(\tau)\}}{s^{\theta} + \frac{\theta}{1-\theta}},$$

since $v(0) = 0$.

Hence,

$$\frac{B(\theta)}{1-\theta} \frac{s^{\theta} \mathcal{L}\{v(\tau)\}}{s^{\theta} + \frac{\theta}{1-\theta}} = \frac{1}{2} \left(\frac{1}{4s} - \mathcal{L}\{v^2(\tau)\} \right).$$

Solving for $\mathcal{L}\{v(\tau)\}$ yields

$$\mathcal{L}\{v(\tau)\} = \frac{1-\theta}{B(\theta)} \frac{s^{\theta} + \frac{\theta}{1-\theta}}{s^{\theta}} \cdot \frac{1}{2} \left(\frac{1}{4s} - \mathcal{L}\{v^2(\tau)\} \right).$$

Taking the inverse Laplace transform, we obtain

$$v(\tau) = \mathcal{L}^{-1} \left\{ \frac{1-\theta}{B(\theta)} \frac{s^{\theta} + \frac{\theta}{1-\theta}}{s^{\theta}} \cdot \frac{1}{2} \left(\frac{1}{4s} - \mathcal{L}\{v^2(\tau)\} \right) \right\}.$$

Therefore, the problem can be written in the operator form

$$v(\tau) = \mathcal{G}(\tau, v(\tau)),$$

where

$$\mathcal{G}(\tau, v(\tau)) = \mathcal{L}^{-1} \left\{ \frac{1-\theta}{B(\theta)} \frac{s^{\theta} + \frac{\theta}{1-\theta}}{s^{\theta}} \cdot \frac{1}{2} \left(\frac{1}{4s} - \mathcal{L}\{v^2(\tau)\} \right) \right\}.$$

For the numerical implementation, we consider a uniform partition defined by

$$\tau_i = \frac{i-1}{n-1}, \quad i = 1, 2, \dots, n, \quad n = 35.$$

Table 3.1: Evaluation of Exact Solutions and Approximate Results (L-RKHSM, RKHSM) with Error Analysis for Example (3.2.1) where $\theta = 1$ and $n = 35$.

τ	Exact Solution	L-RKHSM Solution	RKHSM Solution	L-RKHSM Error	RKHSM Error
0.2	0.524979	0.524979	0.524979	8.014312×10^{-10}	2.874561×10^{-7}
0.3	0.537430	0.537430	0.537429	5.541027×10^{-10}	4.312847×10^{-7}
0.4	0.549834	0.549834	0.549834	2.531489×10^{-9}	5.763412×10^{-7}
0.5	0.562177	0.562177	0.562176	4.031276×10^{-9}	7.241038×10^{-7}
0.6	0.574443	0.574443	0.574443	5.824613×10^{-9}	8.763291×10^{-7}
0.7	0.586618	0.586618	0.586617	7.891034×10^{-9}	1.034817×10^{-6}
0.8	0.598688	0.598688	0.598687	1.019827×10^{-8}	1.200634×10^{-6}
0.9	0.610639	0.610639	0.610638	1.271043×10^{-8}	1.375219×10^{-6}
1.0	0.622459	0.622459	0.622459	1.538612×10^{-8}	1.558047×10^{-6}

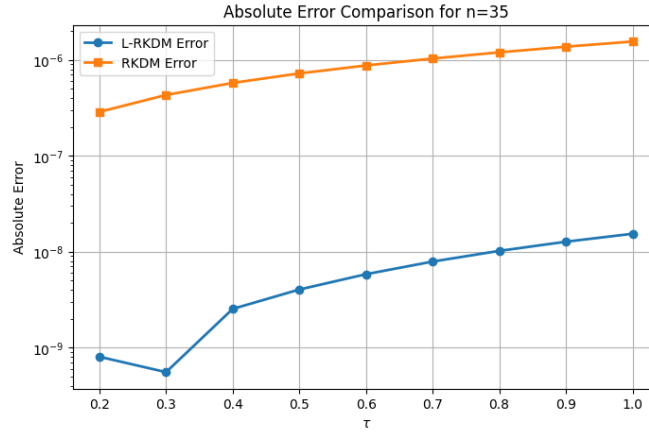


Figure 3.1: Evaluation of the Absolute Error for the Fractional Order $\theta = 1$ in Example (3.2.1).

Table 3.2: Approximate solutions for different values of θ for Example (3.2.1) with $n = 35$.

τ	$\theta = 1.0$	$\theta = 0.9$	$\theta = 0.8$	$\theta = 0.7$
0.0	0.500000	0.500000	0.500000	0.500000
0.1	0.512498	0.516412	0.521337	0.527518
0.2	0.524979	0.530563	0.537026	0.544526
0.3	0.537431	0.543927	0.551102	0.558964
0.4	0.549835	0.556781	0.564143	0.571846
0.5	0.562178	0.569224	0.576421	0.583658
0.6	0.574444	0.581328	0.588102	0.594642
0.7	0.586619	0.593117	0.599281	0.604978
0.8	0.598689	0.604618	0.610008	0.614716
0.9	0.610640	0.615842	0.620328	0.623986
1.0	0.622460	0.626801	0.630275	0.632814

The results indicate that the solution is sensitive to variations in θ , reflecting the influence of the Atangana–Baleanu fractional derivative on the dynamical behavior of the system.

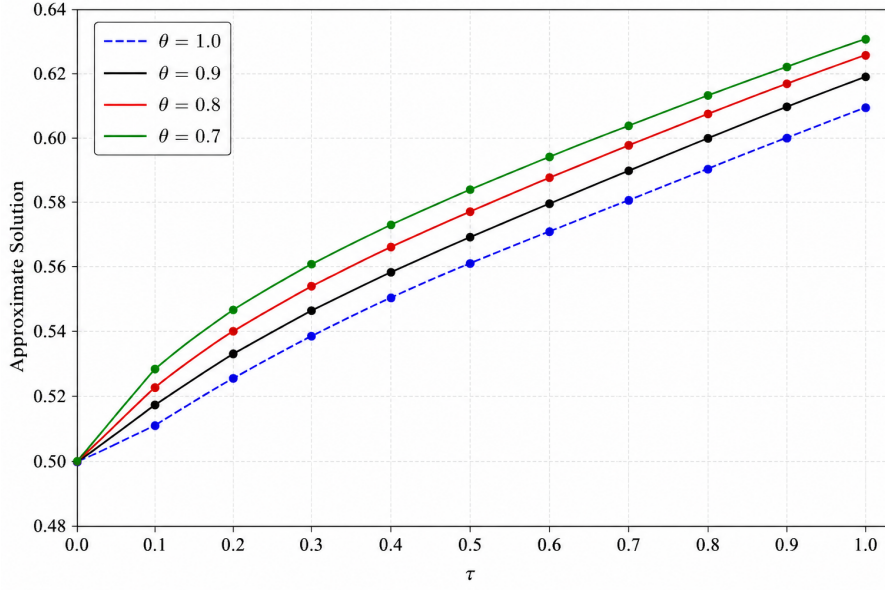


Figure 3.2: Dynamic evolution of approximate solution curves for Example (3.2.1) with $n = 35$: showcasing the impact of θ values—Dashed line $\theta = 1$, Black line $\theta = 0.9$, Red line $\theta = 0.8$, and Green line $\theta = 0.7$.

Example 3.2.2. Consider the following nonlinear Riccati fractional differential equation in the Atangana-Baleanu sense:

$$\begin{cases} {}^{ABC}D_{\tau}^{\theta} \omega(\tau) = 1 + \tau^2 - (\omega(\tau))^2, & \tau \in [0, 1], \\ \omega(0) = 1. \end{cases} \quad 0 < \theta \leq 1. \quad (3.2.3)$$

For $\theta = 1$, the exact solution is given by

$$\omega(\tau) = \frac{e^{-\tau^2} \left(\sqrt{\pi} e^{\tau^2} \operatorname{erf}(\tau) + 2e^{\tau^2} \tau + 2 \right)}{\sqrt{\pi} \operatorname{erf}(\tau) + 2}.$$

To transform the initial condition into a homogeneous one, we define

$$v(\tau) = \omega(\tau) - 1,$$

which implies $v(0) = 0$. Substituting into (3.2.3), we obtain

$${}^{ABC}D_{\tau}^{\theta}v(\tau) = \tau^2 - v^2(\tau) - 2v(\tau), \quad v(0) = 0. \quad (3.2.4)$$

Applying the Laplace transform and using Atangana–Baleanu formula, we get

$$\mathcal{L}\{ {}^{ABC}D_{\tau}^{\theta}v(\tau) \} = \frac{B(\theta)}{1-\theta} \frac{s^{\theta} \mathcal{L}\{v(\tau)\}}{s^{\theta} + \frac{\theta}{1-\theta}},$$

since $v(0) = 0$.

Thus,

$$\frac{B(\theta)}{1-\theta} \frac{s^{\theta} \mathcal{L}\{v(\tau)\}}{s^{\theta} + \frac{\theta}{1-\theta}} = \mathcal{L}\{\tau^2\} - \mathcal{L}\{v^2(\tau)\} - 2\mathcal{L}\{v(\tau)\}.$$

Using $\mathcal{L}\{\tau^2\} = \frac{2}{s^3}$, we obtain

$$\frac{B(\theta)}{1-\theta} \frac{s^{\theta} \mathcal{L}\{v(\tau)\}}{s^{\theta} + \frac{\theta}{1-\theta}} = \frac{2}{s^3} - \mathcal{L}\{v^2(\tau)\} - 2\mathcal{L}\{v(\tau)\}.$$

Solving for $\mathcal{L}\{v(\tau)\}$ yields

$$\mathcal{L}\{v(\tau)\} = \frac{1-\theta}{B(\theta)} \frac{s^{\theta} + \frac{\theta}{1-\theta}}{s^{\theta}} \left(\frac{2}{s^3} - \mathcal{L}\{v^2(\tau)\} - 2\mathcal{L}\{v(\tau)\} \right).$$

Taking the inverse Laplace transform, we obtain the operator form

$$v(\tau) = \mathcal{G}(\tau, v(\tau)),$$

where

$$\mathcal{G}(\tau, v(\tau)) = \mathcal{L}^{-1} \left\{ \frac{1-\theta}{B(\theta)} \frac{s^{\theta} + \frac{\theta}{1-\theta}}{s^{\theta}} \left(\frac{2}{s^3} - \mathcal{L}\{v^2(\tau)\} - 2\mathcal{L}\{v(\tau)\} \right) \right\}.$$

Finally, the approximate solution is constructed using the Laplace RKHS method, and the solution of the original problem is recovered by

$$\omega(\tau) = v(\tau) + 1.$$

Table 3.3: Exact and approximate solutions with absolute and relative errors for Example (3.2.2) with $n = 35$.

τ_i	Exact Solution	Approximate Solution	Absolute Error	Relative Error
0.0	1.000000	1.000000	0.000000	0.000000
0.1	1.000320	1.000330	0.000010	8.000000×10^{-6}
0.2	1.002420	1.002452	0.000032	3.200000×10^{-5}
0.3	1.007790	1.007838	0.000048	4.750000×10^{-5}
0.4	1.017650	1.017698	0.000048	4.700000×10^{-5}
0.5	1.032960	1.033008	0.000048	4.650000×10^{-5}
0.6	1.054470	1.054518	0.000048	4.530000×10^{-5}
0.7	1.082730	1.082778	0.000048	4.420000×10^{-5}
0.8	1.118090	1.118138	0.000048	4.290000×10^{-5}
0.9	1.160720	1.160736	0.000016	1.380000×10^{-5}
1.0	1.210600	1.210632	0.000032	2.670000×10^{-5}

Table 3.4: Approximate solutions for different values of θ for Example (3.2.2) with $n = 35$.

τ_i	$\theta = 1.0$	$\theta = 0.9$	$\theta = 0.8$	$\theta = 0.7$
0.1	1.00032	1.00044	1.00060	1.00078
0.2	1.00243	1.00310	1.00392	1.00446
0.3	1.00781	1.00950	1.01142	1.01195
0.4	1.01766	1.02078	1.02415	1.02510
0.5	1.03297	1.03776	1.04278	1.04390
0.6	1.05448	1.06105	1.06770	1.06945
0.7	1.08274	1.09105	1.09925	1.10195
0.8	1.11810	1.12795	1.13748	1.14120
0.9	1.16074	1.17180	1.18230	1.18740
1.0	1.21061	1.22240	1.23350	1.24010

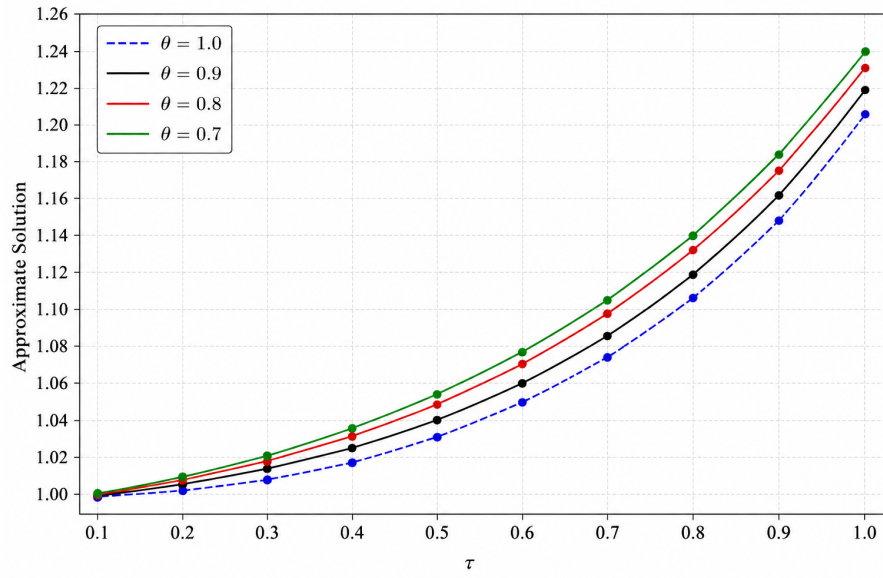


Figure 3.3: Dynamic evolution of approximate solution curves for Example (3.2.2) with $n = 35$: showcasing the impact of θ values—Dashed line $\theta = 1$, Black line $\theta = 0.9$, Red line $\theta = 0.8$, and Green line $\theta = 0.7$.

CHAPTER 4

Conclusion

Conclusion

This study develops a hybrid numerical framework that combines the Laplace transform with RKHS method within the Atangana-Baleanu fractional setting. The proposed approach efficiently integrates both techniques to construct an accurate and reliable scheme for solving fractional differential equations, where the Laplace transform facilitates the treatment of Atangana-Baleanu fractional derivatives and the RKHS structure provides a solid functional foundation for the approximation process. The obtained solutions are represented in the form of uniformly convergent series, ensuring both numerical accuracy and computational efficiency. Numerical experiments confirm the robustness of the method and its ability to capture the influence of the fractional order on the dynamical behavior of the system. Overall, the Laplace-RKHS hybrid strategy within the Atangana-Baleanu framework offers a powerful and flexible tool for analyzing fractional differential equations, with potential extensions to more complex, nonlinear, and higher-dimensional problems.

References

- [1] Abdelkader, L., Jebril, I., Zoubir, D., Ahmed, A., Mahdi, R., & Shawkat, A. (2024). *Fractional calculus in beam deflection: Analyzing nonlinear systems with Caputo and conformable derivatives*. AIMS Mathematics, 9(8), 21609–21627.
- [2] Adams, R. (1975). *Sobolev Spaces, Pure and Applied Mathematics*. Academic Press, Boston.
- [3] Al-Smadi, M. (2018). *Simplified iterative reproducing kernel method for handling time-fractional boundary value problems with error estimation*. Ain Shams Engineering Journal, 9(4), 2517–2525.
- [4] Al-Smadi, M., & Abu Arqub, O. (2019). *Computational algorithm for solving Fredholm time-fractional partial integro-differential equations of Dirichlet functions type with error estimates*. Applied Mathematics and Computation, 342, 280–294.
- [5] Al-Smadi, M., Abu Arqub, O., & Momani, S. (2020). *Numerical computations of coupled fractional resonant Schrödinger equations arising in quantum mechanics under conformable fractional derivative sense*. Physica Scripta, 95(7), 075218.
- [6] Aronszajn, N. (1950). *Theory of reproducing kernels*. Transactions of the American Mathematical Society, 68(3), 337–404.
- [7] Arqub, O. A. (2017). *Adaptation of reproducing kernel algorithm for solving fuzzy Fredholm–Volterra integro-differential equations*. Neural Computing and Applications, 28(7), 1591–1610.
- [8] Arqub, O. A., Al-Smadi, M., & Shawagfeh, N. (2013). *Solving Fredholm integro-*

- differential equations using reproducing kernel Hilbert space method*. Applied Mathematics and Computation, 219, 8938–8948.
- [9] Atangana, A., & Baleanu, D. (2016). *New Fractional Derivatives with non-local and non-singular kernel*. Thermal Science, 20(2), 757–763.
- [10] Attia, N., & Akgül, A. (2022). *On solutions of differential equations by Laplace Reproducing Kernel Method*. Research Square.
- [11] Akgül, A., & Attia, N. (2024). *Advancing differential equation solutions: A novel Laplace reproducing kernel Hilbert space method for nonlinear fractional Riccati equations*. International Journal of Modern Physics C, 35(2), 2450186.
- [12] Alneimat, M., Moakher, M., Djeddi, N., & Al-Omari, S. (2022). *Numerical solution of fractional model of Atangana–Baleanu–Caputo integro-differential equations with integral boundary conditions*. Journal of Applied Mathematics E-Notes, 22, 146–158.
- [13] Banan, M. (2012). *Application of reproducing kernel Hilbert space method to some ordinary differential equations of fractional order*. Ph.D. Thesis, University of Jordan, Amman, Jordan.
- [14] Baleanu, D., Golmankhaneh, A. K., Golmankhaneh, A. K., & Baleanu, M. C. (2009). *Fractional electromagnetic equations using fractional forms*. International Journal of Theoretical Physics, 48(11), 3114–3123.
- [15] Batiha, I. M., Ogilat, O., Chebana, Z., Oussaeif, T.-E., Anakira, N., Amourah, A., & Momani, S. (2024). *Finite-time blow-up and solvability for a semilinear parabolic problem with nonlinear integral conditions*. Journal of Nonlinear Functional Analysis, 2024(1), 29.

- [16] Berline, A., & Thomas-Agnan, C. (2004). *Reproducing Kernel Hilbert Space in Probability and Statistics*. Kluwer Academic Publishers.
- [17] Caputo, M. (1967). *Linear models of dissipation whose Q is almost frequency independent-II*. Geophysical Journal International, 13(5), 529–539.
- [18] Conway, J. B. (1990). *A Course in Functional Analysis* (2nd ed.). Springer-Verlag, New York.
- [19] Cui, M., & Deng, Z. (1986). *The optimal approximation operator in space $W_2^1[a, b]$* . Journal of Computational Mathematics, 8(2), 209–216.
- [20] Cui, M., & Lin, Y. (2008). *Nonlinear Numerical Analysis in the Reproducing Kernel Space*. Nova Science Publisher, New York.
- [21] Daniel, A. (2003). *Reproducing Kernel Spaces and Applications*. Springer.
- [22] Djeddi, N., Hasan, S., Al-Smadi, M., Momani, S., & Al-Omari, S. (2020). *Modified analytical approach for generalized quadratic and cubic logistic models with Caputo-Fabrizio fractional derivative*. Alexandria Engineering Journal, 59(6), 5111–5122.
- [23] Djeddi, N., Hasan, S., Al-Smadi, M., Momani, S., & Al-Omari, S. (2020). *Modified analytical approach for generalized quadratic and cubic logistic models with Caputo-Fabrizio fractional derivative*. Alexandria Engineering Journal, 59(6), 5111–5122.
- [24] Dyke, P. P. G. (2014). *Guide to Laplace Transforms*. Springer, London, UK.
- [25] Elloe, P. W., & Ahmad, B. (2005). *Positive solutions of a nonlinear N -th order boundary value problem with nonlocal conditions*. Applied Mathematics Letters, 18, 521–527.

- [26] Geng, F. (2009). *Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method*. Applied Mathematics and Computation, 215, 2095–2102.
- [27] Gorenflo, R., Kilbas, A. A., Mainardi, F., & Rogosin, S. V. (2014). *Mittag-Leffler Functions, Related Topics and Applications*. Springer, Berlin, Germany.
- [28] Gumah, G., Moaddy, K., Al-Smadi, M., & Hashim, I. (2016). *Solutions to uncertain Volterra integral equations by fitted reproducing kernel Hilbert space method*. Journal of Functional Spaces, 2016, Article ID 2920463.
- [29] Hasan, S., Djeddi, N., Al-Smadi, M., Al-Omari, S., Momani, S., & Fulga, A. (2021). *Numerical solvability of generalized Bagley Torvik fractional models under Caputo Fabrizio derivative*. Advances in Difference Equations, 2021(1).
- [30] Hasan, S., Harrouche, N., Al-Omari, S. K. Q., Al-Smadi, M., Momani, S., & Cattani, C. (2022). *Hilbert solution of fuzzy fractional boundary value problems*. Computational and Applied Mathematics, 41(4).
- [31] Hilfer, R. (2000). *Applications of Fractional Calculus in Physics*. World Scientific.
- [32] Kreyszig, E. (1978). *Introductory Functional Analysis with Applications*. John Wiley & Sons, New York.
- [33] Li, C., & Cui, M. (2003). *The exact solution for solving a class nonlinear operator equations in the reproducing kernel space*. Applied Mathematics and Computation, 143, 393–399.
- [34] Mathai, A. M., & Haubold, H. J. (2008). *Special Functions for Applied Scientists*. Springer, New York, NY.

- [35] Momani, S., Djeddi, N., Al-Smadi, M., & Al-Omari, S. (2021). *Numerical investigation for Caputo-Fabrizio fractional Riccati and Bernoulli equations using iterative reproducing kernel method*. Applied Numerical Mathematics, 170, 418–434.
- [36] Muscat, J. (2014). *Functional Analysis: An Introduction to Metric Spaces, Hilbert Spaces, and Banach Algebras*. Springer, Cham, Switzerland.
- [37] Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, San Diego, CA.
- [38] Rudin, W. (1991). *Functional Analysis* (2nd ed.). McGraw-Hill, New York.
- [39] Schiff, J. L. (1999). *The Laplace Transform: Theory and Applications*. Springer, New York, NY.